# QUANTUM THEORY OF FERMION SYSTEMS: TOPICS BETWEEN PHYSICS AND MATHEMATICS

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A pedagogical introduction to quantum models of fermions is given emphasizing the interplay between physics and mathematics. The aim is to explain physical ideas and mathematical tools by looking at simple examples. The topics discussed include fermion systems with a finite number of degrees of freedom, examples for the quantum field theory limit and regularization, the abstract formalism of quasi-free second quantization and some of its applications like the boson-fermion correspondence, loop algebras, and the Luttinger model.

## Contents

## 1 Introduction

These notes are intended as an introduction to quantum (field) theory and to (some) mathematical topics used to formulate and analyze simple fermion models relevant in physics, both for mathematics- and theoretical physics students. I should stress that this is not a systematic presentation of the subject but rather my personal (subjective!) selection of topics. My selection was guided by my strong belief that a good way to learn a complicated subject is to first look in detail at simple non-trivial examples where one can understand things explicitly.[1] After that it is often much easier to understand the general structure which underlies all these examples. I also hope that these notes reflect my view that the interplay between mathematics and physics can be very fruitful for both disciplines.

In my presentation I try to be mathematically precise and explicit (which I can be by restricting to rather simple examples), and I also try to explain the physical interpretation of the formalism presented. One of my goals is to give a physics- and mathematics introduction to **quasi-free second quantization of fermions**, a formalism which allows a mathematical precise formulation and treatment of certain quantum field theory models and, at the same time, plays a central role in the representation theory of certain infinite dimensional Lie algebras like the affine Kac-Moody algebras and the Virasoro algebra. I try to explain the physical ideas underlying this formalism, and to illustrate them in specific examples also making explicit the mathematical elegance of its general, abstract formulation. To substantiate my claim that this formalism is useful I also include short reviews of several seemingly very different research projects in which I have been involved and in which this formalism plays a central role. I have attempted to write these notes so that a student can work through them and acquire an active knowledge of the material treated. Due to limitations of space, these notes cannot be as detailed as a text book. Instead of writing '...as is easily seen...' and the like as a substitute for a detailed argument (which in fact might be 'easily seen' often only by an expert) I decided to clearly mark as '**Fact(s)**' all statements where beginners probably have to invest work to prove, verify, or at least fully understand them. Readers who do not find such a Fact obvious are strongly encouraged to work out a proof of it and/or look it up in the references given. I sometimes give some '*Hints*' outlining a proof (not so much for those Facts which are more difficult to prove, but rather for those where I believe that the proof is important to understand their significance or illustrates a useful calculation

[1] I learned this several years ago from Harald Grosse who was my PhD supervisor then.

tool). I use the symbol $\square$ to mark the ending of a Fact, Theorem, Remark etc. There are several remarks marked as '**Remark (adv.)**' which are either more advanced than the rest of the notes or intended for experts. I now describe in more detail the topics discussed in these notes. Fermion models are important in many different areas of physics. They are very successfully used to describe electrons in metals, protons in atomic nuclei, elementary particles in particle accelerators (like the ones at CERN in Geneva) to mention only a few examples. These and many other examples belong to the realm of **quantum physics**. Quantum systems with a *finite* number $N$ of degrees of freedom belong to the realm of **quantum mechanics**, and the mathematical framework for this is very well understood since quite some time. **Quantum field theory** models deal with quantum systems with an *infinite* number of degrees of freedom, $N = \infty$. Quantum field theory is mathematically much more challenging than quantum mechanics due to the occurrence of so-called **divergences**: the naive limit $N \to \infty$ often does not make sense, and one has to **regularize** in order to get meaningful answers.

The plan of these these lecture notes is as follows. In Section 2 I collect some prerequisites. Mathematical prerequisites include some facts about $*$-algebras and the theory of Hilbert space operators. The latter subject is essential to fully appreciate the quantum field theory case, but I stress that I tried to write these notes so as to make them accessible also for readers not familiar with the general Hilbert space theory. Section 2 also includes a review of quantum theory intended for mathematicians, and some preliminary discussion of the significance of the quantum field theory limit $N \to \infty$. Fermion quantum mechanics is discussed in Section 3, mainly to introduce the physics ideas needed in the simplest context possible. As an application I mention some basic facts about Hubbard-like models which are important in solid state physics and which can be formulated in the setting of fermion quantum mechanics. I also indicate the additional difficulties arising in the quantum field theory limit. Section 4 is devoted to the general, abstract formalism of quasi-free second quantization from a mathematical point of view. In Section 5 I discuss applications in the representation theory of the loop algebras and loop groups, and I show how the mathematical results in this setting give rise to intriguing identities which are known as the boson-fermion correspondence and which play a central role in the quantum field theory application of this formalism. In Section 6 it is shown how these results can be used to construct and solve the so-called **Luttinger model** which is a model of interacting(!) fermions moving in one dimensional continuum space and describing a one dimensional metal (e.g. electrons in a wire). I chose this example since it is the simplest non-trivial one I know, and it nicely illustrates one source for divergences in quantum field theory: they can be regarded as the price to pay for simplifying the description of a physical system by eliminating the details of its short distance structure. In this example I also show how the mathematical tools developed in the previous sections can be used to construct a quantum field theory model, and I sketch how the boson-fermion correspondence is used to completely solve this model. Finally, in Section 7 I briefly review several other developments.

**Remark:** I stress that limiting my discussion of quantum models to fermions is a severe restriction from the physics point of view: many phenomena in nature are described by **bosons**, e.g. gauge fields (which play a fundamental role in elementary particle physics) are bosons, and conventional quantum mechanics (as discussed in physics text books) can be regarded as quantum theory for bosons with a finite number of degrees of freedom. However, fermion models are, from a mathematical point of view, much simpler and therefore, as I believe, well suited for introducing quantum theory. Moreover, much of what I discuss here for fermions can be quite straightforwardly extended also to boson models (see Ref. $[R, GL1, GL2]$).

**Remark (adv.):** In the mathematical theory of infinite dimensional spaces topology plays an important role. We will encounter several such spaces in the examples discussed in these notes. Still, I will avoid to discuss topological questions (and not even define the topology in many cases) since this would be beyond the scope of these lectures.

## 2 Preliminaries

The mathematical framework of quantum mechanics is Hilbert space theory, and an useful prerequisite for doing quantum mechanics is a good knowledge of the theory of operators on Hilbert spaces. The more fundamental objects in quantum physics are actually $*$-algebras[2] which nevertheless, can in many cases (including all examples I discuss), be represented by operators on a Hilbert space. In quantum field theory (but not quantum mechanics) this view becomes important since there one finds truly different representations of the $*$-algebras of interest.

In this Section I summarize some definitions and facts about $*$-algebras, Hilbert space theory, and quantum physics which I will use later. In particular, some of my notation used later is explained here. Of course, my presentation is too short to be in any way a substitute for systematic courses. I therefore stress that readers with no prior knowledge in Hilbert space theory should be able to understand a large part of these notes with some knowledge in linear algebra corresponding to the special case of finite dimensional Hilbert spaces $\mathbb{C}^N$ and accepting on trust the generalizations to general Hilbert spaces which I mention.

### *2.1 $*$-Algebras*

$*$-Algebras are an important mathematical structure in quantum physics and I therefore recall their definition in this section.

Let $\mathcal{A}$ be a complex algebra (i.e. for $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ there is a definition of $A + B$ (sum), $\lambda A$ (product with scalar) and $AB$ (product) so that the usual axioms hold). Then $\mathcal{A}$ is a $*$-algebra if there is an anti linear map $* : \mathcal{A} \to \mathcal{A}$ called *involution* so that $(AB)^* = B^* A^*$ and $(A^*)^* = A$ for all $A, B \in \mathcal{A}$. In the following we always assume that a $*$-algebra has a unit element which we denote as $\mathbf{1}$ (or $\mathbf{1}_{\mathcal{A}}$ if there is danger of confusion).

A simple example of $*$-algebras are algebras of (complex) $N \times N$ matrices (the natural $*$ is given the matrix adjungation). Other important examples are algebras of operators on Hilbert spaces which will be discussed in the next section.

**Remark (adv.):** Usually the $*$-algebras of interest in quantum physics come with some topology compatible with the algebraic structure, and one often requires that they are $C^*$-algebras, see e.g. $[BrR]$ for definition and discussion.

[2] usually equipped with some suitable topology

Even though the fermion field algebras naturally have the structure of a $C^*$-algebra and thus one can make use of the quite powerful (but mathematically advanced) theory of $C^*$-algebras, I will not do this for two reasons. Firstly, I believe that avoiding $C^*$-algebra arguments makes the subject easier to learn. Secondly, if one insists on $C^*$-algebra techniques, the treatment of quantum theory for bosons becomes very different from the fermion case, whereas the approach which I will discuss can be adapted to bosons in a straightforward manner, see e.g. [$GL1, GL2, L0$].

### *2.2 Operators on Hilbert spaces*

In this paragraph I collect some basic results of Hilbert space operator theory, see e.g. [$RS$]. Readers familiar with this subject can find a summary of my notation in the next paragraph. Readers not familiar with the terms in this paragraph can get acquainted with them in the rest of this section. For the latter readers I recommend to browse the complete section at first reading and go back to it later when needed.

**Notation:** If $\mathcal{H}$ is a separable Hilbert space then $\langle\cdot,\cdot\rangle$ is the inner product in $\mathcal{H}$ which we choose anti-linear in the first- and linear in the second argument.[3] We denote as $\mathcal{L}$ the space of all linear operators $\mathcal{H}\to\mathcal{H}$, and $\mathcal{B}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are the bounded, trace class and Hilbert Schmidt operators on $\mathcal{H}$, respectively. Moreover, Tr is the Hilbert space trace and $*$ the Hilbert space adjoint as usual. Throughout these notes $\mathbf{1}$ will be the identity operator. We will later have several different Hilbert spaces at the same time and then sometimes write $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ instead of $\langle\cdot,\cdot\rangle$ if there is danger of confusion. Similarly we will sometimes write $\mathbf{1}_{\mathcal{H}}$, $\mathrm{Tr}_{\mathcal{H}}$, $\mathcal{L}(\mathcal{H})$ etc. For $A\in\mathcal{L}$ and $f\in\mathcal{H}$ I often write $Af$ instead of $A(f)$. If $f_n$ and $g_n$ are vectors $\mathcal{H}$ and $\lambda_n$ complex numbers I write $A=\sum_n\lambda_n f_n\langle g_n,\cdot\rangle$ short for the operator defined as: $Af=\sum_n\lambda_n f_n\langle g_n,f\rangle$ $\forall f\in\mathcal{H}$.

(As mentioned, the rest of this section is an introduction to Hilbert space theory assuming only knowledge in linear algebra). A *basis* (= complete orthonormal basis) in $\mathcal{H}$ is a set $\{e_n\}_{n=1}^N$, $1\leq N\leq\infty$, of vectors in $\mathcal{H}$ such that $\langle e_m,e_n\rangle=\delta_{m,n}$ ($:=1$ if $m=n$ and $=0$ otherwise) for all $m,n$, and $\sum_{n=1}^N e_n\langle e_n,\cdot\rangle=\mathbf{1}$. **Fact:** $N$ is basis independent[4] and only depends on $\mathcal{H}$.$\square$ **Fact:** Given a basis $\{e_n\}_{n=1}^N$ one can represent all $f\in\mathcal{H}$ as $f=\sum_{n=1}^N f_n e_n$ with $f_n=\langle e_n,f\rangle$.$\square$ **Fact:** If $N<\infty$

[3] This is the convention used by most physicists and which unfortunately is different from what is common in the mathematics literature.

[4] i.e. it is the same for whatever basis one has

one can therefore naturally identify $\mathcal{H}$ with $\mathbb{C}^N$, and we write $\mathcal{H} \cong \mathbb{C}^N$.$\square$ In the following I first discuss the special case $\mathcal{H} = \mathbb{C}^N$ for finite $N$. I then discuss some of the caveats which arise if one tries to generalize the results there to the infinite dimensional case $N = \infty$.

### *2.2.1 Finite dimensional Hilbert spaces*

This section contains a summary of results from linear algebra formulated so that they can be generalized to general Hilbert spaces. The proofs of the 'Facts' should be easy and essentially only require getting used to my notation which is chosen to easily generalize to the infinite dimensional case.

The simplest example (sufficient for our discussion of fermion quantum mechanics) is $\mathcal{H} = \mathbb{C}^N$ with $\langle f, g\rangle_{\mathbb{C}^N} = \sum_{n=1}^N \overline{f_n} g_n$ where we write $\mathbb{C}^N \ni f = (f_1, \ldots, f_N)$, $f_n \in \mathbb{C}$, similarly for $g$, and the bar denotes complex conjugation. In this case, the distinction between the different operator algebras mentioned above is irrelevant, i.e. $\mathcal{L} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2$ if $N < \infty$.

Note that the set $\mathcal{B}$ of all linear operators on $\mathbb{C}^N$ has a natural structure of an algebra, in particular, there is a natural definition of a product of two linear operators on $\mathbb{C}^N$ (i.e. for $A, B \in \mathcal{B}$, $(AB)(f)) := A(B(f))$ for all $f \in \mathcal{H}$).

**Fact:** The *(Hilbert space) adjoint* $A^*$ of $A \in \mathcal{B}$ is defined by $\langle f, A^* g\rangle = \langle Af, g\rangle$ for all $f, g \in \mathcal{H}$. (*Hint:* You need to check that this indeed defines $A^*$.)$\square$ A linear operator $A$ on $\mathcal{H}$ is called *self-adjoint* if $A^* = A$, and it is called *unitary* if $AA^* = A^*A = \mathbf{1}$. **Fact:** With the $*$ operation $A \to A^*$, $\mathcal{B}$ naturally is a $*$-algebra.$\square$

The standard basis in $\mathbb{C}^N$ is defined as follows, $(e_n)_k = \delta_{n,k}$ for all $n, k = 1, \ldots, N$. However, there are many other examples for a basis which can be of interest in applications. **Fact:** Given a basis, we can write all $A \in \mathcal{B}$ as $A = \sum_{m,n} a_{mn} e_m \langle e_n, \cdot\rangle$, and the operator product thereby can be represented by matrix multiplication. Thus we get the **Fact:** $\mathcal{B}(\mathbb{C}^N)$ can be identified with the $*$-algebra of $N \times N$ matrices (with $(a^*)_{mn} = \overline{a_{nm}}$).$\square$

The importance of self-adjoint and unitary operators in quantum physics is closely related to the (generalization of the) following **Fact:** If $A \in \mathcal{B}$ is self-adjoint then there a basis $\{e_n\}_{n=1}^N$ and real numbers $\{\lambda_n\}_{n=1}^N$ such that $A = \sum_{n=1}^N \lambda_n e_n \langle e_n, \cdot\rangle$.$\square$ This result implies that functions of self-adjoint operators are well-defined: $f(A) := \sum_{n=1}^N f(\lambda_n) e_n \langle e_n, \cdot\rangle$ for any bounded $\mathbb{C}$-valued function $f$ on $\mathbb{R}$. Unitary operators have a similar representation, $U = \sum_{n=1}^N \lambda_n e_n \langle e_n, \cdot\rangle$, but here the $\lambda_n$ are *phases* i.e. complex numbers with absolute value equal 1. One can also easily prove the following

**Fact *:** For every self-adjoint operator $A$ on $\mathcal{H}$, $U(t) := e^{itA}$ exists and is unitary for all real $t$. Moreover, $U(0) = \mathbf{1}$ and $U(s)U(t) = U(s+t)$ for all $s, t \in \mathbb{R}$, and $\frac{\partial}{\partial t}U(t)f = iAU(t)f$ for all $f$ in the domain of definition of $A$. $\square$

One often refers to $e^{itA}$ as the *one-parameter family of unitary operators generated by* $A$.

For all $A \in \mathcal{B}$ and an arbitrary basis $\{e_n\}_{n=1}^N$ in $\mathbb{C}^N$ we can define the *trace* $\mathrm{Tr}(A) := \sum_{n=1}^N \langle e_n, Ae_n\rangle$ which is a linear map $\mathcal{B} \to \mathbb{C}$. One can easily prove the following **Facts:** This definition is basis independent, and the trace is *cyclic* i.e. $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for all $A, B \in \mathcal{B}$.$\square$ The generalization of this to infinite dimensions has several caveats which are of central importance in quantum field theory.

### *2.2.2 Infinite dimensional Hilbert spaces*

We now consider the case $N = \infty$. Many of the results above can be generalized to this case but this requires some care. For example, one if often interested in operators $A \in \mathcal{L}$ where $Af$ is not defined for all $f \in \mathcal{H}$ but only for $f$ in some subset $\mathcal{D}(A)$ called the *domain of* $A$. In particular, $A = B$ means not only that $Af = Bf$ for all $f \in \mathcal{D}(A)$ but also that $\mathcal{D}(A) = \mathcal{D}(B)$. Moreover, to define the sum and the operator product of two operators $A, B \in \mathcal{L}$ one then has to be careful about domains, and $\mathcal{L}$ is not an algebra in general. Also the definition of the adjoint $A^*$ of an operator $A \in \mathcal{L}$ etc. requires some care about domains. The notion of a trace also becomes more delicate (our definition of $\mathrm{Tr}(A)$ above now involves an infinite sum which will not converge in general, and even if it converges for some basis, the results will no longer be independent of this basis in general).

The strategy to generalize results valid for $\mathbb{C}^N$, $N$ finite, to general Hilbert spaces $\mathcal{H}$ is to restrict them to subsets of $\mathcal{L}$. **Fact:** One subset for which *all* definitions and results generalize in a simple manner is the *finite rank operators* $\mathcal{B}_0$ which is defined as the set of all operators of the form $A = \sum_{n=1}^M \lambda_n e_n \langle f_n, \cdot\rangle$ as above with $M$ *finite*, $\lambda_n \in \mathbb{C}$, and both $\{e_n\}_{n=1}^\infty$ and $\{f_n\}_{n=1}^\infty$ some basis.$\square$ Unfortunately, this set $\mathcal{B}_0$ is rather small and many operators of interest are not in $\mathcal{B}_0$. To define interesting larger sets of operators containing more operators of interest it is useful to consider *dense subsets* $\mathcal{D}$ of $\mathcal{H}$ which are such that, loosely speaking, every $f \in \mathcal{H}$ can be represented as a limit of a converging sequence $\{f_n\}_{n=0}^\infty$ where all $f_n \in \mathcal{D}$.[5] In particular, the *span* of any basis $\{e_n\}_{n=1}^\infty$ (i.e. the vector space generated by all finite linear combinations of vectors in this set) is such a dense set. Then one can define

[5] for a precise definition see $[RS]$

the set $\mathcal{B}(\mathcal{D})$ as the set of all operators which have $\mathcal{D}$ as a common dense invariant domain i.e. for all $f \in \mathcal{D}$, $Af$ and $A^*f$ exist and are vectors in $\mathcal{D}$ for all $A \in \mathcal{B}(\mathcal{D})$. **Fact:** For every dense domain $\mathcal{D}$, $\mathcal{B}(\mathcal{D})$ is a $*$-algebra (with product given by the operator product and the star operation by the operator adjoint).$\Box$ In particular, $\mathcal{B} = \mathcal{B}(\mathcal{H})$ are the so-called *bounded operators*.[6]

As mentioned, the definition of self-adjoint operators $A \in \mathcal{L}$ is now somewhat delicate but for *all* these operators one can prove the

**Spectral Theorem**: For any bounded function $f : \mathbb{R} \to \mathbb{C}$, the definition of $f(A)$ for self-adjoint finite rank operators $A$ can be naturally extended to all self-adjoint operators $A \in \mathcal{L}$ such that $f(A)^* = \bar{f}(A)$. In particular, $\chi_I(A)$ is well-defined for all characteristic functions $\chi_I$[7], and $\chi_{I_1 \cup I_2}(A) = \chi_{I_1}(A) + \chi_{I_2}(A)$ if $I_1 \cap I_2 = \emptyset$, and $\chi_I(A)^* = \chi_I(A)^2 = \chi_I(A)$.

*Proof:* see e.g. $[RS]$. $\Box$

One other important result for quantum theory is that Fact * above holds true in arbitrary Hilbert spaces $\mathcal{H}$ (this is essentially **Stone's Theorem**, see e.g. $[RS]$).

