HEAT EQUATION AND SPECTRAL GEOMETRY. INTRODUCTION FOR BEGINNERS.

KRZYSZTOF P. WOJCIECHOWSKI
Department of Mathematics
IUPUI (Indiana/Purdue)
Indianapolis IN 46202–3216, U.S.A.
E-mail:kwojciechowski@math.iupui.edu

These notes contain an introduction into the theory of ζ-determinants of Dirac operators. We discuss here the simplest possible case of the operator on S^1. We study in details the necessary facts from the theory of *Heat Equations* concentrating more specifically on the *small time asymptotics* of the trace of the kernel of the *Heat Operator*.

Contents

Introduction

In the beginning of 1999 I was invited to spend the majority of July in Villa de Leyva, a beautiful small town located in Colombia. I was supposed to deliver a set of lectures on spectral geometry of elliptic boundary problems for Dirac operators in the Summer School on *Geometric Methods in Quantum Field Theory* organized by CIMPA and Universidad de Los Andes. However, once in Villa de Leyva I found out that the majority of the audience were graduate students at different stages of development. Therefore, I faced the difficult task of presenting some introductory material on this beautiful subject to beginners, who did not know global analysis and some of whom were not particularly advanced in functional analysis. I decided to present a rigorous introduction into theory of the ζ-determinant for the simplest possible Dirac operator, which is the operator $\mathcal{D}_f = -i\frac{d}{dx} + f(x)$ on the circle. In fact, I tried to keep the presentation open to advanced undergraduate students. The only two prerequisites, beyond the advanced undergraduate level, are the notion of the meromorphic extension to the complex plane of a function $h(s)$, holomorphic on the half-plane $Re\ s > r$ for certain real number r , and some fundamental results on the trace of an operator acting on a separable Hilbert space. I usually tried to organize lectures as follows: first, I outlined a theory for a general Dirac operator on a closed manifold, then I proved the results discussed in the first part in the case of the operator $\mathcal{D}_f$.

1 ζ-determinant

Many fundamental calculations of *Quantum Field Theory* reduce to the computation of the determinant of some *Dirac* operator. In fact, most physicists will tell you that, at the one loop level, any such theory reduces to a theory of determinants. Therefore, the notion of the determinant of the Dirac operator plays a central role in many recent studies in the area of *Quantum Field Theory*, and in the *Spectral Geometry* and *Topology* as well.

We discuss here one of the possible schemes of making precise the notion of the determinant. We use ζ-function regularization. The starting point is a well-known formula, which holds in the case of a linear operator $T : \mathbf{C}^n \to \mathbf{C}^n$, which we also assume to be positive. We denote the eigenvalues of T by $0 < a_1 < ... < a_n$ and then the following equality holds

$$det\ T = \prod_{k=1}^{n} a_k \quad ,$$

which may be further rewritten using an auxiliary complex variable s in the following way

$$det\ T = e^{\sum_{k=1}^{n} ln\ a_k} = e^{-\frac{d}{ds}(\sum_{k=1}^{n} a_k^{-s})|_{s=0}} \quad . \tag{1}$$

Hence, if we introduce ζ-function of the operator T as

$$\zeta_T(s) = \sum_{k=1}^{n} a_k^{-s} = Tr\ T^{-s} \quad ,$$

the equality (1) can be rewritten as

$$det\ T = e^{-\frac{d}{ds}(\zeta_T(s))|_{s=0}} \quad . \tag{2}$$

The important fact here is that formula (2) holds in the infinite-dimensional case as well. More precisely, if we assume that $P : C^\infty(M; S) \to C^\infty(M; S)$ is a positive elliptic differential operator of order $d > 0$ (i.e. Dirac Laplacian) acting on sections of vector bundle S over a closed manifold M of dimension m , then

$$\zeta_P(s) = Tr\ P^{-s} \tag{3}$$

is a well-defined holomorphic function of complex variable s , for $Re\ s > \frac{m}{2}$, which has a unique meromorphic extension to the whole complex plane with simple poles at $s = s_k = \frac{m}{2} - k$. However, $s = 0$ is not a pole and formula (2) makes sense in this context. The proofs of all necessary analytical results can be found for instance in Chapter 1 of $[G]$. We introduce the ζ-determinant of the operator P by the formula

$$det_\zeta P = e^{-\frac{d}{ds}(\zeta_P(s))|_{s=0}} \quad . \tag{4}$$

The goal of these simple notes is to show how things work out in the case of the operator $\mathcal{D}_f$. So far we have discussed only the case of a positive operator. Dirac operator is an elliptic operator of first order and therefore has a spectrum which includes infinitely many positive and negative eigenvalues. Let us assume now that we replace the operator P by a Dirac operator $\mathcal{D} : C^\infty(M; S) \to C^\infty(M; S)$, which has spectrum consisting of positive eigenvalues, which we denote by $\{\lambda_k\}_{k\in\mathbf{N}}$ and negative eigenvalues $\{-\mu_k\}_{k\in\mathbf{N}}$. Now, the ζ-function of $\mathcal{D}$ looks as follows

$$\zeta_{\mathcal{D}}(s) = \sum_k \lambda_k^{-s} + \sum_k (-1)^{-s}\mu_k^{-s}$$

$$= \sum_k \left(\frac{\lambda_k^{-s} - \mu_k^{-s}}{2} + \frac{\lambda_k^{-s} + \mu_k^{-s}}{2}\right) + (-1)^{-s}\sum_k \left(\frac{\lambda_k^{-s} + \mu_k^{-s}}{2} - \frac{\lambda_k^{-s} - \mu_k^{-s}}{2}\right) \quad ,$$

which can be written as

$$\zeta_{\mathcal{D}}(s) = (-1)^{-s}\frac{\zeta_{\mathcal{D}^2}(s/2) - \eta_{\mathcal{D}}(s)}{2} + \frac{\eta_{\mathcal{D}}(s) + \zeta_{\mathcal{D}^2}(s/2)}{2} \quad , \tag{5}$$

where $\eta_{\mathcal{D}}(s) = \sum_k \lambda_k^{-s} - \sum_k \mu_k^{-s}$ is the η-function of the operator $\mathcal{D}$ introduced by Atiyah, Patodi and Singer (see $[APS1]$). Once again it is holomorphic for $Re(s)$ large and has a meromorphic extension to the whole complex plane with only simple poles. There is no pole at $s = 0$ and therefore we can study the derivative of $\zeta_{\mathcal{D}}(s)$ at $s = 0$. We have

$$\zeta'_{\mathcal{D}}(0) = \frac{\zeta'_{\mathcal{D}^2}(0)}{2} + \frac{d}{ds}\{(-1)^{-s}\}|_{s=0} \cdot \frac{\zeta_{\mathcal{D}^2}(0) - \eta_{\mathcal{D}}(0)}{2} \quad .$$