Other important subsets of $\mathcal{L}$ are defined such that they allow for a trace and/or that properties of the matrix trace remain true. To be specific, for $0 < p < \infty$, $\mathcal{B}_p$ is defined as the set of all $A \in \mathcal{L}$ for which[8] $\sum_{n=1}^{\infty} \langle e_n, (AA^*)^{p/2} e_n \rangle$ exists (i.e. is finite) for some basis $\{e_n\}_{n=1}^N$, and one can prove the **Fact** that this definition is basis independent$\Box$. In particular, $\mathcal{B}_1$ are called the *trace class operators* and $\mathcal{B}_2$ the *Hilbert Schmidt operators*. The name of $\mathcal{B}_1$ is due to the following **Fact:** The *Hilbert space trace* $\mathrm{Tr}(A) := \sum_{n=1}^N \langle e_n, Ae_n \rangle$ exists and is basis independent if and only if $A \in \mathcal{B}_1$.$\Box$ The other subsets $\mathcal{B}_p$ are important due to the following **Fact:** If $A \in \mathcal{B}_p$ and $A \in \mathcal{B}_q$ then $AB \in \mathcal{B}_r$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.$\Box$ In particular, the product of two Hilbert Schmidt operators is trace class. We also have the **Fact:** If $A \in \mathcal{B}$, $B \in \mathcal{B}_p$ for some $p < \infty$, and $AB$ and $BA$ are trace class then $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.$\Box$ Note also the **Fact:** $\mathcal{B}_p \subset \mathcal{B}_q$ if $p < q$, and in particular, all trace class operators are Hilbert Schmidt.$\Box$

[6] I abuse notation here: sorry.

[7] i.e. $\chi_I(x) = 1$ $(= 0)$ for $x \in I$ $(x \notin I)$ where $I$ is some open subset of $\mathbb{R}$

[8] One uses here the fact that for all $A \in \mathcal{L}$, $AA^*$ is self-adjoint and non-negative and thus $(AA^*)^{p/2}$ is well-defined.

### 2.2.3 Unitary representations of $*$-algebras

In physics one is usually interested in representations $\pi$ of a $*$-algebra $\mathcal{A}$ by operators on some Hilbert space $\mathcal{H}$ where $*$ is represented by the Hilbert space adjoint, $\pi(A^*) = \pi(A)^*$ for all $A \in \mathcal{A}$, and $\pi(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{H}}$. Such representations are called *unitary*.

**Remark:** Strictly speaking I should use a different symbol, e.g. $\dagger$, for the $*$-operaton in the $*$-algebra to distinguish it from the Hilbert space adjungation which I also denote as $*$. Then I could write the condition for a unitary representation as $\pi(A^\dagger) = \pi(A)^*$ for all $A \in \mathcal{A}$, which is somewhat clearer. This distinction between $*$-operation and Hilbert space adjoint is actually important in the boson case. However, every $*$-algebras discussed in these notes can be indentified with the simplest of its representations $\pi_0$ which is unitary, and thus it is safe to only use one symbol $*$.

**Remark:** In quantum mechanics, all unitary representations of the $*$-algebra of interest (see below) are unitarily equivalent, which essentially means that one can identify the $*$-algebra with (any of) its unitary representations. This is not the case in quantum field theory where truly different unitary representations of the relevant $*$-algebras exist. For pedagogical reasons I will always distinguish between the $*$-algebra and its unitary representation. □

## 2.3 Quantum physics in a nutshell

In this section I summarize the general framework and the postulates in quantum theory.

In a quantum physics model one usually has a separable Hilbert space $\mathcal{H}$ together with an $*$-algebra $\mathcal{A}$ of operators on $\mathcal{H}$ (with the $*$ operation given by the Hilbert space adjoint $*$). (Reader not familiar with Hilbert space theory can assume $\mathcal{H} = \mathbb{C}^N$, and operators on $\mathcal{H}$ are just complex $N \times N$ matrices.)

Vectors[9] in $\mathcal{H}$ represent possible **pure states** of the system, and selfadjoint elements in $\mathcal{A}$ represent **observables** i.e. a possible measurements on the system. We denote the algebra of self-adjoint elements in $\mathcal{A}$ as $\mathcal{O}$. Similarly, the unitary elements represent **(symmetry) transformations** on the system. $\mathcal{A}$ is called the **field algebra** and $\mathcal{O}$ the **observable algebra** of the model.

**Remark:** Even though only $\mathcal{O}$ has a (simple) physical interpretation, it is in many cases very useful to consider the larger algebra $\mathcal{A}$, for example, elements

[9] to be precise: equivalent classes of non-zero vectors which differ by a phase factor

in $\mathcal{O}$ can often be constructed from simpler, non-selfadjoint elements which are not in $\mathcal{O}$ but in $\mathcal{A}$. $\square$

In a specific **model** there is usually one preferred operator $H$ in $\mathcal{O}$ which is called **Hamiltonian** and which plays, at the same time, three special roles: firstly, it represents energy measurements, secondly, it defines the time evolution of the system, and thirdly, it determines the state of the system in thermal equilibrium. Moreover, in many models there is another special element $\mathrm{N} \in \mathcal{O}$ called **particle number operator** and which corresponds to measurements of the number of particles in the system. This interpretation is reasonable only if the eigenvalues of N are integers. A unitary element in $\mathcal{A}$ represents then a **symmetry** of the model if it commutes with $H$ and (if it exists) with N. Often one has symmetries represented by one-parameter families of unitary operators generated by some $A \in \mathcal{O}$ via Stone's theorem (cf. Fact *).

Basic postulates in quantum physics are the following:

**Postulate 1:** If the system is in the pure state represented by a non-zero $\psi \in \mathcal{H}$, then the expectation value of the observable $A \in \mathcal{O}$ (i.e. expected (average) value for the result of the corresponding measurement) is

$$< A >= \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle} \,. \tag{1}$$

In particular, $< \chi_{[a,b]}(A) >$ is the probablity for the measurement corresponding to the operator $A$ to give a result in the interval $[a, b]$ $(-\infty \leq a < b \leq \infty)$.

**Postulate 2:** If $\psi \in \mathcal{H}$ represents the state at time $t = t_0$, then the state at other times $t > t_0$ is

$$\psi = e^{-\mathrm{i}H(t-t_0)}\psi \tag{2}$$

provided the system is isolated in the time interval $[t_0, t]$.

**Postulate 3:** If the system is in thermal equilibrium and has a **temperature** $T \geq 0$, then the expectation value of the observable $A \in \mathcal{O}$ (i.e. expected (average) value for the result of the corresponding measurement) is[10]

$$< A >= \frac{\mathrm{Tr}(e^{-\beta(H-\mu\mathrm{N})}A)}{\mathcal{Z}} \tag{3}$$

[10]Here one implicitly assumes that $e^{-\beta(H-\mu\mathrm{N})}A$ is trace class which should be proven in a specific model.

with

$$\mathcal{Z} = \mathrm{Tr}(e^{-\beta(H-\mu \mathrm{N})}) \tag{4}$$

and $\beta = 1/T$ where $\mu$ is a real parameter which is determined by the requirement that the expectation value $< \mathrm{N} >$ of the particle number has a fixed, given value.

Note that $< \mathbf{1} > = 1$, and the normalization constant $\mathcal{Z}$ is usually referred to as **partition function**. Moreover, the constant $\mu$ is traditionally called **chemical potential**. Note also that

$$< \mathrm{N} > = \frac{1}{\beta} \frac{\partial}{\partial \mu} \log \mathcal{Z} \, . \tag{5}$$

It is important to note that typcially, a necessary condition for $\mathcal{Z}$ to exist is that the Hamiltonian is bounded from below i.e. $\langle \psi, H\psi \rangle \geq E_0 \langle \psi, \psi \rangle$ for all $\psi \in \mathcal{H}$ and some $E_0 > -\infty$ (this is a necessary condition for $e^{-\beta(H-\mu \mathrm{N})}$ to be trace class). If such an $E_0$ is also an eigenvalue of $H$ i.e. for some $\psi_0$, $H\psi_0 = E_0\psi_0$, then this corresponding eigenvector $\psi_0$ is called **ground state** of the model, and $E_0$ is the **ground state energy**. Many models of interest in physics (but not all!!) have a non-degenerate ground state $\psi_0$, i.e., at zero temperature ($\beta \to \infty$), $< A >$ in Eq. (3) reduces to $< A >$ in Eq. (1) with $\psi = \psi_0$.

**Remark:** The physicist's understanding of and intuition about these postulates is mostly based on (many) specific examples of experimental setups and experimental results, together with the corresponding specific quantum models modeling and explaining them. This is the subject of quantum mechanics courses.

**Remark:** My postulate 3 actually is usually only discussed in **statistical physics** i.e. systems with many degrees of freedom at finite temperature.

### *2.4 Quantum field theory limit*

One general aspect of quantum field theory is the occurence of divergencies, i.e. in formal computations one is lead to sums and integrals which do not converge. Over the years physicists have learnt to extract meaningful finite numbers from such expressions. The precedures used in this context are called **regularization** and **renormalization**, the latter being only necessary in more complicated quantum field theory models which are beyond the scope of these notes. Regularization, however, will be important for us, and I will illustrate its basic idea in a simple example.

The following section contains a few 'philosophical' remarks concerning quantum field theory. Section 2.4.2 contains the elementary examples on regularization mentioned, and in Section 2.4.3 I will outline the general strategy for constructing the quantum field theory models which I use in these lectures.

### *2.4.1 General remarks*

There are two different approaches to quantum field theory, one based on path integrals (a good introduction for physicists is e.g. [$Ra$]; for mathematicians I recommend [$S$]) and which today is by far dominating, and a more algebraic approach generalizing the Hilbert space approach to quantum mechanics (see e.g. [$Ha$]). The latter approach is difficult to use in practical computations and thus less popular today. One way to see the difficulty of quantum field theory in this latter approach is as follows. As mentioned, in quantum mechanics all Hilbert space representations of the $*$-algebra $\mathcal{A}$ of the model are essentially the same (i.e. they are unitarily equivalent), and it therefore does not matter which representation one uses. It is therefore easy to get the right setting for a given model, and the only (but in general quite demanding) task is to compute numbers which can be compared with experimental results, for example expectation values of obervables in the ground state etc.

This is different in quantum field theory where one first has to construct an appropriate Hilbert space representation of the algebra $\mathcal{A}$ of the model which is, in general, difficult and can be done only in special cases. But then, in the end, one needs to compute also numbers.

As a mathematician one would like to clearly separate these two steps, $(i)$ construct the appropriate Hilbert space represenation of the field algebra $\mathcal{A}$ of the model, $(ii)$ do computations in that setting. A 'pure' mathematician might (perhaps) not be very much interested in $(ii)$ since he/she would regard it as a problem in 'applied' mathematics (involving expansions, approximations and numerics). However, in many examples which are important in physics, steps $(i)$ and $(ii)$ cannot be clearly separated. To my opinion, this is an important lesson one learns from a very successful approach to quantum field theory called **perturbative renormalization** (see e.g. [$Co, S$]). I believe that this is one fundamental reason why the algebraic approach to quantum field theory is difficult.

The quantum field theory examples discussed in these notes are such that step $(i)$ can be done in a rigorous manner, without doing $(ii)$. These examples already show some of the complications of quantum field theory in a situtation where they can be resolved, and they also show that a Hilbert space approach (which I regard as a special case of the algebraic approach in the spirit of [$Ha$])

to constructing quantum field theory models non-perturbatively is possible and useful, even though the examples discussed (and actually all examples where the approach I use works) are quite simple from a physics point of view. However, I believe that some generalization of this approach should be useful for understanding more complicated quantum field theory models non-perturbatively.

### *2.4.2 Regularizing diverging series*

The basic idea of regularization is quite old and was used already in the mathematical theory of diverging series quite some time before quantum mechanics was discovered. For examples, the formulas[11]

$$\begin{aligned} 1+2+3+4+\cdots &= -\frac{1}{12} \\ 1^3+2^3+3^3+4^3+\cdots &= \frac{1}{120} \end{aligned} \tag{6}$$

can be found already in a famous letter which S. Ramanujan wrote to G.H. Hardy in 1913 (see [$BeR$]). I will now illustrate the basic idea of *regularization* by explaining one elementary method to make sense of these series.

I consider the series

$$S_N = \sum_{n=1}^{N} a_n$$

and allow for cases where the naive limit $N\to\infty$ does not make sense, e.g. $a_n = n^k$ with integer $k \geq -1$. In many such cases it is possible to extract from this limit finite numbers in a well-defined manner. A simple method for that is to introduce a *regularization parameter* $\varepsilon > 0$ and consider

$$S_N(\varepsilon) = \sum_{n=1}^{N} a_n e^{-n\varepsilon} \; .$$

In many cases (including $a_n$ which are polynomial in $n$), the limit $S_\infty(\varepsilon)$ exists for $\varepsilon > 0$, and we can expand $S_\infty(\varepsilon)$ in a series, $S_\infty(\varepsilon) = \sum_{k\in\mathbb{Z}} \hat{S}_\infty(k)\varepsilon^k$. In this way we can determine explicity the **diverging terms** $\sum_{k<0} \hat{S}_\infty(k)\varepsilon^k$ preventing us from taking the limit $\varepsilon \downarrow 0$, and after dropping these terms the

[11] I stress that the sums on the l.h.s. here are *regularized sums* the precise meaing of which has to be defined: the naive interpretation obviously does not make sense.

limit can be taken. We therefore *define*

$$\sum_{n=1}^{\infty}{}' \, a_n := \hat{S}_\infty(0) \tag{7}$$

where the prime here indicates that this defines a regularized sum. In this manner one can derive the following results which give a precise mathematical meaning to the Eqs. in (6):

**Facts:**

$$\sum_{n=1}^{\infty}{}' \, n = -\frac{1}{12}\,, \qquad \sum_{n=1}^{\infty}{}' \, n^3 = \frac{1}{120}\,. \tag{8}$$

*Hint:* To prove these formulas it is convenient to introduce the function $f(\varepsilon) = \sum_{n=1}^{\infty} e^{-n\varepsilon} = e^{-\varepsilon}(1-e^{-\varepsilon})^{-1}$ and obtain the $\varepsilon$-expansion of $\sum_{n=1}^{\infty} n^k e^{-n\varepsilon}$ by taking derivatives of the $\varepsilon$-expansion of $f(\varepsilon)$. □

**Remark:** Note that my definition in Eq. (7) is equivalent to the standard regularization of this limit by analytic continuation, $\hat{S}_\infty(0) = \mathrm{Res}(S_\infty(\varepsilon)\varepsilon^{-1}, \varepsilon = 0)$ (cf. K. Wojciechowski's lectures in the present volume). I also refer the readers to S. Paycha's lectures (also in this volume) concerning the explanation of diverging integrals. □

### *2.4.3 Field algebras and their represenations*

In this section I describe the strategy which I use in this notes to define and construct quantum models. First, a $*$-algebra $\mathcal{A}$ is defined by generators and certain relations amongst them. In all my examples this algebra is generated by fermion field operators obeying anticommutator relations. Even though this algebra is defined without reference to a representation, in all my examples $\mathcal{A}$ has a natural representation $\pi_0$ on some Hilbert space $\mathcal{F}$. A Model is given by a Hamiltonian, i.e., a self-adjoint element $H \in \mathcal{A}$. In case of fermion quantum field theory it is important to consider also other representations $\pi$ of $\mathcal{A}$, and the crucial point in a proper construction of a model is to find one such representation in which $\hat{H} = \pi(H)$ is a self-adjoint operator bounded from below. As I will discuss, there is a general method to find such a representation for models without interactions. This method also allows to construct certain interacting models, namely all those where there is a non-interacting model with a Hamiltonian $H_0$ and for which the proper representation $\pi$ for $H_0$ is such that also $\pi(H)$ is a self-adjoint operator bounded from below. Thus such a model has so 'mild' interactions that it still can be constructed in the Hilbert space of the corresponding non-interacting model.

## 3 Fermion quantum mechanics

### *3.1 The simplest fermion system I know...*

...can be characterized as the $*$-algebra $\mathcal{A}$ generated by the element $a$ obeying the following relations,

$$a^2 = 0, \quad \{a, a^*\} = \mathbf{1}_{\mathcal{A}}. \tag{9}$$

Note that this also implies $(a^*)^2 = 0$.

A simple representation $\pi_0$ of this algebra is given on the Hilbert space $\mathcal{F} = \mathbb{C}^2$ with the usual inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$:

$$a = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^* = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{10}$$

and $\mathbf{1} = \mathbf{1}_{\mathbb{C}^2}$ (= the $2 \times 2$ unit matrix). Note that I abuse notation here and, strictly speaking, I should write $\pi_0(a)$ and $\pi_0(a^*)$ instead of $a$ and $a^*$ in the previous equation. However, for simplicity in notation I identify $A$ and $\pi_0(A)$ for all $A \in \mathcal{A}$ here and in the following.

It is easy to see that this representation $\pi_0$ is unitary, i.e., $*$ in this representation is the matrix adjoint.

Note that the operator $\mathrm{N} = a^*a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is self-adjoint and has two eigenvectors, $e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with corresponding eigenvalues 0 and 1, $\mathrm{N}e_n = ne_n$ for $n = 0, 1$. Moreover, the two eigenvectors provide a basis in $\mathbb{C}^2$.

**Physical interpretation:**

- $e_0$ is the **vacuum state** i.e. the state without particle.
- $e_1$ is the state with one particle.
- N is the **particle number operator**.
- $a$ is the **particle annihilation operator** and $a^*$ is the **particle creation operator**. (Why? Noting that $ae_1 = e_0$ and $ae_0 = 0$ resp. $a^*e_0 = e_1$ and $a^*e_1 = e_0$ and recalling the above interpretation of the states $e_{0,1}$ these names are kind of natural, aren't they).

Note that we only have states with 0 or 1 particle and thus the **Pauli Principle** is obeyed: *no more than one fermion in one state.*

**Models:** In the present simple example there is not much room to define very interesting models. One kind of Hamiltonian we can consider is $H = \omega a^* a$ with real $\omega$. Since $H = \omega \mathrm{N}$, the eigenvectors of $H$ are just the vectors $e_n$ above: $He_0 = 0$ and $He_1 = \omega e_1$. Note that only for positive $\omega$ the zero particle state $e_0$ is also the ground state of our model, whereas for negative $\omega$ the groundstate is $e_1$ (for $\omega = 0$ one has obviously a degenerate groundstate).

It is instructive to compute the partition function Eq. (4) for the above model,

$$\mathcal{Z} = \sum_{n=0,1} \left\langle e_n, e^{-\beta(H-\mu\mathrm{N})} e_n \right\rangle_{\mathbb{C}^2} = \sum_{n=0,1} e^{-\beta(\omega n - \mu n)} = 1 + e^{-\beta(\omega-\mu)} \, .$$

From this we obtain

$$< \mathrm{N} > = \frac{1}{\beta}\frac{\partial}{\partial\mu} \log(1 + e^{-\beta(\omega-\mu)}) = \frac{1}{e^{\beta(\omega-\mu)} + 1}$$

which we can write as $< \mathrm{N} > = f_\beta(\omega-\mu)$ where $f_\beta$ is the **Fermi distribution function**,

$$f_\beta(E) = \frac{1}{e^{\beta E} + 1} \, . \tag{11}$$

**Another representation** of the field algebra $\mathcal{A}$ is given by

$$\pi(a) = \sigma_+ \equiv \psi, \quad \pi(a^*) = \sigma_- \equiv \psi^* \, . \tag{12}$$

Note that

$$\psi = \mathcal{U}^* a \mathcal{U} \tag{13}$$

where $\mathcal{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an unitary operator on $\mathcal{F}$: *the representations $\pi_0$ and $\pi$ are unitarily equivalent.*

### *3.2 Fermion systems with a finite number of degrees of freedom...*

...can be characterized by the $*$-algebra $\mathcal{A}_N$ generated by elements $a_j$, $j = 1, \ldots, N$, obeying the following canonical anticommutator relations (CAR),

$$\left\{a_j, a_k\right\} = 0, \quad \left\{a_j, a_k^*\right\} = \delta_{jk}\mathbf{1} \quad \forall j,k = 1,\ldots,N. \tag{14}$$

where $\mathbf{1}$ is the identity in $\mathcal{A}_N$. Note that this implies $\left\{a_j^*, a_k^*\right\} = 0$. In the following I state several 'Facts' which can be proven by (in general) simple computations which I only outline as 'Hints' but otherwise leave as an exercise to the reader.