The ambiguity in defining $(-1)^{-s} = e^{\pm i\pi s}$ (i.e. a choice of spectral cut) now leads to an ambiguity in the phase of the ζ-determinant. We choose the sign in such a way that the definition of the ζ-determinat of the Dirac operator $\mathcal{D}$ is as follows

$$det_{\zeta}\mathcal{D} = e^{\frac{i\pi}{2}(\zeta_{\mathcal{D}^2}(0)-\eta_{\mathcal{D}}(0))} . e^{-\frac{\zeta'_{\mathcal{D}^2}(0)}{2}} \quad , \tag{6}$$

where the first factor on the right side of (6) is the phase of the determinant and the second factor is the absolute value of the determinant of the Dirac operator.

In these lectures we study the case of the operator $\mathcal{D}_a = -i\frac{d}{dx} + a : C^{\infty}(S^1) \to C^{\infty}(S^1)$. We show that all ingredients in formula (6) are well-defined and that in fact ζ-determinant is the true algebraic determinant.

At the end of this introduction let us explain why it is enough to study operators $\mathcal{D}_a$, when it seems that we should be investigating operators of the form

$$-i\frac{d}{dx} + f(x) \quad ,$$

where $f(x)$ denotes a smooth real-valued function on S^1 . Let us consider two such operators

$$\mathcal{D}_{f_i} = -i\frac{d}{dx} + f_i(x) \quad .$$

and let us introduce functions

$$g_i(x) = \int_0^x f_i(s)ds \quad .$$

We have now the following result

Proposition 1.1 *Assume that*

$$g_1(2\pi) = g_2(2\pi) \quad , \tag{7}$$

then the operators $\mathcal{D}_{f_1}$ and $\mathcal{D}_{f_2}$ are unitary equivalent.

Proof. We define operator U acting on functions on S^1 via formula

$$(Us)(x) = e^{i(g_1(x)-g_2(x))}s(x) \quad .$$

The operator U is a unitary operator on $L^2(S^1)$, and the straightforward computation shows

$$U\mathcal{D}_{f_1}U^{-1} = -i\frac{d}{dx} + f_2(x)$$

hence $\mathcal{D}_{f_1}$ and $\mathcal{D}_{f_2}$ are unitary equivalent, which among other things implies that they have the same spectrum. $\square$

Corollary 1.2 *The operator $-i\frac{d}{dx}+f(x)$ is unitary equivalent to the operator $-i\frac{d}{dx}+a$, where*

$$a = \frac{\int_0^{2\pi} f(s)ds}{2\pi} \quad . \tag{8}$$

Corollary 1.3 *The spectrum of the operator $-i\frac{d}{dx} + f(x)$ is equal to $\{k + a\}_{k\in\mathbf{Z}}$, where a is given by the formula (8).*

This last corollary follows from the fact that we know the spectrum of the operator $\mathcal{D}_a = -i\frac{d}{dx} + a$. It has eigenvalues $k + a$ corresponding to eigenfunctions $\phi_k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}$.

2 ζ-function and η-function of a Dirac operator and *Heat Kernel*.

We study the ζ-determinant of the operator $\mathcal{D}_a$ using the *Heat Operator* determined by $\mathcal{D}_a^2$. We start with a discussion of the situation in the case of a general Dirac operator. Later, we prove all the results for the operator $\mathcal{D}_a$ on S^1 . Let $\mathcal{D} : C^\infty(M;S) \to C^\infty(M;S)$ denote a Dirac operator acting on sections of a bundle of Clifford modules over a closed manifold M . We want to solve the *Heat Propagation* problem for the operator $\mathcal{D}^2$, which means that having $f_0 \in C^\infty(M;S)$, we want to solve the problem:

$$(\frac{d}{dt} + \mathcal{D}^2)f(t,x) = 0 \quad for\ t > 0 \quad with \quad f(0,x) = f_0(x) \ . \tag{9}$$

The problem (9) has a unique solution for each smooth initial data $f_0(x)$ (see for instance $[R, T1, T2]$). The usual way to get the solution is to construct a family of operators $e^{-t\mathcal{D}^2} = E(t) : C^\infty(M;S) \to C^\infty(M;S)$ such that $E(0) = Id$ and for each f_0 and $t > 0$ we have

$$(E(t)f_0)(x) = f(t,x) \quad .$$

It is not difficult to see that $E(t)$ satisfies the semigroup property i.e. for each $s, t > 0$ we have $E(t+s) = E(t)E(s)$. Moreover, operator $E(t)$ has a smooth kernel, which means that there exists a smooth function $e(t;x,y)$, where for each $x, y \in M$ $e(t;x,y)$ is a linear map from S_x to S_y such that

$$(e^{-t\mathcal{D}^2} f_0)(x) = f(t,x) = \int_M e(t;x,y)f_0(y)dy \ . \tag{10}$$

Assuming that we know the spectral decomposition of the operator $\mathcal{D}$ we have a nice abstract formula which gives the kernel of the *Heat Operator*. Let us denote by λ_k an eigenvalue of $\mathcal{D}$, which corresponds to the eigensection ϕ_k. Then we have

$$e(t;x,y) = \sum_{-\infty}^{+\infty} e^{-t\lambda_k^2}\phi_k(x) \otimes \phi_k(y) \ . \tag{11}$$

In other words, we have equality

$$(E(t)s)(x) = \sum_{-\infty}^{+\infty}(\int_M < s(y);\phi_k(y) > dy)\phi_k(x) \quad ,$$

where $< \cdot;\cdot >$ denote an inner product on the fibre S_y . We refer to $[G]$ and $[T1]$, $[T2]$ for more details. In the following we concentrate on a very easy special case. Although representation (11) looks nice, it is not of very much use if we want an explicit formula for the kernel $e(t;x,y)$. We start with the differential operator, for which we know the exact formula for the kernel and then study how the perturbation of the operator affects the *Heat Kernel*. This is what we do in the next Section in order to study ζ-function of the operator $\mathcal{D}_0^2 = -\frac{d^2}{dx^2}$ on the circle. Here we justify the introduction of the *Heat Operator*. We use this operator in order to study ζ-function and η-function of the Dirac operators.