**Fact:** The following defines a representation $\pi_0$ of the algebra $\mathcal{A}_N$ on the Hilbert space $\mathcal{F}_N = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N \text{ times}} \cong \mathbb{C}^{2^N}$,[12]

$$\begin{aligned} a_j &= \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{(j-1) \text{ times}} \otimes \sigma_- \otimes \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{(N-j) \text{ times}} \\ a_j^* &= \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{(j-1) \text{ times}} \otimes \sigma_+ \otimes \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{(N-j) \text{ times}} \end{aligned} \tag{15}$$

and $\mathbf{1} = \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{N \text{ times}}$ where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

*Hint:* Use

$$(b_1 \otimes \cdots \otimes b_N) \cdot (c_1 \otimes \cdots \otimes c_N) = b_1 \cdot c_1 \otimes \cdots \otimes b_N \cdot c_N$$

for $2 \times 2$-matrices $b_j$ and $c_j$. □

**Fact:** The representation $\pi_0$ above is unitary i.e. $*$ is the Hilbert space adjoint.
*Hint:* Recall that the inner product in $\mathcal{F}_N$ is given by

$$\langle f_1 \otimes \cdots \otimes f_N, g_1 \otimes \cdots \otimes g_N \rangle_{\mathcal{F}_N} = \langle f_1, g_1 \rangle_{\mathbb{C}^2} \cdots \langle f_N, g_N \rangle_{\mathbb{C}^2}$$

for all $f_j, g_j \in \mathbb{C}^2$. This implies that

$$(b_1 \otimes \cdots \otimes b_N)^* = b_1^* \otimes \cdots \otimes b_N^*$$

for all $2 \times 2$ matrices $b_j$. □

**Fact:** The vector

$$\Omega := \underbrace{e_0 \otimes \cdots \otimes e_0}_{N \text{ times}} \in \mathcal{F}_N \tag{17}$$

with $e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ obeys

$$a_j \Omega = 0 \quad \forall j = 1, \ldots, N. \tag{18}$$

□

[12] Again I identify $A$ and $\pi_0(A)$ for all $A \in \mathcal{A}_N$.

**Fact:** A basis in $\mathcal{F}_N$ is given by

$$F_{\mathbf{n}} = (a_1^*)^{n_1} \cdots (a_N^*)^{n_N}\Omega \quad \text{ for } \mathbf{n} = (n_1, \ldots, n_N), \quad n_j \in \{0,1\}. \tag{19}$$

*Hint:* Show that

$$F_{\mathbf{n}} = s_{\mathbf{n}} e_{n_1} \otimes \cdots \otimes e_{n_N}$$

with $e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $s_{\mathbf{n}}$ either 1 or $-1$. Use this to show that

$$\langle F_{\mathbf{n}}, F_{\mathbf{m}} \rangle_{\mathcal{F}_N} = \delta_{n_1,m_1} \cdots \delta_{n_N,m_N} \quad \forall \mathbf{n}, \mathbf{m} \in \{0,1\}^N . \tag{20}$$

Check that there are enough independent vectors $F_{\mathbf{n}}$ to span $\mathcal{F}_N$. □

**Fact:** The operators

$$\mathrm{N}_j \ := \ a_j^* a_j \tag{21}$$

obey $\mathrm{N}_j F_{\mathbf{n}} = n_j F_{\mathbf{n}}$ ($F_{\mathbf{n}}$ is defined in Eq. (19)), and the operator

$$\mathrm{N} \ := \ \sum_{j=1}^{N} \mathrm{N}_j \tag{22}$$

therefore obeys

$$\mathrm{N} F_{\mathbf{n}} = \left( \sum_{j=1}^{N} n_j \right) F_{\mathbf{n}} \quad \forall \mathbf{n} \in \{0,1\}^N . \tag{23}$$

□

**Physical interpretation:**

- In our system we have $N$ different 1-particle states. In the state $F_{\mathbf{n}}$ we have $n_j$ fermions in the 1-particle state '$j$', $j = 1, \ldots, N$.

- In particular, $\Omega$ is the **vacuum state** i.e. the 'empty' state without fermions.

- N is the **particle number operator**.

- $a_j$ and $a_j^*$ are the **particle annihilation operator** and the **particle creation operator** corresponding to the 1-particle state '$j$'.

Again we see the **Pauli Principle** at work: *no more than one fermion in one 1-particle state.*

**Fact:** Let $\mathcal{F}_N^{(n)}$ be the subspace of $\mathcal{F}_N$ spanned by the eigenvectors of N with eigenvalue $n$. The complex dimension of $\mathcal{F}_N^{(n)}$ is

$$\frac{N!}{(N-n)!n!} \; .$$

Moreover,

$$\mathcal{F}_N = \mathcal{F}_N^{(0)} \oplus \mathcal{F}_N^{(1)} \oplus \cdots \oplus \mathcal{F}_N^{(N)} = \bigoplus_{n=0}^{N} \mathcal{F}_N^{(n)} \tag{24}$$

i.e. the particle number operator gives a natural grading for the Hilbert space on which our system is represented. □

**Models:** In the present setting we can consider a large class of Hamitonians which are of interest for many physical applications. A very simple class of Hamiltonians is given by

$$H = \sum_{j=1}^{N} \omega_j a_j^* a_j \; , \quad \omega_1 \leq \omega_2 \leq \cdots \leq \omega_N \tag{25}$$

equal to $H = \sum_{j=1}^{N} \omega_j \mathrm{N}_j$. From our discussion above it follows that all the vectors $F_{\mathbf{n}}$ in Eq. (19) are also eigenvectors of $H$,

$$HF_{\mathbf{n}} = \sum_{j=1}^{N} \omega_j n_j F_{\mathbf{n}} \quad \forall \mathbf{n} \in \{0,1\}^N \; . \tag{26}$$

Thus, if all the $\omega_j$ are positive, then $\Omega$ obviously is also the groundstate of our model. However, this is not the case if some of the $\omega_j$ are negative.

**Facts:** If

$$\omega_1, \ldots, \omega_m < 0, \quad \omega_{m+1}, \ldots, \omega_N > 0 \tag{27}$$

then

$$\tilde{\Omega} \; := \; a_1^* \cdots a_m^* \Omega \tag{28}$$

is the ground state of our model, and the ground state energy is

$$E_0 = \sum_{j=1}^{m} \omega_j . \tag{29}$$

Moreover,

$$a_j^* \tilde{\Omega} = 0 \quad \text{for } j = 1, \ldots, m \,, \quad a_j \tilde{\Omega} = 0 \quad \text{for } j = m+1, \ldots, N \,. \tag{30}$$

*Hint:* To see this, convince yourself that the minumum of $\sum_{j=1}^{N} \omega_j n_j$, $\mathbf{n} \in \{0,1\}^N$, is $E_0$ and assumed for $n_1 = \cdots = n_m = 1$ and $n_{m+1} = \cdots = n_N = 0$. □

**Fact:** The partition function Eq. (4) for the model with the Hamiltonian Eq. (25) is given by

$$\mathcal{Z} = \prod_{j=1}^{N} \left(1 + e^{-\beta(\omega_j - \mu)}\right) \tag{31}$$

and < N > Eq. (5) is equal to

$$< \mathrm{N} >= \sum_{j=1}^{N} f_\beta(\omega_j - N) \tag{32}$$

with the Fermi distribution function defined in Eq. (11).
*Hint:* The proof of Eq. (31) is a simple but instructive computation which we therefore outline it here:

$$\mathcal{Z} = \sum_{\mathbf{n} \in \{0,1\}^N} \left\langle F_{\mathbf{n}}, e^{-\beta(\hat{H} - \mu \mathrm{N})} F_{\mathbf{n}} \right\rangle_{\mathcal{F}_N}$$

$$= \sum_{n_1, \ldots, n_N = 0,1} e^{-\beta \sum_{j=1}^{N} (\omega_j n_j - \mu n_j)} = \prod_{j=1}^{N} \sum_{n_j = 0,1} e^{-\beta(\omega_j n_j - \mu n_j)} =$$

$$\prod_{j=1}^{N} \left(1 + e^{-\beta(\omega_j - \mu)}\right) \,.$$

□

**Physical interpretation:** In the groundstate $\tilde{\Omega}$, all the 1-particle states with a negative energy, $\omega_j < 0$, are 'filled' and all 1-particle states with a positive energy, $\omega_j > 0$, are 'empty'. Moreover, for large $\beta$ (corresponding to low temperatures), $< N >$ is close to $m$.

The Hamiltonian Eq. (25) might look rather special. However, as I now show, it has already most of the essential features of a quite large class of

Hamiltonians, namely

$$H = \sum_{j,k=1}^{N} a_j^* D_{jk} a_k \; . \tag{33}$$

**Fact:** $H$ above is self-adjoint iff $D = (D_{jk})_{j,k=1}^{N}$ is a self-adjoint $N \times N$ matrix i.e. $D_{jk} = \overline{D_{kj}}$ for all $j, k$. □

**Remark:** We use the symbol '$D$' since later the corresponding object will be some Dirac operator.

Indeed, using the fact that every self-adjoint matrix $D$ can be diagonalized by a unitary matrix $U$, $D = U^* \omega U$, where $\omega = diag(\omega_1, \ldots, \omega_N)$, we can reduce the present case to what we discussed after Eq. (25): Using the **Fact:** The operators

$$\tilde{\pi}(a_j) = \sum_{k=1}^{N} U_{jk} a_k \; =: \; \tilde{a}_j \; , \quad \tilde{\pi}(a_j^*) = \sum_{k=1}^{N} a_k^* \overline{U_{kj}} \; =: \; \tilde{a}_j^* \tag{34}$$

give a unitary representation of the algebra $\mathcal{A}_N$□, we can write

$$H = \sum_{j=1}^{N} \omega_j \tilde{a}_j^* \tilde{a}_j \; , \tag{35}$$

and our discussion implies that all eigenstates of the model are given by

$$\tilde{F}_{\mathbf{n}} = (\tilde{a}_1^*)^{n_1} \cdots (\tilde{a}_N^*)^{n_N} \Omega \, , \tag{36}$$

with $H\tilde{F}_{\mathbf{n}} = \sum_{j=1}^{N} \omega_j n_j \tilde{F}_{\mathbf{n}}$, and these also form a basis in $\mathcal{F}_N$. Especially if Eq. (27) holds then

$$\tilde{\Omega} = \tilde{a}_1^* \cdots \tilde{a}_m^* \Omega \tag{37}$$

is the ground state of our system. Moreover, we have the **Fact:** Eqs. (31)–(32) also hold true in the present case.□ We thus can also solve models with Hamiltonians of the form Eq. (33) quite easily by diagonalizing the 1-particle Hamiltonian $D$.

As will be discussed in more detail later, it is natural to regard $H$ as the second quantization of the $N \times N$ matrix $D$. More generally, we can define for any $N \times N$ matrix $A$

$$d\Gamma(A) \; := \; \sum_{j,k=1}^{N} a_j^* A_{jk} a_k \tag{38}$$

i.e. $H = d\Gamma(D)$. One can prove several nice relations for these operators $d\Gamma(A)$ (see the Eqs. (56) later). Here we only mention the following interesting **Fact:**

$$\mathrm{Tr}_{\mathcal{F}_N}(e^{id\Gamma(A_1)} \cdots e^{id\Gamma(A_k)}) = \det(\mathbf{1}_{\mathbb{C}^N} + e^{iA_1} \cdots e^{iA_k}) \tag{39}$$

for all $k \in \mathbb{N}$ and self-adjoint $N \times N$ matrices $A_j$ where det is the usual matrix determinant here. (The proof of this is not easy, and we will not use this result later and only mentioned it here since the generalization of this relation can be used to relate our approach to quantum field theory to the one discussed in K. Wojciechowski's lectures).□

**Application in solid state physics. Hubbard-like models:** As we saw, models with Hamiltonians given in Eq. (33) are rather simple, as discussed. In physics one is often interested in more complicated models given by Hamiltonians

$$H = H_0 + V = \sum_{j,k=1}^{N} a_j^* D_{jk} a_k + \sum_{j,k,\ell,m=1}^{N} v_{jklm} a_j^* a_k^* a_\ell a_m \,. \tag{40}$$

with complex numbers $D_{jk}$ and $v_{jklm}$ so that $H$ is self-adjoint. The first term (i.e. the one quadratic in the $a^{(*)}$) is usually referred to as the **free part** of the Hamiltonian, and the second one is called the **interaction term**.

A simple example is the **one dimensional Hubbard model** defined by the Hamiltonian[13] $[Hu]$

$$H = -t \sum_{j=1}^{\ell} \sum_{\sigma=\uparrow,\downarrow} (a_{j+1,\sigma}^* a_{j,\sigma} + a_{j,\sigma}^* a_{j+1,\sigma}) + U \sum_{j=1}^{\ell} n_{j,\uparrow} n_{j,\downarrow}, \tag{41}$$

where $n_{j,\sigma} = a_{j,\sigma}^* a_{j,\sigma}$ , $N = 2\ell$ , $a_{N+1,\sigma}^{(*)} \equiv a_{1,\sigma}^{(*)}$ ,

$$a_{j,\uparrow}^{(*)} \equiv a_j^{(*)}, \quad a_{j,\downarrow}^{(*)} \equiv a_{\ell+j}^{(*)} \quad \text{for } j = 1, \ldots, \ell$$

and $t$ and $U$ are real parameters. The physical interpretation of this model is as follows: One has a one dimensional lattice with $\ell$ sites and periodic boundary conditions. On each site $j = 1, \ldots, \ell$ one has fermions with internal spin i.e. one has two one-particle states '$\uparrow$' and '$\downarrow$'. The first term describes hopping of the fermion, e.g. the term $a_{j+1,\uparrow}^* a_{j,\uparrow}$ corresponds to a $\uparrow$-fermion hopping from site $j$ to site $j+1$ (i.e. a $\uparrow$-fermion on site $j$ is destroyed and a $\uparrow$-fermion on site $j+1$ is created) etc. Moreover, the interaction term describes a strongly screened Coulomb interaction of the fermions: $n_{j,\sigma}$ is the

[13]Our notation above would suggest to write $\mathrm{N}_{j,\sigma}$ rather than $n_{j,\sigma}$. The notation used here is standard in the solid state physics literature.

operator measuring the number of $\sigma$-fermions on the site $j$, thus (for $U > 0$) the last term in Eq. (41) above describes on-site Coulomb repulsion of $\uparrow$- and $\downarrow$-fermions.

For later reference I note the following simple **Fact:** For $U = 0$ (no interactions), the Hubbard Hamiltonian in Eq. (41) can be written as in Eq. (33) with a $2L \times 2L$-matrix $D$ which has the eigenvalues

$$E(p) = -2t\cos(p)\,, \quad p = (2n+1)\frac{\pi}{\ell}\,, n = -\frac{\ell}{2}, -\frac{\ell}{2}+1, \ldots, \frac{\ell}{2}-1 \quad (42)$$

(I assume $\ell$ even for simplicity) which all are double degenerate. (*Hint:* It is an instructive exercise to check that: Using $(\sigma, j)$ as matrix indices ($\sigma =\uparrow,\downarrow$, $j = 1, \ldots, \ell$) the matrix elements of $D$ are $(D)_{\sigma,j;\sigma',j'} = -2t\delta_{\sigma\sigma'}A_{jj'}$ where $A$ is the $\ell \times \ell$ matrix

$$A = \begin{pmatrix} 0\,1\,0 & 0 & \ldots 0 & -1 \\ 1\,0\,1 & 0 & \ldots 0 & 0 \\ 0\,1\,0 & 1 & \ldots 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0\,0\,0 \ldots & 0 & 1 & 0 \\ 0\,0\,0 \ldots & 1 & 0 & 1 \\ -1\,0\,0 \ldots & 0 & 1 & 0 \end{pmatrix}$$

which can be diagonalized by the following ansatz for the components of eigenvectors, $e_j = ae^{ipj} + be^{-ipj}$.)$\square$

Similarly, the Hamiltonian for the two dimensional Hubbard model describes fermions with internal spin on a two dimensional lattice $\{(j,k) \in \mathbb{Z}^2 | j = 1, \ldots, \ell_1, k = 1, \ldots, \ell_2\}$ with $\ell_1\ell_2$ sites, periodic boundary conditions, hopping between nearest neighbor sites and interacting via on-site Coulomb repulsion.

**Remark:** Our discussion above makes clear that conceptually, all the models defined by Hamiltonians $H$ in Eq. (40) are rather trivial: $H$ can be represented by a $2^N \times 2^N$-matrix, and the models can be solved simply by diagonalizing this matrix. However, from a practical point of view, these models can be very complicated, and e.g. for the two dimensional Hubbard model very little is known up to this day (even though this model has been studied very intensively in the past 10 years or so $[D, cond-mat]$). One reason is that one is often interested in such models on rather large lattices, but the size of matrices grows exponentially with the number of lattice sites and their numeric diagonalization very soon is beyond the abilities of the most advanced computers. Our discussion here also illustrates why models with Hamiltonians Eq. (33)

without interactions are so much simpler: as discussed above, if all the $v_{jklm}$ are zero in Eq. (40), the model can be completely solved by diagonalizing the $N \times N$ matrix $(D_{jk})$. Otherwise one has to diagonalize (or at least find the eigenstate with the minimum eigenvalue) of the $2^N \times 2^N$-matrix $H$ which is 'exponentially more difficult'. □

### *3.3 Quantum field theory limit: Heuristics*

In physics one is often interested in fermion systems with an infinite number of degrees of freedom. The simplest example showing already the main problem is the model of **chiral fermions on the circle** formally given by the Hamiltonian (cf. Eq. (25))

$$H = \sum_{n \in \mathbb{Z}} n\, a_n^* a_n \tag{43}$$

where the $a_n^{(*)}$ obey CAR as in Eq. (14), but now for all $j, k \in \mathbb{Z}$.

Applying naively our results in the previous section to the present model we would deduce that the ground state is (cf. Eq. (28))

$$\tilde{\Omega} = \text{``}a_{-1}^* a_{-2}^* \cdots a_{-1001}^* \cdots \Omega\text{''} \tag{44}$$

(which is degenerate with $a_0^* \tilde{\Omega}$, of course, but this degeneracy is not important for what we want to say here and thus will be ignored) where the quotation marks indicate that this is only a formal expression. The corresponding ground state energy should then be (cf. (29))

$$E_0 = -\text{``}(1 + 2 + \ldots + 1001 + \ldots)\text{''} \tag{45}$$

which clearly is divergent.[14] One therefore has a serious problem: the Hamiltonian of the model is not bounded from below, the partition function (cf. Eq. (31))

$$\mathcal{Z} = \text{``} \prod_{n \in \mathbb{Z}} \left(1 + e^{-\beta(n-\mu)}\right) \text{''}$$

does not exist (the infinite product diverges) etc. As will be made clear later in these notes, the main problem is that the ground state of the model (even though formally defined in Eq. (44)) is not a vector in the Hilbert space where we have defined the model.

The physical idea to remedy this problem is due to Dirac: work in a different Hilbert space where one makes sure from the start that the ground

[14] After reading Section 2.4.2 you might guess $E_0 = \frac{1}{12}$, right? Good guess ;-) but to justify this is a longer story; see e.g. [*Le*].

state is present. This can be done by using the fermion field algebra as a starting point and by representing it such that

$$a_n^* \tilde{\Omega} = 0 \quad \text{for } n = -1, -2, \ldots, -1001, \ldots, \quad a_n \tilde{\Omega} = 0 \quad \text{for } n = 0, 1, 2, \ldots .$$

as suggested by Eq. (30). Requiring that $\left\langle \tilde{\Omega}, \tilde{\Omega} \right\rangle = 1$, the full Hilbert space is then generated by the states

$$\prod_{n=1}^{\infty} (a_{n-1}^*)^{k_n} (a_{-n})^{\ell_n} \tilde{\Omega}$$

where $k_n, \ell_n \in \{0, 1\}$ are such that $\sum_{n=1}^{\infty}(k_n + \ell_n) < 0$.