Proposition 2.1 *The following equalities hold for a Dirac operator $\mathcal{D}$*

$$\zeta_{\mathcal{D}^2}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}^2} dt \quad for \quad Re(s) > \frac{dim\ M}{2} \quad , \tag{12}$$

and

$$\eta_{\mathcal{D}}(s) = \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{s-1} Tr\ \mathcal{D}e^{-t\mathcal{D}^2} dt \quad for \quad Re(s) > \frac{1+dim\ M}{2} \quad .$$

where in the discussion of $\zeta_{\mathcal{D}^2}(s)$ we assume that $\mathcal{D}$ is invertible. This assumption is not necessary in the case of $\eta_{\mathcal{D}}(s)$.

Proof. We prove second equality in (12). The proof of the first one is completely analogous. We have

$$\int_0^\infty t^{\frac{s-1}{2}} Tr\ \mathcal{D}e^{-t\mathcal{D}^2} dt = \sum_{-\infty}^{+\infty} \int_0^\infty t^{\frac{s-1}{2}} \lambda_k e^{-t\lambda_k^2} dt =$$

$$= \sum_{-\infty}^{+\infty} \lambda_k (\lambda_k^2)^{-\frac{s+1}{2}} \int_0^\infty (t\lambda_k^2)^{\frac{s-1}{2}} e^{-t\lambda_k^2} d(t\lambda_k^2) =$$

$$= \sum_{-\infty}^{+\infty} sign\ \lambda_k \cdot |\lambda_k|^{-s} \cdot \int_0^\infty r^{\frac{s-1}{2}} e^{-r} dr = \Gamma(\frac{s+1}{2}) \cdot \eta_{\mathcal{D}}(s) \quad .$$

We discuss the suitable domain of s for which (12) is valid only for the operator $\mathcal{D}_f$. $\square$

Remark 2.2 If we assume that $\mathcal{D}$ has eigenvalue 0 , i.e. there exists a non-trivial solution of the equation $\mathcal{D}s = 0$, then of course the first formula in (12) does not hold as the integral on the right side is divergent. To cure this problem we proceed as follows: the operator $\mathcal{D}$ is an elliptic operator on a closed manifold hence $ker\ \mathcal{D}$, *kernel of* $\mathcal{D}$, the space of all solutions of $\mathcal{D}s = 0$, is finite dimensional and consists of only smooth sections. This is obvious for the operator $\mathcal{D}_f$ and we refer to $[G]$, or $[T1]$, $[T2]$ for the proof of this fact for general Dirac operator. We consider the integral in (12) on the orthogonal complement of this space, but to stay consistent with definition (3) we have to add the dimension of $ker\ \mathcal{D}$. More precisely, if we denote by $\Pi_{\mathcal{D}}$ the orthogonal projection onto $ker\ \mathcal{D}$, then the first formula in (12) is replaced by

$$\zeta_{\mathcal{D}^2}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr(e^{-t\mathcal{D}^2} - \Pi_{\mathcal{D}}) dt + dim\ ker\ \mathcal{D} \quad \text{for} \quad Re(s) > \frac{dim\ M}{2} \ . \tag{13}$$

3 Heat kernel and ζ-function on S^1.

We now discuss the ζ-function of the operator $\mathcal{D}_0^2 = \Delta = -\frac{d^2}{dx^2}$ on S^1 . We will use formula (12) so that we need information about the kernel of the operator $e^{-t\Delta}$. It is not difficult to check that the $e_{\mathbf{R}^1}(t;x,y)$ kernel of the corresponding operator on $\mathbf{R}^1$ is given by the formula

$$e_{\mathbf{R}^1}(t;x,y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \quad .$$

Let me repeat again that this means that the function $f(t,x)$ given by the formula

$$f(t,x) = (e^{-t(-\frac{d^2}{dx^2})} f_0)(t,x) = \int_{\mathbf{R}^1} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} f_0(y) dy$$

solves the problem (9) with the initial data $f_0(x)$. Now we define kernel on S^1 using the formula

$$e_{S^1}(t;x,y) = \sum_{n \in \mathbf{Z}} e_{\mathbf{R}^1}(t;x,y+2\pi n) \quad . \tag{14}$$

In (14) we use the representation of S^1 as $\mathbf{R}/2\pi\mathbf{Z}$. Now as an exercise the reader may check the following fact (see for instance $[R]$)

Proposition 3.1 *The kernel of the operator* $e^{-t\Delta}$ *on* S^1 *is equal to* $e_{S^1}(t;x,y)$.

Proposition 3.1 has the following extremely important consequence

Theorem 3.2 *Let us assume that* $0 < t < 1$ *, then there exist positive constants* c_1, c_2 *such that the following equality holds*

$$|e_{S^1}(t:x,y) - e_{\mathbf{R}^1}(t;x,y)| < c_1 \cdot e^{-\frac{c_2}{t}} \quad .$$

Remark 3.3 (1) The statement of the Theorem 3.2 is usually written as

$$e_{s^1}(t;x,y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} + O(e^{-\frac{c}{t}}) \quad \text{for } t \text{ small} \ . \tag{15}$$

(2) In the following we really need (15) for (x,y) close to the diagonal hence the distance between x and y is indeed given by $|x-y|$.

Proof. We have

$$e_{s^1}(t;x,y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} + \frac{1}{\sqrt{4\pi t}} \cdot \sum_{n\neq 0} e^{-\frac{(x-y-2\pi n)^2}{4t}} \quad ,$$

and we estimate the sum as follows

$$\sum_{n\neq 0} e^{-\frac{(x-y-2\pi n)^2}{4t}} = \sum_{n=1}^{\infty} (e^{-\frac{(x-y-2\pi n)^2}{4t}} + e^{-\frac{(x-y+2\pi n)^2}{4t}}) < 2\cdot \sum_{n=1}^{\infty} e^{-\frac{(\pi n)^2}{4t}} <$$

$$2\cdot \sum_{n=1}^{\infty} (e^{-\frac{\pi^2}{4t}})^n = 2\cdot \frac{1}{e^{\frac{\pi^2}{4t}}} \cdot \frac{1}{1-\frac{1}{e^{\frac{\pi^2}{4t}}}} = \frac{2}{e^{\frac{\pi^2}{4t}-1}} < \frac{2}{e^{\frac{\pi^2}{8t}}} = 2\cdot e^{-\frac{\pi^2}{8t}} \quad .$$

Now (15) follows from the elementary estimate

$$\frac{1}{\sqrt{t}} \cdot e^{-\frac{c}{t}} < c_1 \cdot e^{-\frac{c}{2t}} \quad .$$

□

Let $f, g : (0,\infty) \to \mathbf{R}$ be smooth functions, then we write $f \sim g$ if and only if for any natural number m we have

$$\lim_{t\to 0} \frac{f(t)-g(t)}{t^m} = 0 \quad .$$

In particular we have just shown that for each $x, y \in S^1$

$$e_{S^1}(t;x,y) \sim \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{t}} \quad .$$

The next Corollary is the first result, which ties the spectral geometry that we are studying with *Number Theory.* It is also at that point that we introduce the trace of the heat operator.

Corollary 3.4

$$\sum_{n\in\mathbf{N}} e^{-tn^2} \sim \frac{\sqrt{\pi}}{2} \cdot t^{-1/2} - \frac{1}{2} \quad . \tag{16}$$

Proof. We study the trace of the *Heat Operator* $e^{-t\Delta}$ on S^1 . The operator Δ has a complete set of eigenvalues with corresponding eigensections given by the standard basis of $L^2(S^1)$. Namely, for any integer k , k^2 is an eigenvalue with corresponding eigenfunction $\phi_k = \frac{1}{\sqrt{2\pi}} e^{ikx}$. We know that the trace of the operator can be represented by the eigensections and eigenfunctions as follows (see for instance one of our standard references like [G], and for more of related *Functional Analysis* [$RS1$], or [$T1$]). In our particular case we have

$$Tr\ e^{-t\Delta} = \sum_{k\in\mathbf{Z}} (e^{-t\Delta}\phi_k ; \phi_k) = \sum_{k\in\mathbf{Z}} e^{-tk^2} = 1 + 2 \cdot \sum_{n\in\mathbf{N}} e^{-tn^2} \quad ,$$

where

$$(f;g) = \int_{S^1} f(x)\overline{g}(x)dx$$

is standard L^2 product on S^1 . On the other hand the trace of an operator with a smooth kernel is also given by the integral of this kernel over the diagonal

$$Tr\ e^{-t\Delta} = \int_{S^1} e_{s^1}(t;x,x)dx \quad .$$

Therefore, we have

$$\sum_{n=1}^{\infty} e^{-tn^2} = \frac{1}{2}(Tr\ e^{-t\Delta} - 1) = \frac{1}{2}(\int_{S^1} e_{S^1}(t;x,x)dx - 1) =$$

$$= \frac{1}{2}(\frac{1}{\sqrt{4\pi t}}\int_{S^1} dx - 1) + O(e^{-\frac{c}{t}}) \quad ,$$

which finally gives (16). $\square$

Now we investigate $\zeta_\Delta(s)$ the ζ-function of the operator Δ . Let us observe that this analysis is immediately related to *Number Theory*. Namely, if we introduce *Riemann ζ-function*

$$\zeta_{\mathcal{R}}(s) = \sum_{n=1}^{\infty} n^{-s} \quad ,$$

then we have the equality

$$\zeta_\Delta(s) = \sum_{k\in\mathbf{Z}} (k^2)^{-s} = 2\cdot\zeta_{\mathcal{R}}(2s) + 1 \quad . \tag{17}$$

This is the reason that we can formulate the main result of this Section in terms of $\zeta_{\mathcal{R}}(s)$. We have the following Theorem

Theorem 3.5 *Function $\zeta_{\mathcal{R}}(s)$ is a holomorphic function of s for $Re(s) > 1$. It has a meromorphic extension to the complex plane. Point $s = 1$ is the only pole of $\zeta_{\mathcal{R}}(s)$. It is a simple pole and we have*

$$Res_{s=1}\zeta_{\mathcal{R}}(s) = 1 \ \ , \ \ \zeta_{\mathcal{R}}(0) = -\frac{1}{2} \ \ and \ \ \zeta_{\mathcal{R}}(-2l) = 0 \ \ for\ l = 1,2,3,\ldots \ . \tag{18}$$

Theorem 3.5 follows from the corresponding result for the ζ-function of the operator $\Delta = \mathcal{D}_0^2$ on S^1 . We use representation (13) . We also need two properties of the function $\Gamma(s)$. First, let us recall that in the neighborhood of $s = 0$ function $\Gamma(s)$ has the following form

$$\Gamma(s) = \frac{1}{s} + \gamma + sf(s) = \frac{1 + s\gamma + s^2 f(s)}{s} \quad , \tag{19}$$

where γ denotes Euler's constant. We also use the identity

$$s\Gamma(s) = \Gamma(s+1) \ , \tag{20}$$

in order to extend $\Gamma(s)$ from the holomorphic function on $Re(s) > 0$ to a meromorphic functionon on the whole complex plane. Points $s_k = -k \ \ k = 0, 1, 2, ..$ are the only poles and

$$Res_{s=-k}\Gamma(s) = \frac{(-1)^k}{k!} \ , \tag{21}$$

as follows from (19) and (20). Let us also observe that now it is easy to show that function $\frac{1}{\Gamma(s)}$ is a holomorphic function of s on the whole complex plane and the only zeros of $\frac{1}{\Gamma(s)}$ are points $s_k = -k$. Now, we are ready to analyze the function $\zeta_\Delta(s)$.

Proposition 3.6 *Let function $h(s)$ be given by the formula*

$$h(s) = \frac{1}{\Gamma(s)} \cdot \int_1^\infty t^{s-1} Tr(e^{-t\mathcal{D}^2} - \Pi_{\mathcal{D}}) dt \ . \tag{22}$$

Then $h(s)$ is a holomorphic function of s on the whole complex plane.