The physical motivation for remedying the problem with the diverging ground state energy is as follows: all one can ever measure in a physical system are energy **differences**, thus we may equally well work with the Hamiltonian $\hat{H} = H - E_0 \mathbf{1}$. This enforces that $\tilde{\Omega}$ has finite energy (namely zero). It can be achieved in a simple manner by defining **normal ordering** $: \cdots :$ which is linear and such that $: a_n^* a_n := a_n^* a_n$ if $n \geq 0$ and $: a_n^* a_n := -a_n a_n^*$ if $n < 0$. One then can write

$$\hat{H} = \sum_{n \in \mathbb{Z}} n : a_n^* a_n := \sum_{n=0}^{\infty} n \left( a_n^* a_n + a_{-n-1} a_{-n-1}^* \right) \tag{46}$$

which clearly is positive definite. Similarly,

$$\hat{\mathrm{N}} = \sum_{n \in \mathbb{Z}} : a_n^* a_n := \sum_{n=0}^{\infty} \left( a_n^* a_n - a_{-n-1} a_{-n-1}^* \right) \tag{47}$$

is the operator measuring the particle number relative to the number of particles in the ground state.

The state $\tilde{\Omega}$ in this and similar examples is called the **filled Dirac sea** and is interpreted as the state in which an infinite number of negative one particle states is occupied. The fermions operators $a_n^*$ for $n \geq 0$ then create **particles**, whereas the operators $a_n$ for $n < 0$ create **anti-particles** (i.e. they remove negative-energy particles from the filled Dirac sea). Thus the operator $\hat{\mathrm{N}}$ measures the number of particles minus the number of anti-particles. Since particles and anti-particles have opposite charges it is natural to regard $\hat{\mathrm{N}}$ as **charge operator**.

**Remark:** This example is typical: the one particle Hamiltonian for relativistic fermions always is some Dirac operator which is neither bounded from above nor below. Therefore the one particle model of relativistic fermions never is physically acceptable (there is no ground state etc.). In the next

Chapter we will develop a general, abstract formalism allowing to construct a physically acceptable many particle description. This formalism generalizes and gives a precise mathematical meaning to the physical ideas which I tried to explain in this section. $\square$

## 4 Quasi-free second quantization of fermions

In this Chapter we develop the general, abstract formalism for constructing quantum models for fermions. This formalism amounts to the following: Given the one-particle description of the fermions, construct the description of the corresponding many particle system i.e. the system with an arbitrary number of identical fermions each of which is described by the given one particle formalism. In physics this is called **second quantization**.

In the first two sections we describe the general construction of the fermion field algebra, the many particle Hilbert space, and the second quantization of observables and transformations. In Section 4.3 I show how to recover our previous results on fermion quantum mechanics from the general formalism and the assumption that one has $N < \infty$ one particle degrees of freedom. At the end of this section I also indicate the problems one has if one tries to make the limit $N \to \infty$ and thereby motivate the necessity of the quasi-free representations which are discussed in general in Section 4.4.

The material covered in the section is standard. It might be helpful for some readers to consult some other references with a different emphasis, e.g. $[A, CaR, KR, Mi, O, Wu]$.

### *4.1 Fermion field algebras and fermion Fock spaces*

Let $\mathcal{H}$ be a separable Hilbert space. The **fermion field algebra** $\mathcal{A} = \mathcal{A}(\mathcal{H})$ over $\mathcal{H}$ is defined as the $*$-algebra generated by elements $a^*(f)$ and $a(f) \equiv a^*(f)^*$ such that $f \to a^*(f)$ is linear, and the following CAR hold,

$$\{a(f), a(g)\} = 0, \quad \{a(f), a^*(g)\} = \langle f, g\rangle_{\mathcal{H}} \mathbf{1} \quad \forall f, g \in \mathcal{H} \,. \tag{48}$$

Note that this also implies $\{a^*(f), a^*(g)\} = 0$.

**Remark (adv.):** Usually one defines $\mathcal{A}(\mathcal{H})$ as $C^*$-algebra by also defining a norm via $||a^{(*)}(f)|| = \langle f, f\rangle_{\mathcal{H}}$ (see e.g. $[BrR]$). As mentioned, we will not use $C^*$-algebra arguments in the following. □

There is a simple representation $\pi_0$ of this algebra $\mathcal{A}$ on a Hilbert space $\mathcal{F} = \mathcal{F}(\mathcal{H})$ which is completely determined by the following conditions (as will be explained below): $(i)$ the representation $\pi_0$ is unitary, $(ii)$ there is a vector $\Omega$ in $\mathcal{F}$ such that

$$\langle \Omega, \Omega\rangle = 1 \tag{49}$$

and[15]

$$a(f)\Omega = 0 \quad \forall f \in \mathcal{H} \,. \tag{50}$$

The Hilbert space $\mathcal{F}(\mathcal{H})$ is called the **fermion Fock space** over $\mathcal{H}$, and $\pi_0$ is the so-called **Fock-Cook representation** of the fermion field algebra $\mathcal{A}(\mathcal{H})$.

I now sketch how one can reconstruct $\mathcal{F}$ from the conditions above: The $a(f)$ and $a^*(f)$ are operators on $\mathcal{F}$, thus all vectors[16]

$$f_1 \wedge \cdots \wedge f_n \,:=\, a^*(f_1)\cdots a^*(f_n)\Omega \tag{51}$$

with $f_j \in \mathcal{H}$, $n$ an arbitrary non-zero integer, are in $\mathcal{F}$. Then one can show that these vectors span $\mathcal{F}$, and the Hilbert space structure of $\mathcal{F}$ is determined by the relations above. In particular, one can prove the **Fact:** The above conditions imply

$$\langle f_1 \wedge \cdots \wedge f_n, g_1 \wedge \cdots \wedge g_m \rangle = \delta_{n,m} \det \begin{pmatrix} \langle f_1, g_1\rangle_{\mathcal{H}} & \ldots & \langle f_1, g_n\rangle_{\mathcal{H}} \\ \vdots & \ddots & \vdots \\ \langle f_n, g_1\rangle_{\mathcal{H}} & \ldots & \langle f_n, g_n\rangle_{\mathcal{H}} \end{pmatrix} \,. \tag{52}$$

*Hint:* Your can prove Eq. (52) by induction. One important step is the following computation (where one uses repeatedly $a(f_1)a(g_j)^* = \langle f_1, g_j\rangle_{\mathcal{H}}\, \mathbf{1} - a(g_j)^* a(f_1)$ to move $a(f_1)$ to the right until one can use $a(f_1)\Omega = 0$ and thus get rid of it)

$$\begin{aligned} \langle \Omega, a(f_n)\cdots a(f_1)a^*(g_1)\cdots a^*(g_m)\Omega\rangle &= \\ \langle f_1, g_1\rangle_{\mathcal{H}} \langle \Omega, a(f_n)\cdots a(f_2)a(g_2)^*\cdots a(g_m)^*\Omega\rangle &- \\ - \langle \Omega, a(f_n)\cdots a(f_2)a(g_1)^* a(f_1) a(g_2)^*\cdots a^*(g_m)\Omega\rangle &= \\ \ldots = \sum_{j=1}^{m} (-1)^{j-1} \langle f_1, g_j\rangle_{\mathcal{H}} \left\langle \Omega, a(f_n)\cdots a(f_2)a^*(g_1)\cdots \widehat{a(g_j)^*}\cdots a^*(g_m)\Omega \right\rangle & \end{aligned}$$

where $\widehat{a(g_j)^*}$ means that this factor is missing. $\square$

From this one can reconstruct the inner product in $\mathcal{F}$. It is also useful to observe the **Fact:** The conditions above imply

$$\begin{aligned} a^*(f) f_1 \wedge \cdots \wedge f_n &= f \wedge f_1 \wedge \cdots \wedge f_n \\ a(f) f_1 \wedge \cdots \wedge f_n &= \sum_{j=1}^{n} (-1)^{j-1} \langle f, f_j\rangle_{\mathcal{H}}\, f_1 \wedge \cdots \widehat{\wedge f_j} \wedge \cdots \wedge f_n \quad (n \geq 1)\,. \end{aligned} \tag{53}$$

[15]As before, I should write $\pi_0(a(f))$ in the following equation, but for simplicity in notation I identify $A$ and $\pi_0(A)$ for all $A \in \mathcal{A}$ in the following.
[16]You can regard the following equation as definition of a useful and suggestive notation.

□

**Remark (adv.):** I am a bit sloppy here, but if you know more about Hilbert space theory you can fill in the missing details (see e.g. $[BrR]$): denoting the vector space spanned by the elements $f_1 \wedge \cdots \wedge f_n$ Eq. (51) as $\wedge\mathcal{H}$, one can take Eq. (52) as a definition of a sesquilinear form on $\wedge\mathcal{H}$ and then prove that it has the properties of an inner product making $\wedge\mathcal{H}$ into a pre-Hilbert space (see e.g. $[RS]$). Then $\mathcal{F}$ is recovered as the norm completion of $\wedge\mathcal{H}$. □

**Remark (adv.):** My discussion of the representation $\pi_0$ here is similar to the one in most physics texts. A more elegant argument is as follows (see e.g. $[CaR]$): first construct the Hilbert space $\mathcal{F}$, and especially the vectors $\Omega$ and $f_1 \wedge \cdots \wedge f_n$, without reference to the fermion field algebra $\mathcal{A}$ and such that Eqs. (52) and (49) provide the definition of the inner product in $\mathcal{F}$. One then takes Eqs. (53) and (50) as a definition and proves that this defines a unitary representation $\pi_0$ of $\mathcal{A}$ on $\mathcal{F}$. □

### *4.2 Free second quantization of observables and symmetries*

Let $A$ be a bounded operator on the one particle Hilbert space $\mathcal{H}$. Then

$$d\Gamma(A)\Omega \,=\, 0 \tag{54}$$

and

$$d\Gamma(A) f_1 \wedge \cdots \wedge f_n \,:=\, \sum_{j=1}^{n} f_1 \wedge \cdots \wedge A f_j \wedge \cdots \wedge f_n \tag{55}$$

define an operator on $\mathcal{F}$ (since the vectors $\Omega$ and $f_1 \wedge \cdots \wedge f_n$ span $\mathcal{H}$). By simple computations one can prove the **Facts:**

$$\begin{aligned} [d\Gamma(A), d\Gamma(A)] &= d\Gamma([A,B]) \\ d\Gamma(A)^* &= d\Gamma(A^*) \end{aligned} \tag{56}$$

for all $A, B \in \mathcal{B}$. In particular, if $A$ is self-adjoint then $d\Gamma(A)$ is also self-adjoint.□

Let $U$ be an invertible, bounded operator on $\mathcal{H}$. Then

$$\Gamma(U)\Omega \,=\, \Omega \tag{57}$$

and

$$\Gamma(U) f_1 \wedge \cdots \wedge f_n \,:=\, U f_1 \wedge \cdots \wedge U f_n \tag{58}$$

define an operator on $\mathcal{F}$, and similarly as above one proves the **Facts:**

$$\begin{aligned}\Gamma(U)\Gamma(V) &= \Gamma(UV)\\ \Gamma(U)^{-1} &= \Gamma(U^{-1})\\ \Gamma(U)^* &= \Gamma(U^*)\end{aligned} \tag{59}$$

for all invertible $U, V \in \mathcal{B}$. In particular, if $U$ is unitary on $\mathcal{H}$ then $\Gamma(U)$ is unitary on $\mathcal{F}$. Moreover,

$$\Gamma(e^{iA}) = e^{id\Gamma(A)} \tag{60}$$

and

$$\Gamma(U)d\Gamma(A)\Gamma(U^{-1}) = d\Gamma(UAU^{-1}) \tag{61}$$

for all $A, U \in \mathcal{B}$, $U$ invertible. □

We also observe the **Facts:** We have

$$[d\Gamma(A), a^*(f)] = a^*(Af) \tag{62}$$

and

$$\Gamma(U)a^*(f)\Gamma(U)^{-1} = a^*(Uf) \tag{63}$$

for all $f \in \mathcal{H}$, $A, U \in \mathcal{B}$, $U$ invertible. Moreover, Eqs. (54) and (62) completely determine $d\Gamma(A)$, and Eqs. (57) and (63) completely determine $\Gamma(U)$.□

As will be demonstrated in specific examples later, for self-adjoint $A$ it is natural to regard $d\Gamma(A)$ as the many particle observable associated with the one particle observable $A$. Similarly, for unitary $U$, $\Gamma(U)$ is regarded as the many particle (symmetry) transformation associated with the one particle (symmetry) transformation $U$. $d\Gamma$ and $\Gamma$ are therefore often called the **second quantization map** for observables and (symmetry) transformations, respectively.

**Remark:** Note the **Fact:** The algebra $\mathcal{B}(\mathcal{H})$ is naturally a Lie algebra with involution $*$ and the Lie bracket given by the commutator (with a topology given by the operator norm).□ Since $A \to d\Gamma(A)$ obviously is linear, Eq. (56) shows that $d\Gamma$ provides a $*$ preserving representation of this Lie algebra $\mathcal{B}(\mathcal{H})$ on the fermion Fock space over $\mathcal{H}$. Similarly, one has the **Fact:** The set $\mathcal{I}$ of invertible operators in $\mathcal{B}$ is a Lie group with involution $*$□, and Eqs. (59) show that $\Gamma$ is a $*$-preserving representation of this Lie group. Moreover, Eq. (60) establishes a nice and natural relation between the former Lie algebra and the latter Lie group which motivates the notation $d\Gamma$ vs. $\Gamma$. For self-adjoint operators $A$, Eq. (60) also gives a nice relation of Stone's theorem on $\mathcal{H}$ and on $\mathcal{F}(\mathcal{H})$. $\square$

**Remark (adv.):** For infinite dimensional Hilbert spaces $\mathcal{H}$ the full story is a little more complicated since the $d\Gamma(A)$ are unbounded operators on $\mathcal{F}$ even if the operators $A$ are bounded on $\mathcal{H}$. However, it is quite safe to regard this as technicality which one can ignore since it is easy to construct a common, dense invariant domain for all operators $d\Gamma(A)$ and to establish Eq. (56) on this domain. Moreover, in physics application, one is often interested in $d\Gamma(A)$ for unbounded operators $A$ on $\mathcal{H}$. It is quite straightforward to extend the construction $d\Gamma$ to this situation. In particular, if $\mathcal{L}'$ is some algebra of (in general) unbounded operators on $\mathcal{H}$ with common, dense invariant domain $\mathcal{D}$ then Eq. (55) for $f_j \in \mathcal{D}$ makes sense and defines $d\Gamma(A)$ for all $A \in \mathcal{L}'$ on some common, dense invariant domain in $\mathcal{F}$ such that Eq. (56) etc. hold on this latter domain (for details see [$GL2$]). In particular, Eq. (60) holds true also for unbounded self-adjoint operators $A$ on $\mathcal{H}$. $\square$

### *4.3 Fermion quantum mechanics revisited*

We now show how to recover our results in Section 3.2 from the general formalism above.

**Facts:** Fermion quantum mechanics with $N$ one-particle degrees of freedom corresponds to the special case $\mathcal{H} = \mathbb{C}^N$, i.e. $\mathcal{A}_N = \mathcal{A}(\mathbb{C}^N)$ and $\mathcal{F}_N \cong \mathcal{F}(\mathbb{C}^N)$. Moreover, $d\Gamma$ as defined in Eq. (38) is the same as the one defined in Eq. (55).
*Hint:* To see this, write $\mathbb{C}^N \ni f = \sum_{j=1}^N f_j e_j$ with $f_j \in \mathbb{C}$ and $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$ etc., and check that with $a^*(f) = \sum_{j=1}^N f_j a_j^*$ and $a(f) = \sum_{j=1}^N \overline{f_j} a_j$ the Eqs. (14) and (48) are equivalent. Moreover, the equivalence of the definitions of $d\Gamma(A)$ in Eqs. (38) and (38) can be proven by checking that both obey the relations (54) and (62). $\square$

Our solution of the model with the Hamiltonian in Eq. (33) can now be written as follows: write $D = \sum_{j=1}^N \omega_j \tilde{e}_j \langle \tilde{e}_j, \cdot \rangle$ where $\tilde{e}_j$ are the orthonormal eigenvectors and $\omega_j$ the corresponding eigenvalues of $D$. Then $H = \sum_{j=1}^N \omega_j a^*(\tilde{e}_j) a(\tilde{e}_j)$ i.e. $\tilde{a}_j = a(\tilde{e}_j)$. Thus the ground state of the model can be written as

$$\tilde{\Omega} = a^*(\tilde{e}_1) \cdots a^*(\tilde{e}_m)\Omega. \tag{64}$$

The eigenvectors $\{\tilde{e}_j\}_{j=1}^m$ corresponding to negative eigenvalues $\omega_j$ of $D$ obviously span a subspace $\mathcal{H}_-$ of $\mathcal{H} = \mathbb{C}^N$. One can interpret $\mathcal{H}_-$ as the **negative energy subspace** corresponding to the operator $D$. We now observe the

**Fact:** The ground state of the model only depends on the negative energy subspace $\mathcal{H}_-$ of $\mathbb{C}^N$, i.e. for any basis $\{E_j\}_{j=1}^m$ in $\mathcal{H}_-$, the vector

$$\tilde{\Omega}' = a^*(E_1) \cdots a^*(E_m)\Omega$$

is equal to $\tilde{\Omega}$ in Eq. (64) up to a phase.
*Hint:* Write $E_j = \sum_{k=1}^m V_{jk}\tilde{e}_k$ where $V$ is some unitary $m \times m$ matrix. Then

$$\tilde{\Omega}' = \sum_{k_1,\ldots,k_m=1}^{m} V_{1k_1} \cdots V_{mk_m} a^*(\tilde{e}_{k_1}) \cdots a^*(\tilde{e}_{k_m})\Omega = \det(V)\tilde{\Omega}$$

where in the last step one uses that the $a^*(\tilde{e}_{k_j})$ anticommute with each other. $\square$

This last fact will be an important motivation for the construction in the next section.

### *4.4 Quasi-free representations*

Let $\mathcal{H}_-$ be some closed subspace of $\mathcal{H}$ such that the operator $P_-$ projecting onto $\mathcal{H}$ obeys $P_-^2 = P_-^* = P_-$. We write $P_+ = \mathbf{1} - P_-$ and $\mathcal{H}_+ = P_+\mathcal{H}$. Note that this gives a splitting of $\mathcal{H}$ in two orthogonal subspaces, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, i.e. every $f \in \mathcal{H}$ can be written as $f = f_+ + f_-$ with $f_\pm = P_\pm f \in \mathcal{H}_\pm$, and $\langle f_-, f_+\rangle_{\mathcal{H}} = 0$ (the latter follows from $P_-P_+ = P_+P_- = 0$).

As discussed, in the examples we have in mind $\mathcal{H}_-$ is the negative energy subspace corresponding to some Dirac operator, but the following construction is completely general.