Proof. We estimate $Tr(e^{-t\mathcal{D}^2} - \Pi_{\mathcal{D}})$ for $1 < t$ as follows

$$Tr(e^{-t\mathcal{D}^2} - \Pi_{\mathcal{D}}) = 2\sum_{k\in\mathbf{N}} e^{-tk^2} = 2\sum_{k\in\mathbf{N}} e^{-\frac{t}{2}k^2} \cdot e^{-\frac{t}{2}k^2} <$$

$$2 \cdot e^{-\frac{t}{2}} \cdot \sum_{k\in\mathbf{N}} e^{-\frac{t}{2}k^2} < 2 \cdot e^{-\frac{t}{2}} \cdot \sum_{k\in\mathbf{N}} e^{-\frac{k^2}{2}} < c \cdot e^{-\frac{t}{2}} \ ,$$

and the result follows from elementary complex ananlysis. □

It is the integral on the interval $0 < t < 1$, which determines singularities of $\zeta_\Delta(s)$. We have

$$\frac{1}{\Gamma(s)} \cdot \int_0^1 t^{s-1} Tr(e^{-t\mathcal{D}^2} - \Pi(\mathcal{D})) dt = \frac{1}{\Gamma(s)} \cdot \int_0^1 t^{s-1} (\int_{S^1} (\frac{1}{\sqrt{4\pi t}} - 1 + O(e^{-\frac{c}{t}}) dx) dt$$

$$= \frac{\sqrt{\pi}}{\Gamma(s)} \cdot \int_0^1 t^{s-\frac{1}{2}} dt + g(s) = \frac{\sqrt{\pi}}{\Gamma(s)} \cdot \frac{1}{s-\frac{1}{2}} - \frac{1}{\Gamma(s)} \cdot \int_0^1 t^{s-1} dt + \frac{1}{\Gamma(s)} \cdot g(s) \quad ,$$

where $g(s)$ is yet another function holomorphic on the whole complex plane and we use the fact that

$$Tr \ \Pi(\mathcal{D}) = dim \ ker \ \mathcal{D} = 1 \quad .$$

We have proved the following Theorem

Theorem 3.7 *There exists a function $g(s)$ holomorphic on the whole complex plane such that $\zeta_\Delta(s)$ has the following representation*

$$\zeta_\Delta(s) = \frac{\sqrt{\pi}}{\Gamma(s)} \cdot \frac{1}{s-\frac{1}{2}} - \frac{1}{\Gamma(s)} \cdot \frac{1}{s} + 1 + \frac{1}{\Gamma(s)} \cdot g(s) \quad . \tag{23}$$

We combine this result with

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \ \ and \ \ \frac{1}{\Gamma(-k)} = 0 \ \ for \ k = 0, 1, 2.. \quad ,$$

in order to obtain the next result

Corollary 3.8 *The only pole of $\zeta_\Delta(s)$ is located at $s = \frac{1}{2}$. It is a simple pole and the reesidue of $\Gamma(s)$ at $s = 1$ is equal to 1 . Moreover $\zeta_\Delta(0) = 0$ and $\zeta_\Delta(-k) = 1$ for $k = 1, 2, 3, ..$.*

Theorem 3.5 follows immediately from Corollary 3.8 since now we can use (23) to represent $\zeta_{\mathcal{R}}(s)$ as

$$\zeta_{\mathcal{R}}(s) = \frac{1}{2}(\zeta_\Delta(\frac{s}{2}) - 1) = \frac{\sqrt{\pi}}{\Gamma(\frac{s}{2})} \cdot \frac{1}{s-1} - \frac{1}{\Gamma(\frac{s}{2})} \cdot \frac{1}{s} + \frac{1}{2\Gamma(\frac{s}{2})} \cdot g(\frac{s}{2}) \quad . \tag{24}$$

4 Duhamel's Principle and ζ-function of the operators Δ_f.

In this section we study the ζ-function of the perturbations of the operator $\Delta = \Delta_0 = \mathcal{D}_0^2$. We discuss in order of increasing technical difficulty the operators $\Delta_a = \Delta + a$, $\Delta_f = \Delta + f(x)$, where $f(x)$ is a smooth function on S^1 , and the operator $\mathcal{D}_a^2 = -\frac{d^2}{dx^2} - 2ia\frac{d}{dx} + a^2$. The easiest example here is of course operator Δ_a . It is natural to expect the equality

$$e^{-t\Delta_a} = e^{-t\Delta - ta} = e^{-ta}e^{-t\Delta}$$

to hold, especially because the bounded operator $(Bf)(x) = af(x)$ commutes with the operator Δ and the equality

$$e^{-t(A+B)} = e^{-tA}e^{-tB}$$

holds for commuting matrices A , B and for bounded commuting operators in a Hilbert space. The operator Δ is an unbounded operator on $L^2(S^1)$, which creates technical problems. We discuss the standard way of getting the *Heat Operator* for the perturbation of the given (Dirac) operator. We introduce now *Duhamel's Principle* . We offer a formal formulation without precise assumptions, and later on we will show that everything can be made rigorous in the case of a Dirac operators on S^1 .

Duhamel's Principle Let $\mathcal{A}$ and $\mathcal{A} + B$ be operators acting on a separable Hilbert space such that the operators $e^{-t\mathcal{A}}$ and $e^{-t(\mathcal{A}+B)}$ exist. Then

$$e^{-t(\mathcal{A}+B)} = e^{-t\mathcal{A}} - \int_0^t e^{-s(\mathcal{A}+B)}Be^{-(t-s)\mathcal{A}}ds \ . \tag{25}$$

Proof. The *Heat Operator* is equal to the identity at $t = 0$, therefore

$$e^{-t(\mathcal{A}+B)} - e^{-t\mathcal{A}} = \int_0^t \frac{d}{ds}(e^{-s(\mathcal{A}+B)}e^{-(t-s)\mathcal{A}})ds =$$

$$= \int_0^t e^{-s(\mathcal{A}+B)}(-(\mathcal{A}+B) + \mathcal{A})e^{-(t-s)\mathcal{A}}ds. \qquad \square$$

Now we apply formula (25) to $e^{-s(\mathcal{A}+B)}$ in $\int_0^t e^{-s(\mathcal{A}+B)}Be^{-(t-s)\mathcal{A}}ds$ and we obtain

$$e^{-t(\mathcal{A}+B)} =$$

$$= e^{-t\mathcal{A}} - \int_0^t e^{-s\mathcal{A}}Be^{-(t-s)\mathcal{A}}ds + \int_0^t ds \int_0^s dr e^{-r(\mathcal{A}+B)}Be^{-(s-r)\mathcal{A}}Be^{-(t-s)\mathcal{A}} \quad .$$

The discussion above shows a way of constructing the operator $e^{-t(\mathcal{A}+B)}$. We express operator $e^{-t(\mathcal{A}+B)}$ as an infinite series in terms of the operators $e^{-t\mathcal{A}}$ and B . We introduce some notation before getting more specific. Let $\mathcal{B}(t)$ and $\mathcal{C}(t)$ denote two 1-parameter families of bounded operators in a Hilbert space. We introduce $(\mathcal{B}*\mathcal{C})(t)$ *convolution* of $\mathcal{B}(t)$ and $\mathcal{C}(t)$ as the operator

$$(\mathcal{B}*\mathcal{C})(t) = \int_0^t \mathcal{B}(s)\mathcal{C}(t-s)ds \quad .$$

We take

$$\mathcal{B}(t) = e^{-t\mathcal{A}} \quad , \quad \mathcal{C}(t) = Be^{-t\mathcal{A}} \quad and \quad \mathcal{C}_n(t) = (\mathcal{C}*\mathcal{C}*..*\mathcal{C})(t) \quad ,$$

where we convoluted $\mathcal{C}(t)$ $n-$times in the last formula. Then the following formal equation gives the *Heat Operator* of $\mathcal{A}+B$

$$e^{-t(\mathcal{A}+B)} = e^{-t\mathcal{A}} + \sum_{n=1}^{\infty}(-1)^n(\mathcal{B}*\mathcal{C}_n)(t) \quad . \tag{26}$$

At this point it is a formal expression, which becomes the rigorous identity whenever we are able to show that the series on the right side of (26) is absolutely convergent. Actually, in our case, we will show even more. We use (26) in order to construct the kernel of the operator $e^{-t\Delta_f}$. We take $\mathcal{A}=\Delta$ and $\mathcal{B}=f(x)$ then $e_{n,f}(t;x,y)$ kernel of the operator $(\mathcal{B}*\mathcal{C}_n)(t)$ is equal to

$$e_{n,f}(t;x,y) = \int_0^t ds_1 \int_{S^1} du_1 \int_0^{s_1} ds_2 \int_{S^1} du_2 ..... \int_0^{s_{n-1}} ds_n \int_{S^1} du_n \tag{27}$$

$$e_\Delta(s_n;x,u_n)f(u_n)e_\Delta(s_{n-1};u_n,u_{n-1}).....f(u_1)e_\Delta(t-s_1;u_1,y) \quad .$$

We use this representation in order to prove the absolute and uniform convergence of the series which represents kernel of the operator $e^{-t\Delta_f}$.

Theorem 4.1 *There exists t_0 such that the series*

$$e_\Delta(t;x,y) + \sum_{n=1}^{\infty}(-1)^n e_{n,f}(t;x,y) \tag{28}$$

converges uniformly on S^1 for $0 < t \leq t_0$.

Remark 4.2 1. Although we prove Theorem 4.1 for small t_0 the choice of t_0 is unsignificant. We can easily prove the following variant of the Theorem:

For any t_0 there exists a constant M_{t_0} such that for any $0 < t \leq t_0$ and any $x, y \in S^1$

$$|e_{\Delta_f}(t;x,y) - e_\Delta(t;x,y)| = |\sum_{n=1}^{\infty}(-1)^n e_n(t;x,y)| < M_{t_0} \quad . \tag{29}$$

We leave the estimates leading to (29) to the willing reader.

2. We do not present the optimal result. In fact the estimates in the proof were written down half an hour before the lecture. This is not a very important matter. We want to show to the uninitiated reader an unpolished way of proving that the existence of the *Heat Operator* for the operator Δ_f follows from the perturbation of the operator $e^{-t\Delta}$. We will use extra information on the kernel $e_{S^1}(t;x,y)$ later on.

Proof. There exist positive constants c_1, c_2 such that for any $x, y \in S^1$ and for any $0 < t \leq 1$ we have

$$|e_\Delta(t;x,y)| < \frac{c_1}{\sqrt{t}} \quad , \quad |f(x)| < c_2 \quad .$$

We start with the term $e_{1,f}(t;x,y)$

$$|e_1, f(t;x,y)| = |\int_0^t ds \int_{S^1} du \; e_\Delta(s;x,u) f(u) e_\Delta(t-s;u,y)| <$$

$$\frac{c_1^2 c_2}{4\pi} \int_0^t ds \int_{S^1} du \; \frac{1}{\sqrt{s(t-s)}} = \frac{c_1^2 c_2}{2} \int_0^t \frac{ds}{\sqrt{s(t-s)}} \leq c_1^2 c_2 \int_0^{\frac{t}{2}} \frac{ds}{\sqrt{s \cdot \frac{t}{2}}} =$$

$$c_1^2 c_2 \cdot \sqrt{\frac{t}{2}} \cdot \int_0^{\frac{t}{2}} \frac{ds}{\sqrt{s}} = 2 c_1^2 c_2 \quad .$$

The second inequality is even easier. Let $K > 0$ denote a constant, then we have

$$|\int_0^t ds \int_{S^1} du \; K f(u) e_\Delta(t-s;u,y)| < 2\pi c_1 c_2 K \cdot \int_0^t \frac{ds}{\sqrt{t-s}} = 4\pi c_1 c_2 K \sqrt{t} \quad .$$