We now define a non-trivial representation of the fermion field algebra $\mathcal{A}$ on the fermion Fock space $\mathcal{F}$ corresponding to 'filling up the Dirac sea' i.e. the states in the subspace $\mathcal{H}_-$, as discussed in Sections 3.3 and 4.3 above. For that we need an antilinear operator $J : \mathcal{H} \to \mathcal{H}$ commuting with $P_-$ and obeying $J^2 = \mathbf{1}$ and $\langle f, g\rangle_{\mathcal{H}} = \langle g, f\rangle_{\mathcal{H}}$ for all $f, g \in \mathcal{H}$. (In the following the explicit form of $J$ will play no role, but for the sake of completeness we note the following **Fact:** Let $\{e_n\}_{n\in\mathbb{Z}}$ be any basis in $\mathcal{H}$ such that $\{e_n\}_{n=-\infty}^0$ is a basis in $\mathcal{H}_-$, i.e. $P_- = \sum_{-\infty}^0 e_n \langle e_n, \cdot\rangle$. Then

$$Jf = \sum_{n\in\mathbb{Z}} e_n \overline{\langle e_n, f\rangle} \quad \forall f \in \mathcal{H}$$

defines an operator $J$ with the desired properties.□) We then define

$$\psi^*(f) := a^*(P_+f) + a(JP_-f)\ , \quad \psi(f) := a(P_+f) + a^*(JP_-f) \quad \forall f \in \mathcal{H} \tag{65}$$

where $a^{(*)}(f)$ are the fermion field operators in the Fock-Cook representation as defined above. We observe the following important

**Fact:** For every projection operator $P_-$, the operators in Eq. (65) define a unitary representation $\pi_{P_-}$ of the fermion field algebra $\mathcal{A}$ on the fermion Fock space $\mathcal{F}$,

$$\psi^{(*)}(f) = \pi_{P_-}(a^{(*)}(f))\ , \tag{66}$$

which is such that

$$\psi(f_+)\Omega = \psi^*(f_-)\Omega = 0 \quad \forall f_\pm \in \mathcal{H}_\pm\ . \tag{67}$$

*Hint:* One needs to check that $f \to \psi^*(f)$ is linear, $\psi(f) = \psi^*(f)^*$, and that the CAR in Eq. (48) are obeyed for the operators $\psi^{(*)}(f)$. This can be proved by simple computations using the corresponding properties of the operators $a^{(*)}(f)$. Eq. (67) follows trivially from Eq. (50). □

**Remark:** One can prove that the representations $\pi_{P_-}$, $P_-$ projection operators on $\mathcal{H}$, are irreducible (see e.g. [$BrR$]). □

**Remark:** As mentioned, in a specific model with a 1-particle Hamiltonian (Dirac operator) $D$, a natural choice is $P_- = \theta(-D)$ (defined via the spectral theorem) with $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. Then the state $\Omega$ above represents the **filled Dirac sea** of the model, and one can prove that it is indeed a ground state of the many particle Hamiltonian corresponding to $D$.

### *4.5 Quasi-free second quantization of observables*

We now discuss the second quantization of 1-particle operators in the quasi-free representation $\pi_{P_-}$ defined above.

In case of the Fock-Cook representation, the second quantization of 1-particle observables and transformations was always possible. In a quasi-free representation this is no longer the case: loosely speaking, the second quantization of a 1-particle operator exists only if it does not mix the subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ above 'too much'. To be more specific: the decomposition

$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of our one particle Hilbert space provides a natural decomposition of all operators $A$ on $\mathcal{H}$ as follows,

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} \tag{68}$$

where $A_{++} = P_+ A P_+$ maps $\mathcal{H}_+$ to $\mathcal{H}_+$, $A_{+-} = P_+ A P_-$ maps $\mathcal{H}_-$ to $\mathcal{H}_+$, etc. As we will explain in more detail below, the second quantization of a (bounded[17]) operator $A$ exists if and only if $A_{-+}$ and $A_{+-}$ are Hilbert-Schmidt operators (which loosely speaking means that these operators are 'not too big'). This is the celebrated **Hilbert-Schmidt condition** which we will encounter in several places.

**Remark:** The notation introduced in Eq. (68) is very useful in calculations. Note that in this notation,

$$P_+ = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} , \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} , \tag{69}$$

and the multiplication of operators is simply by the rules of $2 \times 2$ matrix multiplication.

For $A$ a bounded operator on $\mathcal{H}$, one would like to construct a self-adjoint operator $d\hat{\Gamma}_{P_-}(A) = d\hat{\Gamma}(A)$ on $\mathcal{F}$ such that

$$\left[ d\hat{\Gamma}(A), \psi^*(f) \right] = \psi^*(Af) \tag{70}$$

for all $f \in \mathcal{H}$, and

$$\left\langle \Omega, d\hat{\Gamma}(A)\Omega \right\rangle_{\mathcal{F}} = 0\,. \tag{71}$$

It turns our that if this operator exists, the relations in Eqs. (70)–(71) determine it uniquely, and a necessary and sufficient condition for existence is the above mentioned Hilbert-Schmidt condition. Thus the set of all operators allowing such a second quantization is

$$g_1 := \{ A \in \mathcal{B} \,|\, A_{-+}, A_{+-} \in \mathcal{B}_2 \} \,. \tag{72}$$

We now summarize the important properties of the quantization map $d\hat{\Gamma}$ in a

**Theorem:** Let $A \in \mathcal{B}$. Then a unique operator $d\hat{\Gamma}(A)$ on $\mathcal{F}$ obeying the relations (70)–(71) exists if and only if $A \in g_1$. Moreover, $A \to d\hat{\Gamma}(A)$ is linear, and the following relations hold true,

$$d\hat{\Gamma}(A)^* = d\hat{\Gamma}(A^*) \tag{73}$$

[17] This condition can be relaxed but then one has to cope with some additional technicalities.

and

$$\left[d\hat{\Gamma}(A), d\hat{\Gamma}(B)\right] = d\hat{\Gamma}([A,B]) + \hat{S}(A,B)\mathbf{1}_{\mathcal{F}} \tag{74}$$

where

$$\hat{S}(A,B) = \mathrm{Tr}_{\mathcal{H}}(A_{-+}B_{+-} - B_{-+}A_{+-}) , \tag{75}$$

and

$$d\hat{\Gamma}(A)\Omega = 0 \Leftrightarrow A_{-+} = 0 . \tag{76}$$

**Remark:** Note that Eqs. (74)–(75) show that the Hilbert-Schmidt condition is not just a technicality: the term $\hat{S}(A,B)$ has to be finite (otherwise one has a serious problem with the relation in Eq. (74)) and for that it is crucial that the operator $(A_{-+}B_{+-} - B_{-+}A_{+-})$ has a finite trace — this is (essentially) just the Hilbert-Schmidt condition. □

*Outline of Proof:* Instead of giving a full mathematical proof of this Theorem (which can be found e.g. in Ref. [$CaR$]) I present a more down-to-earth construction of $d\hat{\Gamma}(A)$ for a restricted set of operators $A$ which will (hopefully) give a more intuitive understanding of $d\hat{\Gamma}$, and especially of how the Schwinger term in Eq. (74) appears. For that I use a basis $\{e_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ such that $\{e_n\}_{n=-\infty}^{0}$ is a basis in $\mathcal{H}_-$, i.e. $P_- = \sum_{-\infty}^{0} e_n \langle e_n, \cdot\rangle_{\mathcal{H}}$. We can then write bounded operator $A$ on $\mathcal{B}$ as $A = \sum_{m,n\in\mathbb{Z}} a_{mn} e_m \langle e_n, \cdot\rangle_{\mathcal{H}}$ where $a_{mn} = \langle e_m, Ae_n\rangle_{\mathcal{H}}$. For simplicity (i.e. to avoid convergence problems in the following) we assume that $A$ is such that only a finite number of the $a_{mn}$ are non-zero. I will refer to such operators as *finite rank operators.*

I then define

$$\mathcal{Q}(A) := \sum_{m,n} a_{mn}\psi^*(e_m)\psi(e_n) ,$$

and by a simple computation (using the CAR) one can check the **Fact:**

$$[\mathcal{Q}(A), \psi^*(f)] = \psi^*(Af)$$

for all $f \in \mathcal{H}$.□ This shows that $\mathcal{Q}(A)$ is quite close to $d\hat{\Gamma}(A)$ but exactly so due to the following **Fact:**

$$\langle\Omega, \mathcal{Q}(A)\Omega\rangle = \mathrm{Tr}(A_{--})$$

with $\mathrm{Tr} = \mathrm{Tr}_{\mathcal{H}}$ the Hilbert space trace in $\mathcal{H}$. (*Hint:* This can be shown by a straightforward computation using that for all $f, g \in \mathcal{H}$,

$$\langle\Omega, \psi^*(f)\psi(g)\Omega\rangle = \langle\Omega, \psi^*(f_-)\psi(g_-)\Omega\rangle = \langle\Omega, \{\psi^*(f_-), \psi(g_-)\}\,\Omega\rangle = \langle g_-, f_-\rangle_{\mathcal{H}}$$

where I used Eqs. (67) and (49)). □

We thus see that the operator

$$d\hat{\Gamma}(A) \;:=\; \mathcal{Q}(A) - \mathrm{Tr}(A_{--})\mathbf{1}_{\mathcal{F}}$$

obeys both relations Eq. (70) and (71). A simple computations (using again the CAR) proves the **Fact:**

$$[\mathcal{Q}(A), \mathcal{Q}(B)] = \mathcal{Q}([A, B])$$

(similarly as for the operator $d\Gamma(A)$ is the Fock-Cook representation).□ With that we obtain $\left[d\hat{\Gamma}(A), d\hat{\Gamma}(B)\right] = [\mathcal{Q}(A), \mathcal{Q}(B)] = \mathcal{Q}([A, B])$, which implies Eq. (74) with

$$\hat{S}(A, B) = \mathrm{Tr}([A, B]_{--}) = \mathrm{Tr}(A_{-+}B_{+-} + A_{--}B_{--} - B_{-+}A_{+-} - B_{--}A_{--})\,.$$

The second and the forth term on the r.h.s. in this equation cancel due to cyclicity of the trace, and we obtain Eq. (75). It is also easy to check that Eq. (73) holds true.

We now have proven the Theorem for a (small) subset of $g_1$. As will be explained below, if we were only interested in this subset of operators, the Schwinger term $\hat{S}$ would be irrelevant. However, the construction of $d\Gamma(A)$ can be extended to all $A \in g_1$ so that all the relations derived here remain true, and then $\hat{S}$ becomes important. In particular, there are operators $A \in g_1$ such that neither $\mathrm{Tr}(A_{--})$ nor $\mathcal{Q}(A)$ exist (we will give a simple example for this below). Note that the argument above does not apply to such operators since $\mathrm{Tr}([A, B]_{--})$ will not exist in general, even though $\hat{S}(A, B)$ in Eq. (75) is well-defined (i.e. $[A_{--}, B_{--}]$ is not trace class, thus the equation $\mathrm{Tr}([A_{--}, B_{--}]) = 0$) used above does not make sense.)

As mentioned already above, Eq. (75) indicates that the Hilbert-Schmidt condition is necessary, so the non-trivial step in the proof of the Theorem is to show that it is also sufficient for $d\Gamma(A)$ to exist. The proof of this (see e.g. [$CaR$]) is beyond the scope of the present notes.

**Remark:** Note that if $\dim(\mathcal{H})$ is finite, the Hilbert Schmidt condition is empty: all operators can be second quantized. The same is true if $P_- = 0$ which obviously corresponds to the Fock-Cook representation: $d\hat{\Gamma}_0 = d\Gamma$. □

**Remark (adv.):** Again, the operators $d\Gamma(A)$ all are unbounded, but one can construct a common, dense invariant domain in $\mathcal{F}$ so that all the relations above hold on this domain [$CaR$]. Moreover, the construction of $d\hat{\Gamma}(A)$ can also be extended to certain Lie algebras of unbounded operators $A$ on $\mathcal{H}$ obeying the Hilbert-Schmidt condition [$GL2$], and this extension actually is required in several important applications. □

**Remark:** Observe the **Fact:** $g_1$ is a Lie algebra with the Lie bracket given by the commutator and involution $*$ .□ The Theorem above claims that $d\hat{\Gamma}$ is a $*$-preserving so-called **projective representation** of this Lie $*$-algebra on $\mathcal{F}$: it is not a representation due to the appearance of the term $\hat{S}(A,B)\mathbf{1}_{\mathcal{F}}$ which is called **Schwinger term** by physicists. From a mathematical point of view, $\hat{S}$ is a 2-cocycle of the Lie algebra $g_1$: The relations $\left[d\hat{\Gamma}(A), d\hat{\Gamma}(B)\right] = -\left[d\hat{\Gamma}(B), d\hat{\Gamma}(A)\right]$ and $\left[d\hat{\Gamma}(A), \left[d\hat{\Gamma}(B), d\hat{\Gamma}(C)\right]\right] + cycl. = 0$ (= Jacobi identity valid for commutators) imply the **Fact:**

$$\begin{aligned} \hat{S}(A,B) + \hat{S}(B,A) = 0 \\ \hat{S}(A,[B,C]) + \hat{S}(B,[C,A]) + \hat{S}(C,[A,B]) = 0 \quad \square \end{aligned} \tag{77}$$

which are called **2-cocycle relations**. Finally, the relation in Eq. (76) is called **highest weight condition**. In fact, $g_1$ is a prominent example of an infinite dimensional Lie algebra in which many of the well-known infinite dimensional Lie algebras can be naturally embedded, for example the loop algebras. One reason for the importance of the Theorem above is that it immediately gives interesting (and in fact very important) representations of infinite dimensional Lie algebras like the affine Kac-Moody algebras, the Virasoro-algebra[18] etc. We will come back to this in the next Chapter. □

**Remark:** There is one important conceptual point in the extension of $d\Gamma$ from finite rank operators to $g_1$ which I would like to stress: The normal ordering prescription I used to define $d\hat{\Gamma}(A)$ was just one amongst many possible, and another possible prescription would lead to a different second quantization map

$$d\hat{\Gamma}'(A) = d\hat{\Gamma}(A) - \hat{b}(A)\mathbf{1}_{\mathcal{F}} \tag{78}$$

with $\hat{b}$ some linear function $g_1 \to \mathbb{C}$ so that $\hat{b}(A^*) = \overline{\hat{b}(A)}$ for all $A \in g_1$. For example, we could have a model with a ground state $\Omega' \in \mathcal{F}$ different from $\Omega$, and then we would rather normal order with respect to $\Omega'$ i.e. use $d\hat{\Gamma}'$ above with $\hat{b}(A) = \left\langle \Omega', d\hat{\Gamma}(A)\Omega' \right\rangle$. **Fact:** $d\hat{\Gamma}'$ obeys relations as in Eqs. (73) and (74) with $\hat{S}$ replaced by $\hat{S}'$ where

$$\hat{S}'(A,B) = \hat{S}(A,B) + \hat{b}([A,B]) \; . \tag{79}$$

Moreover, $\hat{S}$ is a 2-cocycle if and only if $\hat{S}'$ is.□ It is thus natural to regard the two quantization maps $d\Gamma$ and $d\Gamma'$ as equivalent, and therefore the two

[18] This actually requires the extension of the Theorem to unbounded operators mentioned above.

2-cocycles $\hat{S}$ and $\hat{S}'$ related as in Eq. (79) are also *equivalent 2-cocycles*. Thus the Schwinger term we discuss here is interesting only because of the following **Fact:** The 2-cocycle $\hat{S}$ in Eq. (75) is non-trivial: there is no linear function $\hat{b}$ so that $\hat{S}'(A,B)$ Eq. (79) vanishes for all $A,B \in g_1$. (*Hint:* Convince yourself that this is proven if one can find two commuting operators $A_\pm$ in $g_1$ such that $\hat{S}(A_+,A_-)$ is different from zero. Show then that $A_\pm = \sum_{n\in\mathbb{Z}} e_{n\pm1}\langle e_n,\cdot\rangle_{\mathcal{H}}$ (notation as above) are two such operators.)□ It should be obvious now why the Schwinger term $\hat{S}$ is irrelevant for finite rank operators. This also explains why Schwinger terms are irrelevant in fermion quantum mechanics and becomes important only in quantum field theory. □

**Remark:** What we discussed in the previous two remarks is a special case of Lie algebra cohomology (see e.g. [$ChD$] for a concise introduction). □

**Remark:** In the physics literature one often writes

$$d\hat{\Gamma}(A) \;=\; \sum_{m,n} a_{mn} :\psi^*(e_m)\psi(e_n): , \tag{80}$$

where $:\cdots:$ is the normal ordering symbol mentioned already in Section (3.3) and which (more generally) is defined as the linear map such that

$$:\psi^*(e_m)\psi(e_n): := \begin{cases} -\psi(e_n)\psi^*(e_m) & \text{if } m=n<0 \\ \psi^*(e_m)\psi(e_n) & \text{otherwise} \end{cases} \tag{81}$$

equivalent with $:A: = A - \langle\Omega, A\Omega\rangle\, \mathbf{1}_{\mathcal{F}}$ (i.e. subtraction of the vacuum expectation value). As already mentioned, the Theorem above shows that $:A:$ can be well-defined even if $A$ and $\langle\Omega, A\Omega\rangle$ do not exist separately. A simple and generic example for this is for the operator

$$D = \sum_{n\in\mathbb{Z}} \lambda_n e_n \langle e_n,\cdot\rangle_{\mathcal{H}} \;, \quad \lambda_n \lesseqgtr 0 \text{ for } n \lesseqgtr 0 \tag{82}$$

and $\lambda_m \le \lambda_n$ for $m<n$. In this case, $\mathrm{Tr}(D_{--}) = \sum_{n<0}\lambda_n$ does not exist in general (the sum does not converge). However, $D_{-+} = D_{+-} = 0$ (i.e. both are trivially Hilbert Schmidt), and our theorem above[19] therefore implies that $d\hat{\Gamma}(D)$ is well-defined. Moreover,

$$d\hat{\Gamma}(D) = \sum_{n\in\mathbb{Z}} \lambda_n :\psi^*(e_n)\psi(e_n): = \sum_{n\ge0} |\lambda_n|\psi^*(e_n)\psi(e_n) + \sum_{n<0} |\lambda_n|\psi(e_n)\psi^*(e_n)$$

which shows explicitly that this operator is non-negative and $d\hat{\Gamma}(D)\Omega = 0$. □

[19]To be precise: its extension to unbounded operators; see [$GL2$].

**Remark:** The example in the last remark is actually very important from a physics point of view and leads us back to my discussion in Section 3.3: We can now see explicitly that the quasi-free representation can be chosen such that it obeys an essential physical requirement on any reasonable quantum model, namely the existence of a ground state. As discussed, in a specific model, the 1-particle Hamiltonian is given by self-adjoint $D$ on $\mathcal{H}$ and $d\hat{\Gamma}(D)$ represents the corresponding many particle observable Hamiltonian. Generically, $D$ is some Dirac operator on a compact spin manifold, and such an operator can always be represented as in Eq. (82) with some complete orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ and $\lambda_n \to -\infty$ for $n \to -\infty$. Such an operator does not have a ground state, and $d\Gamma(D)$ in the Fock-Cook representation does not have a ground-state, either. However, our discussion in the previous remark makes explicit that in the quasi-free representation $\pi_{P_-}$ with $P_- = \theta(-D) = \sum_{n<0} e_n \langle e_n, \cdot\rangle_{\mathcal{H}}$, the Hamiltonian $d\hat{\Gamma}(D)$ has a ground-state, namely $\Omega$.

I assumed here that $D$ has a pure point spectrum for simplicity, but the result holds actually true for all self-adjoint operators $D$. $\square$

### *4.6 Quasi-free second quantization of symmetries*

I now discuss the Lie group version of the results in the previous section. Much of the structure there is already contained in the Lie algebra, (which can be regarded as the infinitesimal version of the Lie group), but (as I will only indicate) there are a few features reflecting some interesting topology of the Lie group which is lost when going to the Lie algebra.

For unitary operators $U$ on $\mathcal{H}$, we are interested in a unitary operator $\hat{\Gamma}(U)$ on $\mathcal{F}$ such that

$$\hat{\Gamma}(U)\psi(f)\hat{\Gamma}(U)^* = \psi(Uf) \quad \forall f \in \mathcal{H}\,. \tag{83}$$

Using the results in the previous section it is easy to obtain such an operator $\hat{\Gamma}(U)$ if $U = e^{iA}$ with a self-adjoint $A \in g_1$: In this case, standard Hilbert space arguments imply the **Fact:** [20]

$$\hat{\Gamma}(e^{iA}) = e^{id\hat{\Gamma}(A)}\,. \tag{84}$$

(*Hint:* Existence of the r.h.s. of this follows from Stones Theorem since $d\hat{\Gamma}(A)$ is a self-adjoint operator on $\mathcal{F}$. Moreover, the identity

$$e^{itd\hat{\Gamma}(A)}\psi^*(f)e^{-itd\hat{\Gamma}(A)} = \psi^*(e^{itA}f)$$

[20]The results also holds for a large class of unbounded, self-adjoint operators $A$ obeying the Hilbert-Schmidt condition [$GL2$].

can be proven by expanding the l.h.s. in a Taylor series (in $t \in \mathbb{R}$) using repeatedly Eq. (70) which shows equality with the Taylor series of the r.h.s.)□ However, there are interesting operators $U$ for which $\hat{\Gamma}(U)$ exists but which are not of the form $e^{iA}$, $A \in g_1$. I will not go into this too much and only state the

**Theorem:** Let $U \in \mathcal{B}$ be unitary. Then a unitary operator $\hat{\Gamma}(U)$ obeying Eq. (83) exists if and only if $U$ is in the set

$$G_1 := \{U \text{ unitary on } \mathcal{H} \,| U_{-+}, U_{+-} \in \mathcal{B}_2 \} \ . \tag{85}$$

*Instead of a Proof:* I believe that this result is quite plausible by now, and for a proof I refer to Ref. [$R$] which also contains an explicit (and to my opinion very instructive) construction of the operators $\hat{\Gamma}(U)$ for all $U \in G_1$.