Now, we can deal with $e_{n,f}(t;x,y)$

$$|e_{n,f}(t;x,y)| = |\int_0^t ds_1 \int_{S^1} du_1 \int_0^{s_1} ds_2 \int_{S^1} du_2 ..... \int_0^{s_{n-1}} ds_n \int_{S^1} du_n$$

$$e_\Delta(s_n;x,u_n) f(u_n) e_\Delta(s_{n-1}-s_n;u_n,u_{n-1}) ..... f(u_1) e_\Delta(t-s_1;u_1,y)| <$$

$$2c_1^2 c_2 \int_0^t ds_1 \int_0^t ds_1 \int_{S^1} du_1 \int_0^{s_1} ds_2 \int_{S^1} du_2 ... \int_0^{s_{n-2}} ds_{n-1} \int_{S^1} du_{n-1}$$

$$|f(u_{n-1}) e_\Delta(s_{n-2}-s_{n-1};u_{n-1},u_{n-2}) ..... f(u_1) e_\Delta(t-s_1;u_1,y)| <$$

$$(2c_1^2 c_2)(4\pi c_1 c_2) \int_0^t ds_1 \int_{S^1} du_1 \int_0^{s_1} ds_2 \int_{S^1} du_2 ... \int_0^{s_{n-3}} ds_{n-2} \int_{S^1} du_{n-2}$$

$$\sqrt{s_{n-2}} \cdot |f(u_{n-2}) e_\Delta(s_{n-3}-s_{n-2};u_{n-2},u_{n-3}) ..... f(u_1) e_\Delta(t-s_1;u_1,y)| <$$

$$(2c_1^2c_2)(4\pi c_1c_2)\int_0^t ds_1\int_{S^1}du_1\int_0^{s_1}ds_2\int_{S^1}du_2...\sqrt{s_{n-3}}\int_0^{s_{n-3}}ds_{n-2}\int_{S^1}du_{n-2}$$

$$|f(u_{n-2})e_\Delta(s_{n-3}-s_{n-2};u_{n-2},u_{n-3}).....f(u_1)e_\Delta(t-s_1;u_1,y)| <$$

$$(2c_1^2c_2)(4\pi c_1c_2)^2\int_0^t ds_1\int_{S^1}du_1.....\int_0^{s_{n-4}}ds_{n-3}\int_{S^1}du_{n-3}s_{n-3}$$

$$|f(u_{n-3})e_\Delta(s_{n-4}-s_{n-3};u_{n-3},u_{n-4}).....f(u_1)e_\Delta(t-s_1;u_1,y)| <$$

$$< (2c_1^2c_2)(4\pi c_1c_2)^{n-1}\cdot t^{\frac{n-1}{2}} = (2c_1^2c_2)(4\pi c_1c_2\sqrt{t})^{n-1} \quad .$$

Now, finally we can estimate the whole series.

$$|\sum_{n=1}^{\infty}(-1)^n e_n(t;x,y)| < \sum_{n=1}^{\infty}(2c_1^2c_2)(4\pi c_1c_2\sqrt{t})^{n-1} =$$

$$= (2c_1^2c_2)(4\pi c_1c_2\sqrt{t})\cdot\frac{1}{1-4\pi c_1c_2\sqrt{t}} \quad .$$

We have proved that the series (28) is uniformly convergent for $0 < t < \frac{1}{(4\pi c_1c_2)^2}$. □

The terms in the series are expressed in the succesive powers of $\sqrt{t}$, which leads to the following fact

Corollary 4.3 *There exist a family $\{u_n(x)\}_{n\in\mathbf{N}}$ of smooth functions on S^1 such that for any natural number N we have*

$$e_{\Delta_f}(t;x,x) - \sum_{n=0}^{N} t^{\frac{n-1}{2}}u_n(x) = O(t^N) \quad . \tag{30}$$

The last result is not really the best we can get. We know (see for instance $[G]$,$[T2]$) that actually we have an expansion in powers of t . This result is actually quite difficult to establish in the case of a general Dirac operator. In the case of S^1 we can use the fact that, for any $0 < t$ and any $x, y \in S^1$ $e_\Delta(t; x, y) > 0$ (see (14)). We also use the fact that identity $e^{-s\Delta}e^{-(t-s)\Delta} = e^{-t\Delta}$ reads as follows on the level of the kernel of the operators

$$\int_{S^1} e_\Delta(s; x, u) e_\Delta(t-s; u, y) du = e_\Delta(t; x, y) \ . \tag{31}$$

Theorem 4.4 *There exists a family* $\{v_n(x)\}_{n \in \mathbf{N}}$ *of smooth functions on* S^1 *such that for any natural number* N *we have*

$$e_{\Delta_f}(t; x, x) - \sum_{n=0}^{N} t^{n-\frac{1}{2}} v_n(x) = O(t^N) \ . \tag{32}$$

Proof.

We show that each next term in the series has one more power of t . Let us explain this on the level of the first term

$$|\int_0^t ds \int_{S^1} du \ e_\Delta(s; x, u) f(u) e_\Delta(t-s; u, y)| <$$

$$c_2 \cdot \int_0^t ds \int_{S^1} du \, e_\Delta(s; x, u) e_\Delta(t-s; u, y) = c_2 \cdot \int_0^t e_\Delta(t; x, y) ds = c_2 t \cdot e_\Delta(t; x, y) \ .$$

The estimate on the $n-th$ term of the expansion follows in the same way. $\square$

Corollary 4.5 *For any smooth function* $f(x)$ *on* S^1 *we have*

$$\zeta_{\Delta_f}(0) = 0 \ .$$

Proof. The result follows from the corresponding result for $\Delta = \Delta_0$, as we have just proved that there exists a positive constant c such that the following estimate holds

$$|e_{\Delta_f}(t;x,y) - e_{\Delta}(t;x,y)| < c\sqrt{t} \quad ,$$

for any $x, y \in S^1$ and any $0 < t \leq 1$. Let Π_{Δ_f} denote the orthogonal projection onto the kernel of the operator Δ_f . Then we have

$$\zeta_{\Delta_f}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} Tr(e^{-t\Delta_f} - \Pi_{\Delta_f}) dt + dim\ ker\ \Delta_f =$$

$$\frac{1}{\Gamma(s)} \int_0^{1} t^{s-1} Tr(e^{-t\Delta_f} - \Pi_{\Delta_f}) dt + dim\ ker\ \Delta_f + \frac{1}{\Gamma(s)} \cdot h(s) \quad ,$$