To understand the significance of this result it is useful to convince oneself by direct computations of the following **Facts:** If $A \in g_1$ is self-adjoint then $e^{iA} \in G_1$. Moreover, a simple example of an operator in $G_1$ not of this latter form is $U = \sum_{n\in\mathbb{Z}} e_{n+1} \langle e_n, \cdot\rangle_{\mathcal{H}}$.□ This later example actually is the simplest of its kind and the only one which I will later use (see [$R$] for the full story). □

**Remark:** Observe the **Facts:** The set $G_1$ is a group closed under $*$, and $\hat{\Gamma}(U)$ is uniquely determined by Eq. (83) up to a phase. In particular, if $\hat{\Gamma}(U)$ and $\hat{\Gamma}(V)$ exist also $\hat{\Gamma}(U^*)$ and $\hat{\Gamma}(UV)$ exist, and

$$\hat{\Gamma}(U^*) = \hat{\Gamma}(U)^* \ . \tag{86}$$

Moreover,

$$\hat{\Gamma}(U)\hat{\Gamma}(V) = \hat{\sigma}(U,V)\hat{\Gamma}(UV) \tag{87}$$

with $\hat{\sigma}$ a phase valued function on $G_1 \times G_1$ satisfying

$$\begin{aligned} \hat{\sigma}(U,V)\hat{\sigma}(UV,W) &= \hat{\sigma}(U,VW)\hat{\sigma}(V,W) \\ \hat{\sigma}(U^*,V^*) &= \overline{\hat{\sigma}(V,U)} \end{aligned} \tag{88}$$

for consistency.
*Hint:* To prove the group property of $G_1$, use that $A \in \mathcal{B}_2$ and $B \in \mathcal{B}$ implies $AB, BA \in \mathcal{B}_2$. To prove uniqueness up to a phase, show that if $\hat{\Gamma}(U)$ and $\hat{\Gamma}'(U)$ both obey Eq. (83), then $A = \hat{\Gamma}(U)^*\hat{\Gamma}'(U)$ commutes with all $\psi^*(f)$ and $\psi(f)$, $f \in \mathcal{H}$. Moreover, show that such operators $A$ necessarily are

proportional to the identity. A similar argument can be used to prove Eq. (87). $\square$

To give an explicit formula for $\hat\sigma$ one needs to fix the phase of $\hat\Gamma(U)$. A convenient choice is

$$\left\langle \Omega, \hat\Gamma(U)\Omega \right\rangle \quad \text{real and positive} \tag{89}$$

which, however, does not work for all $U \in G_1$ but only in some neighborhood $G_1^0 \subset G_1$ of the identity $[R]$. An explicit formula for the resulting 2-cocycle $\hat\sigma : G_1^0 \times G_1^0 \to \mathrm{U}(1)$ was derived in $[L1]$. We also note that $\hat\Gamma(e^{iA})$ is equal to $e^{id\hat\Gamma(A)}$ only up to some phase $e^{ib(A)}$, and an explicit formula for this phase (which has interesting physics applications) was also derived in Ref. $[L1]$.

**Remark:** As already mentioned, Eq. (83) determines the operator only up to a phase, i.e. up to to changes

$$\hat\Gamma(U) \to \hat\Gamma'(U) = \hat\beta(U)\Gamma(U) \tag{90}$$

where $\beta$ a phase valued function on $G_1$ such that $\beta(U^*) = \overline{\beta(U)}$. It is easy to see that such a transformation changes $\sigma$ in Eq. (87)

$$\hat\sigma(U,V) \to \hat\sigma'(U,V) = \hat\sigma(U,V)\frac{\hat\beta(U)\hat\beta(V)}{\hat\beta(UV)}\ . \tag{91}$$

Similarly as in the Lie algebra case, two 2-cocycles $\hat\sigma$ and $\hat\sigma'$ related as in Eq. (91) are called equivalent, and the 2-cocycle $\hat\sigma$ appearing in Eq. (87) is interesting only because it is non-trivial i.e. not equivalent to one which is identical to 1. Moreover, what I discussed here and in the previous remark is a special case of Lie group cohomology $[ChD]$.

## 5 Loop algebras and loop groups

The set of maps from the circle to some Lie algebra of matrices are infinite dimensional Lie algebras with a beautiful representation theory which is treated in several excellent text books, see e.g. $[K, Mi, PS]$. Historically, the developments of this theory went in parallel with interesting activities in theoretical physics, namely string theory and progress in the understanding of certain two dimensional quantum field theory models, and there was a fruitful interplay between these two subjects (see e.g. $[GO, CaR]$ for discussion and reference). In this section I discuss a few simple example from these developments.

In the following, $x \in [-\pi, \pi]$ is a coordinate on a circle of length $2\pi$. For $V = \mathbb{C}, \mathbb{R}, \mathrm{U}(1), \ldots$, we denote as $\mathcal{L}(V)$ the set of smooth (i.e. $C^\infty$) maps $S^1 \to V$, and we denote such maps as **loops**.

### *5.1 Loop algebra of complex valued maps on the circle*

The set $\mathcal{L}(\mathbb{C})$ of smooth maps $S^1 \to \mathbb{C}$ has a natural Lie algebra structure i.e. the linear structure and the Lie bracket can all be defined pointwise (e.g. the Lie bracket is $[\alpha_1, \alpha_2](x) := [\alpha_1(x), \alpha_2(x)] = 0$ for all $\alpha_j \in \mathcal{L}(\mathbb{C})$ etc.). This Lie algebra has an interesting central extension $\mathcal{L}(\mathbb{C})'$ which is the simplest examples of an **affine Kac-Moody algebra** and which I now define. As a set, $\mathcal{L}(\mathbb{C})' = \mathcal{L}(\mathbb{C}) \oplus \mathbb{C}$, i.e. elements in this set are pairs $(\alpha, a)$ with $\alpha \in \mathcal{L}(\mathbb{C})$ and complex numbers $a$. **Fact:** This set has a natural Lie algebra structure: the linear structure is obvious, and

$$[(\alpha_1, a_1), (\alpha_2, a_2)] := ([\alpha_1, \alpha_2], S(\alpha_1, \alpha_2)) = (0, S(\alpha_1, \alpha_2)) \qquad (92)$$

with

$$S(\alpha_1, \alpha_2) = \frac{1}{4\pi i} \int_{-\pi}^{\pi} dx \, \left( \frac{d\alpha_1(x)}{dx} \alpha_2(x) - \alpha_1(x) \frac{d\alpha_2(x)}{dx} \right) . \qquad (93)$$

defines a Lie bracket.□

**Remark:** What is interesting here is that $S$ defined in Eq. (93) is a non-trivial 2-cocylce of the Lie algebra $\mathcal{L}(\mathbb{C})$ (see my discussion on 2-cocycles in Section 4.4).□

#### *5.1.1 Wedge representation*

I now describe how the mathematical results in Section (4) can be used to construct interesting representations of the Lie algebra $\mathcal{L}(\mathbb{C})'$. We take as one particle Hilbert space $\mathcal{H} = L^2(S^1)$ with the basis

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{i(n+\frac{1}{2})x}, \quad n \in \mathbb{Z} \tag{94}$$

and define $P_- := \sum_{n<0} e_n \langle e_n, \cdot \rangle$.

**Remark:** Note that $P_- = \theta(-D)$ where $D$ is a self-adjoint extension of $-i\partial_x$ which is the (chiral) Dirac operator on the circle $S^1$. We will come back to this further below.

The loop algebra $\mathcal{L}(\mathbb{C})$ has a natural representation $r$ on $\mathcal{H}$. **Fact:** For every $\alpha \in \mathcal{L}(\mathbb{C})$, $(r(\alpha)f)(x) = \alpha(x)f(x)$ for all $f \in \mathcal{H}$, defines a bounded operator $r(\alpha)$ on $\mathcal{H}$. Moreover, $r([\alpha_1, \alpha_2]) = [r(\alpha_1), r(\alpha_2)]$.$\square$ In the following I will simplify notation and write $\alpha$ for $r(\alpha)$ (this is a common abuse of notation). It will also be convenient for us to decompose such maps $\alpha$ into positive, negative and zero Fourier components,

$$\alpha(x) = \alpha^+(x) + \alpha^-(x) + \bar{\alpha}$$
$$\alpha^\pm(x) = \frac{1}{2\pi} \sum_{\pm p > 0} \hat{\alpha}(p) e^{ipx}, \quad \bar{\alpha} = \frac{1}{2\pi} \hat{\alpha}(0) \tag{95}$$

where I use the following convention for Fourier transformation of loops,

$$\hat{\alpha}(p) = \int_{-\pi}^{\pi} dx\, \alpha(x) e^{-ipx} \quad p \in \mathbb{Z}\,.$$

A central result in the theory of loop groups is the following Corollary of Theorem 1 in Section 4.4:

**Theorem:** The map $(u, a) \to d\hat{\Gamma}(u) + a\mathbf{1}_{\mathcal{F}}$ provides a highest weight representation of $\mathcal{L}(\mathbb{C})'$ on the Fermion Fock space $\mathcal{F}(L^2(S^1))$, i.e.

$$\left[ d\hat{\Gamma}(\alpha_1), d\hat{\Gamma}(\alpha_2) \right] = S(\alpha_1, \alpha_2)\mathbf{1}_{\mathcal{F}} \tag{96}$$

with $S$ in Eq. (93), and

$$d\hat{\Gamma}(\alpha^-)\Omega = d\hat{\Gamma}(\alpha^+)^*\Omega = 0 \tag{97}$$

for all $\alpha, \alpha_j \in \mathcal{L}(\mathbb{C})$.

*Proof:* One needs to show that all $\alpha \in \mathcal{L}(\mathbb{C})$ define operators in $g_1$, and that

$$\mathrm{Tr}((\alpha_1)_{-+}(\alpha_2)_{+-} - (\alpha_2)_{-+}(\alpha_1)_{+-}) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} dx \left( \frac{d\alpha_1(x)}{dx} \alpha_2(x) \right)\,.$$

To prove these results it is convenient to use Fourier transformation. The crucial step is to show that

$$\mathrm{Tr}((\alpha_1)_\mp(\alpha_2)_\pm) = \sum_{m,n\in\mathbb{Z}} \theta(\mp(m+\tfrac{1}{2}))\theta(\pm(n+\tfrac{1}{2}))\hat{\alpha}_1(m-n)\hat{\alpha}_2(n-m) =$$
$$= \sum_{p\in\mathbb{Z}} \theta(\pm p)|p|\hat{\alpha}_1(-p)\hat{\alpha}_2(p)$$

which follows by straightforward (and instructive) computations. □

**Remark:** All results discussed here can be extended in a straightforward manner to the loop algebras $\mathcal{L}(\mathrm{gl}_N)$ of maps from the circle to the complex $N\times N$ matrices $\mathrm{gl}_N$. In this case, the Lie bracket is $[\alpha_1,\alpha_2](x) = \alpha_1(x)\alpha_2(x) - \alpha_2(x)\alpha_1(x)$ and is non-zero in general, and the 2-cocycle is given by

$$S(\alpha_1,\alpha_2) = \frac{1}{4\pi i}\int_{-\pi}^{\pi} dx\, \mathrm{tr}_N \left(\frac{d\alpha_1(x)}{dx}\alpha_2(x) - \alpha_1(x)\frac{d\alpha_2(x)}{dx}\right) \qquad (98)$$

where $\mathrm{tr}_N$ is the usual trace of $N\times N$ matrices (i.e. sum of diagonal elements). It is an instructive exercise to extend our discussion above in all detail to all these loop algebras. Note that one needs to show that $S$ obeys the 2-cocycle relation in Eq. (77). □

### *5.2 Loop group of maps from the circle to* U(1)

I now discuss the Lie group $\mathcal{L}(\mathrm{U}(1))$ of smooth U(1)-valued maps on $S^1$ which has $\mathcal{L}(\mathbb{R}) \subset \mathcal{L}(\mathbb{C})$ as Lie algebra. Much of the structure of $\mathcal{L}(\mathrm{U}(1))$ is a simple consequence of the structure of $\mathcal{L}(\mathbb{C})$, but there are some interesting additional features reflecting non-trivial topology appearing only on the Lie group level. **Fact:** Point-wise multiplication of loops, $(\varphi_1 \cdot \varphi_2)(x) := \varphi_1(x)\varphi_2(x)$, defines a natural group structure on $\mathcal{L}(\mathrm{U}(1))$. □ **Fact:** Each loop $\varphi \in \mathcal{L}(\mathrm{U}(1))$ can be written as

$$\varphi = e^{iu}, \quad u(x) = wx + \alpha(x) \qquad (99)$$

where $w = [u(\pi) - u(-\pi)]/2\pi$ is an integer called the **winding number**, and $\alpha \in \mathcal{L}(\mathbb{R})$ is a periodic function.□ The winding numbers $w \neq 0$ correspond to the structure of $\mathcal{L}(\mathrm{U}(1))$ which is not contained in $\mathcal{L}(\mathbb{C})$ (more precisely: $w$ labels the connected components of $\mathcal{L}(\mathrm{U}(1))$).

It is known that $\mathcal{L}(\mathrm{U}(1))$ has an interesting central extension $\mathcal{L}(\mathrm{U}(1))'$ which I now define: as a set, $\mathcal{L}(\mathrm{U}(1))'$ equals $\mathcal{L}(\mathrm{U}(1)) \times \mathrm{U}(1)$ i.e. its elements are pairs $(e^{iu},\gamma)$ where $\gamma$ is a phase and $e^{iu}\in\mathcal{L}(\mathrm{U}(1))$. Moreover, multiplication in $\mathcal{L}(\mathrm{U}(1))'$ is defined as follows,

$$(e^{iu_1},\gamma_1)\cdot(e^{iu_2},\gamma_2) \,:=\, (e^{iu_1}\cdot e^{iu_2}, e^{S(u_1,u_2)/2}\gamma_1\gamma_2) \tag{100}$$

with

$$\begin{aligned} S(u_1,u_2) = {} & \frac{1}{4\pi i}\left[u_1\left(\pi\right)u_2\left(-\pi\right)-u_1\left(-\pi\right)u_2\left(\pi\right)\right]+ \\ & +\frac{1}{4\pi i}\int_{-\pi}^{\pi}dx\left(\frac{du_1(x)}{dx}u_2(x)-u_1(x)\frac{du_2(x)}{dx}\right)\,. \end{aligned} \tag{101}$$

Note that this equals the 2-cocycle in Eq. (93) for loops with winding number zero, but for other loops there is a non-trivial boundary contribution. It is therefore not completely obvious that Eq. (100) defines a proper multiplication. However, one can convince oneself of the following important **Fact:** The product defined in Eq. (100) makes $\mathcal{L}(\mathrm{U}(1))'$ into a group. (*Hint:* One needs to check that 2-cycle relations similar to the ones in Eq. (88) hold true.) $\square$

We now sketch how the wedge representation of $\mathcal{L}(\mathbb{C})'$ discussed above can be extended to $\mathcal{L}(\mathrm{U}(1))'$. We first note the simple **Fact:** $(r(\varphi)f)(x) = \varphi(x)f(x)$ provides a representation $r$ of $\mathcal{L}(\mathrm{U}(1))$ on $L^2(S^1)$, and the $r(\varphi)$ all are unitary operators, $r(\varphi)^* = r(\overline{\varphi})$ for all $\varphi\in\mathcal{L}(\mathrm{U}(1))$.$\square$ Again, we simplify notation in the following and write $\varphi$ short for $r(\varphi)$.

It is easy to check the **Fact:** $(e^{i\alpha},\gamma)\to\gamma e^{id\hat{\Gamma}(\alpha)}$ provides a representation of the zero winding number loops in $\mathcal{L}(\mathrm{U}(1))'$.$\square$ To extend this representation to all of $\mathcal{L}(\mathrm{U}(1))'$ we only need to represent additionally the simplest non-zero winding number loop, i.e. $\varphi_1(x) = e^{ix}$.

**Facts:** $\varphi_1\in G_1$, and $R := \hat{\Gamma}(\varphi_1)$ therefore exists as unitary operator on $\mathcal{F}$. Moreover,

$$R^{-w}d\hat{\Gamma}(\alpha^{\pm})R^{w} = d\hat{\Gamma}(\alpha^{\pm}), \quad R^{-w}QR^{w} = Q + w\mathbf{1} \tag{102}$$

for all integer $w$ where

$$Q \,:=\, d\hat{\Gamma}(\mathbf{1})\,. \tag{103}$$

*Hint:* Check that $\varphi_1 = \sum_{n\in\mathbb{Z}} e_{n+1}\left\langle e_n,\cdot\right\rangle$, which is just the unitary operator discussed after the Theorem 2 in Section 4. Eq. (102) can be checked by using $R\psi^*(e_n)R^* = \psi^*(e_{n+1})$ and the notation introduced in Section (4), for example,

$$Q = \sum_{n\geq 0}\left(\psi^*(e_n)\psi(e_n) - \psi(e_{-n-1})\psi^*(e_{-n-1})\right)$$

etc. $\square$

From that we obtain the

**Theorem:** The map $(e^{iu},\gamma)\to\gamma\hat{\Gamma}(e^{iu})$ with

$$\hat{\Gamma}(e^{iu}) \; := \; e^{i\bar{\alpha}Q/2}R^{w}e^{i\bar{\alpha}Q/2}e^{id\hat{\Gamma}(\alpha^{+}+\alpha^{-})} \tag{104}$$

provides a unitary representation of the Lie algebra $\mathcal{L}(\mathrm{U}(1))'$ on the fermion Fock space $\mathcal{F}(L^2(S^1))$, i.e.

$$\hat{\Gamma}(e^{iu_1})\hat{\Gamma}(e^{iu_2}) = e^{S(u_1,u_2)/2}\hat{\Gamma}(e^{i(u_1+u_2)}) \tag{105}$$

with $S$ given by Eq. (101). Moreover,

$$\left\langle \Omega,\hat{\Gamma}(e^{iu})\Omega\right\rangle = \delta_{w,0}e^{-S(\alpha^{-},\alpha^{+})/2} \tag{106}$$

where

$$S(\alpha^{-},\alpha^{+}) = \sum_{p=1}^{\infty}\frac{p}{(2\pi)^2}\hat{\alpha}(-p)\hat{\alpha}(p)$$

is positive definite. □

*Proof:* Using

$$e^{a_1}e^{a_2} = e^{[a_1,a_2]/2}e^{a_1+a_2} \tag{107}$$

for $a_j = id\hat{\Gamma}(\alpha_j)$ together with Eq. (96), and $R^n e^{irQ}R^{-n} = e^{-inr}e^{irQ}$ for real $r$ and integer $n$, one obtains Eq. (105) with

$$S(u_1,u_2) = (w_1\bar{\alpha}_2 - \bar{\alpha}_1 w_2) + S(\alpha_1,\alpha_2)$$

identical with $S$ in Eq. (101). A similar computation implies Eq. (106). □

### *5.3 Boson-fermion correspondence*

The wedge representations of loop groups and algebras discussed above provide powerful mathematical tools to construct, analyze, and in some cases even solve non-trivial 1+1 dimensional quantum field theory models. In this section I discuss some results which play an important role in these developments.

I first recall that the quasi-free representation used to construct the wedge representations of $\mathcal{L}(\mathbb{C})'$ and $\mathcal{L}(\mathrm{U}(1))'$ has a simple physical interpretation: it describes a quantum field theory of chiral fermions in the circle $S^1$ with the 1-particle Hamiltonian $D=-i\partial_x$ which is the chiral Dirac operator. Formally, this model can be defined as follows,

$$\hat{H} = \int_{-\pi}^{\pi} dx\,\psi^{*}(x)(-i\partial_x)\psi(x) \tag{108}$$

with (formal) CAR $\{\psi(x), \psi^*(y)\} = \delta(x-y)$ etc., and anti-periodic boundary conditions, $\psi(-\pi) = -\psi(\pi)$. The quasi-free representation we discuss provides a rigorous construction of this model.

I now show that this quantum theory model of fermions is in fact equivalent to a quantum field theory model of **bosons**, i.e., quantum field operators with commutator (rather than anti-commutator) relations. This is actually a simple consequence of Eq. (96) but an important result for quantum field theory applications.

#### *5.3.1 Bosons from fermions*

We consider the representations of $\mathcal{L}(\mathbb{C})'$ and $\mathcal{L}(\mathrm{U}(1))'$ on the fermion Fock space $\mathcal{F} = \mathcal{F}(L^2(S^1))$ discussed above. Defining

$$\hat{\rho}(p) = d\hat{\Gamma}(\epsilon_p)\,, \quad \epsilon_p(x) = e^{-ipx}, \quad p \in \mathbb{Z}\,, \tag{109}$$

it is easy to see that the Theorem in Section 5.1.1 implies the **Facts:** $\hat{\rho}(-p) = \hat{\rho}(p)^*$,

$$[\hat{\rho}(p), \hat{\rho}(p')] = p\delta_{-p,p'}\,, \tag{110}$$

for all $p, p' \in \mathbb{Z}$, and

$$\hat{\rho}(p)\Omega = 0 \quad \forall p \geq 0\,.\Box \tag{111}$$

Note that $\hat{\rho}(0)$ is identical with $Q$ defined in Eq. (103). Due to the relations in Eq. (110) physicists interpret the $\hat{\rho}(p)$ as **boson field operators**.

**Remark:** Formally one can write

$$\rho(p) = \int_{-\pi}^{\pi} dx\, e^{-ipx} : \psi^*(x)\psi(x) :\,, \tag{112}$$

and since $\rho(x) =: \psi^*(x)\psi(x) :$ is what physicists call a **fermion current**, the result above means that the Fourier modes of the fermion currents are bosons.

The following seemingly technical result has then several important consequences:

**Lemma:** The vectors

$$B_\nu(\{n_p\}_{p\in\mathbb{N}}) := \left( \prod_{p=1}^{\infty} \frac{\hat{\rho}(-p)^{n_p}}{\sqrt{n_p! p^{n_p}}} \right) R^\nu \Omega\,,$$

$$n_p \in \mathbb{N}_0\,, \quad \nu \in \mathbb{Z}, \quad \sum_{p=1}^{\infty} n_p < \infty \tag{113}$$

provide a basis in the fermion Fock space $\mathcal{F}$.

*Comments:* I will not try to prove this result (see e.g. [$CaR$] and references therein) and only give a few comments. This lemma means that the loop operators $\hat{\Gamma}(\varphi)$, $\varphi \in \mathcal{L}(\mathrm{U}(1))$, span $\mathcal{F}$, or, put differently, that any operator on $\mathcal{F}$ can be expressed also in terms of the operators $\hat{\rho}(p)$ and $R$. This Lemma also provides a convenient tool to prove that two operators on $\mathcal{F}$ are equal: one only needs to prove that the two operators are the same on all vectors given in Eq. (113). I also note that the vectors defined by Eq. (113) are convenient in practical computations and an alternative to the fermion basis in the Fock space $\mathcal{F}$:

$$F(\{m_n^+, m_n^-\}_{n=0}^\infty) = \left( \prod_{n=1}^\infty \psi^*(e_n)^{m_n^+} \psi^*(e_{-n-1})^{m_n^-} \right) \Omega \,, \qquad m_n^\pm \in \{0,1\} \,, \ \sum_{n=0}^\infty (m_n^+ + m_n^-) < \infty \,. \tag{114}$$

I will give examples for this further below. □

An important result which can be proven using this Lemma is that the fermion Hamiltonian

$$\hat{H} = \sum_{n=1}^\infty (n + \tfrac{1}{2}) \left( \psi^*(e_n)\psi(e_n) - \psi^*(e_{-1-n})\psi(e_{-1-n}) \right) \tag{115}$$

can be expressed in a simple manner in terms of the boson operators. **Fact:**

$$\hat{H} = \tfrac{1}{2} Q^2 + \sum_{p=1}^\infty \hat{\rho}(-p)\hat{\rho}(p) \,. \tag{116}$$

(*Hint:* Check that the operators on the r.h.s. of Eqs. (115) and (116) both obey

$$[\hat{H}, \hat{\rho}(p)] = -p\hat{\rho}(p), \text{ and } \hat{H} R^\nu \Omega = \tfrac{1}{2}\nu^2 R^\nu \Omega \quad \forall p, \nu \in \mathbb{Z} \,,$$

which implies that they are equal on all vectors in Eq. (113).□) This striking result is usually referred to as **Kronig identity**, and it is a special case of the so-called Sugawara construction (see e.g. [$GO$]).

### *5.3.2 Fermions from bosons*

My discussion above implies that it should be possible to express the fermion field operators in terms of the boson operators $\hat{\rho}(p)$ and $R$. I now sketch how to obtain explicit formulas for that.

The idea is to construct loops $e^{iu_{y,\varepsilon}} \in \mathcal{L}(\mathrm{U}(1))$ so that $u_{y,\varepsilon}(x)$ approximates the step function interpolating between $-\pi$ and $\pi$ and jumping at the point $y \in S^1$. The role of $\varepsilon > 0$ is to provide a regularization (smoothing), i.e., $u_{y,0}(x) = \mathrm{sgn}(x-y)/\pi$ is the sharp (i.e. non-continuous) step function and $u_{y,\varepsilon}(x)$ is smooth for $\varepsilon > 0$.
**Fact:** A convenient choice for the $u_{y,\varepsilon}$ is

$$u_{y,\varepsilon}(x) = (x-y) + \alpha^+_{y,\varepsilon}(x) + \alpha^-_{y,\varepsilon}(x) \tag{117}$$

with

$$\alpha^{\pm}_{y,\varepsilon}(x) = \pm i \log(1 - e^{\pm i(x-y)-\varepsilon}) = \pm \sum_{n=1}^{\infty} \frac{1}{in} e^{\pm in(x-y)} e^{-\varepsilon n} \, . \tag{118}$$

(*Hint:* Check that $\partial_x u_{y,\varepsilon}(x)/(2\pi)$ equals $\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{in(x-y)-\varepsilon|n|}$ which is obviously an approximate (and periodized) $\delta$-function, and $u_{y,\varepsilon}(y) = 0$.)□ Note that the winding number of the $u_{y,\varepsilon}$ equals 1. By a straightforward computation one can check the **Fact:**

$$S(u_{y,\varepsilon}, u_{y',\varepsilon'}) = i\pi \mathrm{sgn}(y-y';\varepsilon+\varepsilon') \tag{119}$$

where $\mathrm{sgn}(y-y';\varepsilon+\varepsilon') = u_{y',\varepsilon}(y)/\pi$ is an approximate sign function.□ We thus obtain from Eq. (105) the **Fact:**

$$\hat{\Gamma}(e^{iu_{y,\varepsilon}})\hat{\Gamma}(e^{iu_{y',\varepsilon'}}) = e^{i\pi \mathrm{sgn}(y-y';\varepsilon+\varepsilon')}\hat{\Gamma}(e^{iu_{y',\varepsilon'}})\hat{\Gamma}(e^{iu_{y,\varepsilon}}) \, , \square$$

i.e., the operators $\hat{\Gamma}(e^{iu_{y,\varepsilon}})$ in the limit $\varepsilon \downarrow 0$ formally behave like fermions! This latter limit, however, is quite delicate: one can prove that $c_\varepsilon \hat{\Gamma}(e^{iu_{y,\varepsilon}})$ for

$$c_\varepsilon = \frac{1}{\sqrt{2\pi(1-e^{-2\varepsilon})}} \tag{120}$$

converges to $\psi^*(y)$ as $\varepsilon \downarrow 0$. To be more precise:

$$\psi^*(f) = \lim_{\varepsilon\downarrow 0} \int_{-\pi}^{\pi} dy \, f(y) \, c_\varepsilon \hat{\Gamma}(e^{iu_{y,\varepsilon}}) \tag{121}$$

for all $f$ in a dense subset of $L^2(S^1)$.

**Remark (adv.):** This latter equation holds in the sense of strong convergence on a dense domain, for all $f$ which have a Fourier transform with compact support. Note that one can regard $\psi_\varepsilon(y) = c_\varepsilon \hat{\Gamma}(e^{iu_{y,\varepsilon}})$ as approximative fermion field operators: For $\varepsilon > 0$ they are proportional to unitary operators and thus 'nice' operators on $\mathcal{F}$, and in the limit they converge to operator valued distributions.

For more details and a proof close in spirit of my presentation I refer to Ref. $[CaH]$.

### 5.4 Bosons, fermions, and elliptic functions

As an application I show in this section how the boson fermion correspondence can be used to prove identities for elliptic functions. I discuss the simplest example I know.

I compute

$$\mathcal{Z}(z) = \mathrm{Tr}_{\mathcal{F}}(e^{-\beta\hat{H}+\mathrm{i}zQ}) , \quad z \in \mathbb{C} \tag{122}$$

for the operators given in Eqs. (115) and (103). Note that $\mathcal{Z}(z)$ has a simple physical interpretation: it is the the partition function of the free fermion model formally defined in Eq. (108). I have thrown in the variable $z$ since it does not make the computation more difficult but the result becomes more interesting. We will obtain an identity by computing $\mathcal{Z}(z)$ first using the fermion basis in Eq. (114) and then by using the boson basis in Eq. (113). Due to the boson-fermion correspondence, all these vectors are eigenstates of $\hat{H}$ and $Q$.

Using $[\hat{H}, \psi^*(e_n)] = (n+\frac{1}{2})\psi^*(e_n)$ and $[Q, \psi^*(e_n)] = \psi^*(e_n)$ etc. it is not difficult to prove the **Fact:** Every vector in Eq. (114) is a common eigenvectors of $\hat{H}$ and $Q$, and the corresponding eigenvalues are $\sum_{n=1}^\infty (n-\frac{1}{2})(m_n^+ + m_n^-)$ and $\sum_{n=1}^\infty (m_n^+ - m_n^-)$, respectively.□ Recalling that the vectors in (114) provide a basis in $\mathcal{F}$, we can therefore compute $\mathcal{Z}(z)$ as follows,

$$\begin{aligned}\mathcal{Z}(z) &= \sum_{m_1^+, m_1^-, m_2^+, m_2^-, \ldots = 0,1} e^{-\beta\sum_{n=1}^\infty (n-\frac{1}{2})(m_n^+ + m_n^-) + \mathrm{i}z\sum_{n=1}^\infty (m_n^+ - m_n^-)} = \\ &= \prod_{n=1}^\infty \left( \sum_{m_n^+ = 0,1} e^{-(\beta(n-\frac{1}{2}) - \mathrm{i}z)m_n^+} \right) \left( \sum_{m_n^- = 0,1} e^{-(\beta(n-\frac{1}{2}) + \mathrm{i}z)m_n^-} \right) = \\ &= \prod_{n=1}^\infty (1 + e^{-\beta(n-\frac{1}{2}) + \mathrm{i}z})(1 + e^{-\beta(n-\frac{1}{2}) - \mathrm{i}z}) .\end{aligned}$$

Using the notation

$$q = e^{-\beta/2} \tag{123}$$

we can write this result as

$$\mathcal{Z}(z) = \prod_{n=1}^{\infty} \left(1 + q^{2n-1}\cos(z) + q^{4n-2}\right) \ . \tag{124}$$

Using $[\hat{H}, \hat{\rho}(-p)] = p\hat{\rho}(-p)$ it is not difficult to prove the **Fact:** Every vector in Eq. (113) is a common eigenvectors of $\hat{H}$ and $Q$, and the corresponding eigenvalues are $\frac{1}{2}\nu^2 + \sum_{p=1}^{\infty} pn_p$ and $\nu$, respectively.$\square$ Using that the vectors in (113) provide a basis in $\mathcal{F}$, we can therefore compute $\mathcal{Z}(z)$ also as follows,

$$\mathcal{Z}(z) = \sum_{\nu\in\mathbb{Z}} \sum_{n_1,n_2,\ldots=0}^{\infty} e^{-\beta\sum_{p=1}^{\infty} pn_p - \frac{1}{2}\beta\nu^2 + \mathrm{i}\nu z} =$$

$$= \sum_{\nu\in\mathbb{Z}} e^{-\frac{1}{2}\beta\nu^2+\mathrm{i}\nu z} \prod_{p=1}^{\infty}\sum_{n_p=0}^{\infty} e^{-\beta p n_p} = \sum_{\nu\in\mathbb{Z}} e^{-\frac{1}{2}\beta\nu^2+\mathrm{i}\nu z} \prod_{p=1}^{\infty} \frac{1}{(1-e^{-\beta p})} \ .$$

Recalling the definition in Eq. (123) we can write this result as

$$\mathcal{Z}(z) = \sum_{\nu\in\mathbb{Z}} q^{\nu^2} e^{\mathrm{i}\nu z} \frac{1}{\prod_{p=1}^{\infty}(1-q^{2p})} \ . \tag{125}$$

Equating $Z(z)$ in Eqs. (124) and (125) and replacing $z$ by $2\pi v$ we obtain

$$\sum_{\nu\in\mathbb{Z}} q^{\nu^2} e^{\mathrm{i}\pi 2\nu v} = \prod_{p=1}^{\infty}(1-q^{2p}) \prod_{n=1}^{\infty} \left(1 + q^{2n-1}\cos(2\pi v) + q^{4n-2}\right) \tag{126}$$

for all $v \in \mathbb{C}$, which is a well-known identity in the theory of elliptic functions: the l.h.s. in this Eq. is the definition of the theta function $\theta_3(v)$, and the r.h.s. is the product representation of this very function (see e.g. $[EMOT]$, *pp.* 355 and 357).

## 6 The Luttinger model

In this section we discuss quantum field theory models for interacting fermions in one dimensional space. To be specific I concentrate on the Luttinger model $[ML]$, a simple model for a one dimensional metal. My purpose is to illustrate how the mathematical results summarized above can be used to construct and solve 1+1 dimensional quantum field theory models.

### *6.1 Physical interpretation*

We start with a physical motivation for this model. We consider spin-less fermions in a one dimensional metal (wire) which is a lattice with lattice constant $a$ and $\ell$ sites and which can be characterized by a so-called **band relation** $E(p) = E(-p)$ describing the energy as a function of the (pseudo-) momentum $p$. For example, if we would consider the Hubbard Hamiltonian in Eq. (41) for $U = 0$, we would take the band relation in Eq. (42).

We are interested in the continuum limit where one keeps the length $L = a\ell$ of the system fixed while the lattice constant $a$ is sent to zero. A physical motivation for this is as follows: the lattice model (finite $a$) has obviously more structure (i.e. is more complicated) since any physical quantity in the model will depend on $a$. Physicists are mainly interested in properties of systems which do not depend on this **short-distance structure** i.e. properties which are (essentially) independent of $a$. Instead of doing computations at finite $a$ and then taking the limit $a \to 0$, it seems easier to directly do computations in a model where the short-distance structure (i.e. dependence on $a$) has been eliminated from the start: such a model has less structure and therefore should be easier to treat. I will outline how such a model can be found in our example. As we will see, this model is indeed simpler, but one has to pay a price: in contrast to the finite-$a$ model, it is a quantum field theory model with all the difficulties discussed (and solved) in previous sections.

A formal argument leading to the continuum model is as follows. We consider a metal, i.e. there is a finite density of fermions. We therefore assume that the energy eigenstates are all filled for energies $E(p)$ less then the chemical potential $\mu$. The so-called **Fermi surface** separating filled from empty states consists in our example of the two points $p = \pm p_F$ where $E(p) - \mu$ vanishes. Physical intuition suggests that in the continuum limit only states close to the Fermi surface are relevant, whereas states far away from the Fermion surface (i.e. where $|E(p) - \mu|$ is large) should not have much influence on the physical properties of the system and only provide some inert background. To obtain a non-trivial continuum limit, we label these states as $p = \pm p_F + ak$ where $k$

is $\frac{2\pi}{L}$ times a half-integer (cf. Eq. (42)), and then we Taylor expand the band relation about the Fermi surface,

$$E(\pm p_F + ak) - \mu = \pm v_F k + \dots \tag{127}$$

where $v_F = aE'(p_F)$, for example, $v_F = 2ta\sin(p_F)$ in case of the band in Eq. (41). We thus obtain two branches corresponding to the excitation close to the two points $p = \pm p_F$ of the Fermion surface. The constant $v_F$ characterizes the slope of the band at the Fermi surface and is usually called **Fermi velocity** in the physics literature. If $v_F$ has a non-zero limit as $a \to 0$ (which for the band in Eq. (42) is achieved by tuning $t = t(a)$ and $\mu = \mu(a)$ carefully) it is plausible that we can neglect the higher order terms '...' in the Taylor expansion above. In the continuum limit we then obtain $E_\pm(k) = \pm v_F k$ with $k \in \Lambda_0^*$ where

$$\Lambda_0^* = \left\{ k = \frac{2\pi}{L}(n + \tfrac{1}{2}) \,|n \in \mathbb{Z} \right\} \tag{128}$$

the Fourier space for anti-periodic functions on a circle of length $L$.

**Remark:** The upshot of my discussion above is: I replaced the energies for states 'far away from the Fermi surface' by something simpler, and justified this by arguing that the precise properties of these states should not matter much.

Note that the energy band $E_\pm(k)$ above is precisely the energy spectrum of the 1-particle Hamiltonian

$$D = -i\partial_x v_F \sigma_3\,, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{129}$$

(with anti-periodic boundary conditions) acting on the 1-particle Hilbert space

$$\mathcal{H} = L^2(S^1_L) \otimes \mathbb{C}^2 \tag{130}$$

where $S^1_L$ is the circle of length $L$, and $x \in [-L/2, L/2]$ is a coordinate on $S^1_L$. Formally, the second quantization of this Hamiltonian is

$$H_0 = v_F \int_{-L/2}^{L/2} dx \left( \psi^*_+(x)(-i\partial_x)\psi_+(x) - \psi^*_-(x)(-i\partial_x)\psi_-(x) \right) \tag{131}$$

where $\left\{\psi_\pm(x), \psi^*_\pm(y)\right\} = \delta(x-y)$ etc. The Hamiltonian of the Luttinger model is obtained by adding an interaction terms $H'$ to $H_0$,

$$H = H_0 + H' \tag{132}$$

with

$$H' = \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy\ \rho_+(x) V(x-y) \rho_-(y), \quad \rho_\pm(x) = \psi_\pm^*(x)\psi_\pm(x) \quad (133)$$

describing a 4-point interaction of the fermions. The real-valued function $v(r)$ describing the spatial dependence of the interaction is arbitrary up to some (minor) restrictions which we will discuss and specify further below.

The mathematical methods discussed in these notes obviously allow a construction and solution of the model formally defined by the free Hamiltonian $H_0$. We now argue that this framework allows also to construct the interacting model with the Hamiltonian $H$.

**Remark:** It is interesting to note that $H_0$ for $v_F = 1$ is identical with the Hamiltonian of free relativistic Dirac fermions in 1+1 dimensions. Therefore $H_0$ is also the free part of the Hamiltonian of 1+1 dimensional relativistic interacting fermion models, for example, in the Schwinger model $[Ma]$ which is electrodynamics with massless fermions. This and other models can therefore be treated by similar methods as the ones discussed below.

**Remark:** Above we discussed an example for the so-called **continuum limit** motivated by the physicists' expectation that many properties of a system should not depend of its short-distance structure. One also expects that many properties should not dependent on the size of a system (as long as it is sufficiently large that surface effects play no role) and therefore often performs the so-called **thermodynamic limit** where the system size $L$ is sent to $\infty$. This latter limit is also very interesting (and non-trivial) in many models but its discussion is beyond the scope of the present notes. I only mention that the divergences associated with a continuum- and thermodynamic limit are often called **ultra violet**- and **infra red divergences** in the physics literature.

### *6.2 Construction and solution of the model*

Note that the Hamiltonian $H_0$ in Eq. (131) is the sum of two terms, $H_0 = v_F(H_0^+ + H_0^-)$, where $H_0^+$ is precisely the chiral Hamiltonian in Eq. (108) which we used to construct the wedge representation of the loop spaces $\mathcal{L}(\mathbb{C})$ and $\mathcal{L}(\mathrm{U}(1))$ in Section 5 (up to a rescaling $x \to 2\pi x/L$ etc.). Moreover, $H_0^-$ is the same as $H_0^+$ up to some sign change. It is thus quite obvious that the results which we derived in Section 5 for $H_0^+$ should also hold for $H_0^+$ (up to some minor changes). As we now show, these results are essential for constructing and solving the Luttinger model.

To simplify notation we set

$$v_F = 1 \tag{134}$$

from now on (this corresponds to assuming $v_F > 0$ and measuring energies in units $\propto v_F$).

We can construct the model in the quasi-free representation defined by the 1-particle Hilbert space in Eq. (130) and the projection $P_- = \theta(-D)$ with $D$ given in Eq. (129) (anti-periodic boundary conditions). Denoting as $\hat{\psi}_\pm(k) = \psi(e_k^\pm)$ with $e_k^\pm(x) = \frac{1}{\sqrt{2\pi}} e^{-ikx} E^\pm$ and $E^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $E^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we can completely characterize this representation by the following equations,

$$\hat{\psi}_\pm(\pm k)\Omega = \hat{\psi}^*_\pm(\mp k)\Omega = 0 \quad \forall k > 0 \tag{135}$$

and $\left\{ \hat{\psi}_\sigma(k), \hat{\psi}^*_{\sigma'}(k') \right\} = \delta_{\sigma\sigma'} \frac{L}{2\pi} \delta_{k,k'}$ etc. In this representation we can give a precise meaning to the free part of the Luttinger Hamiltonian i.e. $\hat{H}_0 = d\hat{\Gamma}(D)$. In a physicist's notation,

$$\hat{H}_0 = \frac{2\pi}{L} \sum_{k \in \Lambda_0^*} k : \left( \hat{\psi}^*_+(k)\hat{\psi}_+(k) - \hat{\psi}^*_-(k)\hat{\psi}_-(k) \right) : \tag{136}$$

where $: \cdots :$ is normal ordering with respect to the free Dirac sea $\Omega$.

To give a precise meaning to the interaction term of the Luttinger Hamiltonian we construct $\hat{\rho}_\pm(p) = d\hat{\Gamma}(\varepsilon_p \frac{1}{2}(\mathbf{1} \pm \sigma_3))$ where $\varepsilon_p(x) = e^{ipx}$ with $p \in \Lambda^*$,

$$\Lambda^* = \left\{ p = \frac{2\pi}{L} n \,|n \in \mathbb{Z} \right\} \tag{137}$$

the Fourier space for periodic functions on $S^1_L$. The arguments explained in Section 5 can be easily adapted to prove existence of these operators together with the following relations,

$$\begin{aligned} \left[\hat{\rho}^\pm(p), \hat{\rho}^\pm(p')\right] &= \pm p \frac{L}{2\pi} \delta_{p,-p'} \\ \left[\hat{\rho}^\pm(p), \hat{\rho}^\mp(p')\right] &= 0\,, \\ \left[\hat{H}_0, \hat{\rho}^\pm(p)\right] &= \pm p \hat{\rho}^\pm(p) \end{aligned} \tag{138}$$

for all $p, p' \in \Lambda^*$ and

$$\hat{\rho}^+(p)\Omega = \hat{\rho}^-(-p)\Omega = 0 \quad \forall p > 0\,. \tag{139}$$

Formally, $\rho^{\pm}(x) \;=: \psi^*_{\pm}(x)\psi_{\pm}(x):$, suggesting to try to make sense of the interaction term in Eq. (133) using Fourier transformation,

$$\hat{H}' = \frac{2\pi}{L}\sum_{p\in\Lambda^*} W_p \hat{\rho}^+(p)\hat{\rho}^-(-p) \tag{140}$$

where

$$W_p = \frac{1}{8\pi}\int_{-L/2}^{L/2} dr\, V(r)e^{-ipr} = W_{-p} = \overline{W_p}\,. \tag{141}$$

As I will discuss in more detail in a remark below, if (and only if) the potential function $V(r)$ is such that

$$|W_p| < 1 \quad \forall p \quad \text{ and } \quad \sum_{p>0} pW_p^2 < \infty \tag{142}$$

does $\hat{H} = \hat{H}_0 + \hat{H}'$ define a self-adjoint operator on $\mathcal{F}$ which is bounded from below, i.e., is the quasi-free representation appropriate for the free model $H_0$ also appropriate for the Luttinger model.

The relation which allows a solution of the Luttinger model is the Kronig identity in Eq. (116) which adapted to $\hat{H}_0$ becomes

$$\hat{H}_0 = \frac{2\pi}{L}\left(\tfrac{1}{2}(Q_+^2 + Q_-^2) + \sum_{p>0}\left(\hat{\rho}_+(-p)\hat{\rho}_+(p) - \hat{\rho}_-(p)\hat{\rho}_-(-p)\right)\right)\,. \tag{143}$$

As discussed, this relation makes explicit that the free fermion Hamiltonian is identical with a free boson Hamiltonian. The simple but crucial observation now is that the interaction term $H'$ is also quadratic in the boson fields, and therefore the Luttinger Hamiltonian $H$ is identical with a free boson Hamiltonian. Hamiltonian of this latter kind can be diagonalized explicitly, and all what is needed for that is some simple facts about the harmonic oscillator problem which one learns in an introductory quantum mechanics courses. In the following we outline this diagonalization.

We write

$$H = \frac{2\pi}{L}\sum_{p\geq 0} h_p \tag{144}$$

and introduce the notation

$$c(p) = \begin{cases} \frac{1}{\sqrt{|p|}}\hat{\rho}^+(p) & \text{for } p > 0 \\ \frac{1}{\sqrt{|p|}}\hat{\rho}^-(p) & \text{for } p < 0 \end{cases} \tag{145}$$

so that

$$h_p = p[c^*(p)c(p) + c^*(-p)c(-p)] - W_p[c^*(p)c^*(-p) + c(-p)c(-p)] \quad \forall p > 0\,,$$

and

$$h_0 = \tfrac{1}{2}(Q_+^2 + Q_-^2) + W_0 Q_+ Q_-\,. \tag{146}$$

Note that

$$[c(p), c^*(p')] = \frac{L}{2\pi}\delta_{p,p'} \quad \text{etc.} \tag{147}$$

for all $p, p' \neq 0$, which are precisely the relations of the creation- and annihilation operators in a harmonic oscillator problem. Moreover, we see that we can diagonalize the Luttinger Hamiltonian by diagonalizing the $h_p$ separately, and each $h_p$ is a harmonic oscillator-type Hamiltonian. For that we introduce new operators

$$C(\pm p) = \cosh(\lambda_p)c(\pm p) + \sinh(\lambda_p)c^*(\mp p) \quad \forall p \neq 0 \tag{148}$$

obeying the same relations as the operators $c(p)$, and we determine the real parameters $\lambda_p$ so that $h_p$ can be written in the following form,

$$h_p = \omega_p\,(C^*(p)C(p) + C^*(-p)C(-p)) - \eta_p \mathbf{1} \tag{149}$$

with some constants $\omega_p, \eta_p$ to be determined. By straightforward computation one finds that $h_p$ has the desired form if we choose

$$\tanh(2\lambda_p) = W_p \tag{150}$$

which has a solution $\lambda_p$ only if the first condition in Eq. (142) is fulfilled, and in this case,

$$\omega_p = p\sqrt{1 - W_p^2}, \quad \eta_p = p(1 - \sqrt{1 - W_p^2})\,. \tag{151}$$

With that we can now find all eigenstates of the Luttinger Hamiltonian: the ground-state $\Omega$ is determined by the following conditions,

$$C(p)\tilde{\Omega} = 0 \quad \forall p \neq 0, \quad Q_\pm \tilde{\Omega} = 0, \tag{152}$$

and the ground state energy is

$$E_0 = -\frac{2\pi}{L}\sum_{p>0} p(1 - \sqrt{1 - W_p^2})\,. \tag{153}$$

Note that the ground-state energy $E_0$ in Eq. (153) is finite if and only if $\sum_{p>0} pW_p^2$ is finite, and we see that the second condition in Eq. (142) is necessary for the construction to make sense (otherwise there cannot be a ground-state $\tilde{\Omega}$ in $\mathcal{F}$).

Once $\tilde{\Omega}$ is constructed it is easy to also find all other eigenstates of the Luttinger model: they are given by

$$\prod_{p\neq 0} C^*(p)^{n_p} R_+^{\nu_+} R_-^{\nu_-} \tilde{\Omega}, \quad n_p \in \mathbb{N}_0 \,, \nu_\pm \in \mathbb{Z}$$

where $R_\pm = \hat{\Gamma}(e^{\mathrm{i}x2\pi/L\frac{1}{2}(\mathbf{1}\pm\sigma_3)})$, and the corresponding energy eigenvalues are

$$E_0 + \frac{2\pi}{L}\left(\tfrac{1}{2}(\nu_+^2 + \tfrac{1}{2}\nu_-^2) + W_0\nu_+\nu_- + \sum_{p>0}\omega_p(n_p + n_{-p})\right)$$

showing that also $|W_0| < 0$ is required for $\hat{H}$ to have a ground-state. From that we can compute the partition function etc., and we have completely solved the model.

I finally mention that physicists are also interested in the so-called **Green's functions** in quantum field theory models which are (in general) distributions allowing to predict results of measurements. In case of the Luttinger model, these Green's functions are

$$\left\langle \tilde{\Omega}, \Psi_\tau^\sigma(\mathbf{t},\mathbf{x})\tilde{\Omega} \right\rangle$$

where $\Psi_\tau^\sigma(\mathbf{t},\mathbf{x})$ stands for an arbitrary product of finitely many fermion field operators $\psi_\pm(t,x)$ and $\psi_\pm^*(t,x)$, and the $t$ dependence of these latter operators is given by by $A(t) = e^{\mathrm{i}\hat{H}t}Ae^{-\mathrm{i}\hat{H}t}$. We note that there are formulas similar to Eq. (121) allowing to express the fermion field operators $\psi_\pm(x)$ in terms of the boson fields $\hat{\rho}_\pm(p)$, and with the results summarized above one can compute all Green's functions of the Luttinger model explicitly (see $[HSU]$ for more details).

**Remark:** The construction and solution for the Luttinger models described here was for zero temperature. A similar construction and solution of the Luttinger model at finite temperature was given in $[CaHa]$.

**Remark:** Another prominent quantum field theory model is the the so-called **massless Thirring model** which is obtained from the Luttinger model in the limits where space becomes infinite, $L \to \infty$, and the interaction becomes local, i.e.

$$W_p = g \quad \text{independent of } p, \quad |g| < 1 \,. \tag{154}$$

These limits can be quite easily performed in the Green's functions. However, they are highly non-trivial and interesting on the operator level and provide an example for multiplicative regularization in a fully solvable model (see e.g. [$GLR$] for more details).

## 7 Further developments

In these notes I have tried to give a pedagogical introduction to the theory of quasi-free second quantization of fermions, a formalism which has many applications and which, to my opinion, provides a fine example of a fruitful interplay between mathematics (functional analysis, Lie group theory, differential geometry, theory of special functions...) and theoretical physics (quantum field theory models in elementary particle physics and solid state physics, string theory, integrable models...). I therefore hope that these notes can be useful for mathematicians who want to learn some physics and for theoretical physicists interested in the mathematics discussed here.

As I will describe in this final section, this formalism has been playing also a central role in my research. Again, my selection of topics here is personal, and I restrict my discussion to projects in which I have been directly involved.

### *7.1 Boson-anyon correspondence and the Sutherland model*

In 1+1 dimensions it is possible to have quantum fields $\varphi(x)$ which neither are bosons nor fermions but which obey relations

$$\varphi(x)\varphi(y) = e^{\pm i\pi\alpha \operatorname{sgn}(x-y)}\varphi(y)\varphi(x) \quad \forall x, y \in S^1, x \neq y \tag{155}$$

with $\alpha > 0$ a so-called **statistics parameter**. For $\alpha$ an odd integer these fields are fermions (i.e. they anti-commute), for $\alpha$ an even integer they are bosons (i.e. they commute), and since $\alpha$ can have *any* value such fields often are called **anyons**. Anyons have received quite some attention in theoretical physics during recent years. For example, they are known to be relevant in the theory of the fractional quantum Hall effect.

We recently gave a construction of a quantum field theory model of anyons based on the representation theory of the loop group $\mathcal{L}(\mathrm{U}(1))$ and generalizing the boson-fermion correspondence discussed in Section (5.3) to a boson-anyon correspondence $[CaL]$. Moreover, we showed that this model of anyons naturally provides a second quantization of the so-called **Calogero-Sutherland model** which is defined by the Hamiltonian

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{1\leq j<k\leq N} \frac{2\alpha(\alpha-1)}{\sin^2((x_j-x_k)/2)} \tag{156}$$

and describes a system of $N$ interacting particles moving on the circle. Using this anyon model we were able to derive in a simple manner an algorithm allowing to solve the CS model for arbitrary particle number $N$ and coupling $\alpha$ $[CaL]$, and we showed that this algorithm allows to recover the full solution of the CS model originally found by different methods by Sutherland about 30 years ago.

Recently I generalized this construction to the elliptic CS model which is the generalization of the Hamiltonian in Eq. (156) where $\sin^{-2}((\cdot)/2)$ is replaced by the Weierstrass elliptic function $\wp(\cdot)$ with periods $2\pi$ and $\mathrm{i}\beta$, $\beta > 0$ (the CS model corresponds to the limiting case $\beta \to \infty$). The idea underlying this construction is to construct the anyon model at finite temperature $1/\beta$. In this way I obtained an algorithm for solving the elliptic CS model [$L4$]. This can be regarded as a continuation of the theme illustrated in Section 5.4: use relations of operators on the fermion Fock-space $\mathcal{F}(L^2(S^1))$ to derive non-trivial identities involving elliptic function.

### *7.2 Other 1+1 dimensional quantum field theory models*

Other interacting quantum field theory models which can be constructed and solved by methods similar to the ones we described in our discussion of the Luttinger model include the **Schwinger model** [$Ma$], i.e. 1+1 dimensional quantum electrodynamics with massless fermions, and the Luttinger-Schwinger model, i.e. the gauged Luttinger model [$GLR$]. A similar construction of **massless** $\mathbf{QCD}_{1+1}$, i.e. the non-Abelian version of the Schwinger model, was given in [$LS$].

### *7.3 Quasi-free second quantization and non-commutative geometry*

There is an interesting relation of quasi-free second quantization of fermions and Connes' non-commutative geometry [$Con$]. The cocycles in Eq. (98) and (75) offer the simplest non-trivial example of this. This can be seen by introducing some simple notation: using integration by parts we can write Eq. (98) as follows,

$$S(\alpha_1, \alpha_2) = -\frac{1}{2\pi i}\int_{S^1} \mathrm{tr}_N(\alpha_1 d\alpha_2) \tag{157}$$

where the r.h.s. is proportional to the integral of the de Rham 1-form $\alpha_1 d\alpha_2 = \alpha_1 \frac{\partial \alpha_2}{\partial x} dx$ over the manifold $S^1$. We can also write Eq. (75) as follows,

$$\hat{S}(A, B) = -\tfrac{1}{2}\mathrm{Tr}_C(A[F, B]) \tag{158}$$

where $F = P_+ - P_-$ and $\mathrm{Tr}_C(A) = \frac{1}{2}\mathrm{Tr}(A + FAF) = \mathrm{Tr}(A_{++} + A_{--})$ is the so-called **conditional trace**. (This can be checked easily by using $F^2 = \mathbf{1}$ and the notation introduced in Eq. (68).) According to Connes, $\hat{d}A := \mathrm{i}[F, A]$ can be regarded as a natural definition of an exterior differentiation on algebras of Hilbert space operators. Thus we can interpret the equation $\hat{S}(\alpha_1,\alpha_2) = S(\alpha_1,\alpha_2)$ naturally as follows: $\hat{S}$ is the natural generalization of $S$ from de Rham form on the circle to an algebra of operators on a Hilbert space according to the following rules,

$$
\begin{aligned}
&\text{0-forms} \quad \alpha \in Map(S^1, \mathrm{gl}_N) \to \hat{g}_1 = \{A \in \mathcal{B} | A_{-+}, A_{+-} \in \mathcal{B}_2\} \\
&\text{1-forms} \quad \alpha_1 d\alpha_2 = \alpha \frac{\partial \alpha_2}{\partial x} dx \to A\hat{d}B = \mathrm{i}A[F,B] \\
&\qquad\quad \text{integration} \quad \int_{S^1} \to \frac{1}{\pi}\mathrm{Tr}_C(\cdot)\,. \qquad (159)
\end{aligned}
$$

This is a simple example of Connes' quantized calculus [$Con$] and the starting point of a quite long and nice story in which the following topics play a central role: current algebras and anomalies in higher dimensions (e.g. the analog of the 2-cocycles above to 3+1 dimensions etc.), quantum gauge theories, non-commutative geometry, .... I have described this story in more detail in previous reviews [$L3, L2$] and will not repeat it here. As an appetizer, I only state the following identity which is a generalization of the correspondence in Eq. (159) to de Rham forms on higher dimensional manifolds,[21]

$$
\begin{aligned}
\mathrm{i}^n \mathrm{Tr}_C(\alpha_0\,[F,\alpha_1]\cdots[F,\alpha_n]) = c_n \int_{\mathbb{R}^n} \mathrm{tr}_N(\alpha_0 \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n) \\
\forall \alpha_j \in Map(\mathbb{R}^n, \mathrm{gl}_N)\,, n = 1,3,5\ldots \qquad (159a)
\end{aligned}
$$

where d is the usual exterior differentiation, $F$ is the sign of the chiral Dirac operator on $\mathbb{R}^n$, and

$$
c_n = (2\mathrm{i})^{(n-1)/2} \frac{1}{n(2\pi)^n} \frac{2\pi^{n/2}}{\Gamma(n/2)}. \qquad (159b)
$$

are normalization constants [$L1$].

### 7.4 Bosons and the super-version of quasi-free second quantization

In these notes I have restricted myself to fermions for simplicity. However, many results discussed can be adapted to bosons. Formally the boson version can be obtained by replacing anti-commutators $\{\cdot,\cdot\}$ by commutators $[\cdot,\cdot]$. For example, the boson quantum mechanics with $N$ degrees of freedom can

[21] for simplicity I write the formula only for odd dimensions $n$, but there is a similar one for $n$ even [$L1$].

be characterized by the following relations,

$$\left[a_j, a_k\right] = 0, \quad \left[a_j, a_k^\dagger\right] = \delta_{jk}\mathbf{1} \quad \forall j,k = 1,\ldots,N \tag{160}$$

with the usual representation on the Hilbert space $L^2(\mathbb{R}^N)$. This is usual quantum mechanics but, as mentioned, mathematically more difficult than the fermion version since the $a_j$ can only be represented by unbounded operators on an infinite dimensional Hilbert space. Still, there is a natural boson version of the formalism discussed in Section (4.4), and there is even a **superversion** which is a generalization corresponding to a model where one has bosons and fermions and also allows for transformations mixing bosons and fermions [$GL1, GL2$]. Surprisingly, most results obtained in this formalism have a natural interpretation as a $\mathbb{Z}_2$-graded generalization of the fermion results discussed in Section (4.4): all algebras etc. in the formalism can be given a natural $\mathbb{Z}_2$-grading, and all one has to do is to replace (anti)-commutators by graded commutators. As an application, this formalism naturally yields representations of the Lie super-algebra generalizations of the affine Kac Moody algebra and the Virasoro algebra which play an important role in string theory.


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