where $h(s)$ is a function holomorphic on the whole complex plane. Trace of Π_{Δ_f} is equal to $dim\ ker\ \Delta_f$ and therefore we have

$$\zeta_{\Delta_f}(s) =$$

$$\frac{1}{\Gamma(s)} \int_0^{1} t^{s-1} Tr\ e^{-t\Delta_f} dt + (dim\ ker\ \Delta_f) \cdot (1 - \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} ds) + \frac{1}{\Gamma(s)} \cdot h(s) =$$

$$\frac{1}{\Gamma(s)} \int_0^{1} t^{s-1} (\frac{\sqrt{\pi}}{\sqrt{t}} + O(\sqrt{t})) dt + (dim\ ker\ \Delta_f) \cdot (1 - \frac{s}{\Gamma(s)}) + \frac{1}{\Gamma(s)} \cdot h(s) =$$

$$\frac{\sqrt{\pi}}{\Gamma(s)} \cdot \frac{1}{s - \frac{1}{2}} + \frac{1}{\Gamma(s)} \int_0^{1} t^{s-1} O(\sqrt{t}) dt + (dim\ ker\ \Delta_f) \cdot (1 - \frac{s}{\Gamma(s)}) + \frac{1}{\Gamma(s)} \cdot h(s) \quad .$$

We take $\lim_{s \to 0}$ and obtain 0 .

□

As an exercise reader may use *Duhamel's Principle* to check that

$$e_{\Delta + a}(t;x,y) = e^{-ta} e_{\Delta}(t;x,y) \quad . \tag{33}$$

5 Heat kernel of the operator $\mathcal{D}_a^2$.

In this section we use *Duhamel's Principle* to construct the kernel of the operator $e^{-t\mathcal{D}_a^2}$ on S^1 . Our goal is to prove the analogue of Theorem 4.4 in this new situation. We have

$$\mathcal{D}_a^2 = -\Delta - 2ia\frac{d}{dx} + a^2 = -\Delta + 2a\mathcal{D}_0 + a^2 \quad .$$

We know that there is no problem with a^2 . We simply have:

$$e_{-\Delta+2a\mathcal{D}_0+a^2}(t;x,y) = e^{-ta^2}e_{-\Delta+2a\mathcal{D}_0}(t;x,y) \quad .$$

Now we have to face the perturbation of order 1, which is not a bounded operator on $L^2(S^1)$. Still we can study the kernel of $e^{-\Delta+2a\mathcal{D}_0}$ the way we did it in a previous Section. We have to show the absolute convergence of the series

$$e_\Delta(t;x,y) + \sum_{n=1}^{\infty} e_{n,\mathcal{D}_a}(t;x,y) \quad , \tag{34}$$

where

$$e_{n,\mathcal{D}_a}(t;x,y) = \int_0^t ds_1 \int_{S^1} du_1 \int_0^{s_1} ds_2 \int_{S^1} du_2 \ldots\ldots \int_0^{s_{n-1}} ds_n \int_{S^1} du_n$$

$$e_\Delta(s_n;x,u_n)2a\mathcal{D}_0 e_\Delta(s_{n-1};u_n,u_{n-1})\ldots\ldots 2a\mathcal{D}_0 e_\Delta(t-s_1;u_1,y) \quad .$$

Let us first figure out the straightforward estimates on $e_{n,\mathcal{D}_a}(t;x,y)$. We have

$$|e_{1,\mathcal{D}_a}(t;x,y)| = 2a\cdot\left|\int_0^t ds \int_{S^1} e_\Delta(s;x,u)(\frac{d}{du})e_\Delta(t-s;u,y)\right| \leq$$

$$2a\frac{1}{4\pi}\cdot\int_0^t \frac{ds}{\sqrt{s(t-s)}}\int_{S^1} e^{-\frac{(x-u)^2}{4t}}\cdot\frac{|u-y|}{2(t-s)}e^{-\frac{(u-y)^2}{4t}} \leq$$

$$2a\frac{1}{4\pi}\cdot\int_0^t \frac{ds}{\sqrt{s(t-s)}}\int_{S^1} \frac{|u-y|}{2(t-s)}e^{-\frac{(u-y)^2}{4t}} <$$

$$\frac{2}{4\pi}\cdot\int_0^t \frac{ds}{\sqrt{s(t-s)}}\int_0^\infty \frac{z}{2(t-s)}e^{-\frac{z^2}{4t}} =$$

$$2a\frac{1}{2\pi}\int_0^t \frac{ds}{\sqrt{s(t-s}} \leq 2a\frac{1}{2\pi}\frac{2}{t}\cdot\int_0^t ds = 2a\frac{1}{\pi} \quad .$$

More generally we have

$$|e_{n,\mathcal{D}_a}(t;x,y)| < \frac{1}{\pi}(2a)^n \frac{1}{\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot\frac{n-1}{2}}(\frac{1}{\sqrt{\pi}})^{n-1}t^{\frac{n-1}{2}} \quad , \tag{35}$$

and as in Section 4, we have proved the uniform and absolute convergence of the series which formally gives the kernel of the operator $e^{-\Delta+2a\mathcal{D}_0}$; hence, we have just constructed this kernel. Again we expect that terms with odd n should disappear. This follows from the general theory (see for instance $[G]$ Chapter 1, or $[T2]$). However, in our case we can offer a simple argument which shows that $Tr\ e^{-t\mathcal{D}_a^2}$ expands in powers of t rather than $\sqrt{t}$.

Theorem 5.1 *There exists a sequence of real numbers* $\{r_k\}_{k=0}^{\infty}$ *such that for any natural number* N *we have*

$$Tr\ e^{-t\mathcal{D}_a^2} - \sum_{k=0}^{N} t^{k-\frac{1}{2}} r_k = O(t^N) \quad . \tag{36}$$

Remark 5.2 Working a little harder we would be able to prove that there exist a sequence of smooth functions $\{f_k(x)\}$ such that

$$e_{\mathcal{D}_a^2}(t;x,x) - \sum_{k=0}^{N} t^{k-\frac{1}{2}} f_k(x) = O(t^N) \quad .$$

This *"Local Variant"* of Theorem 5.1 for the general Dirac operator is proved in $[G]$,$[T2]$.

Proof. We use the fact that operators Δ and $\mathcal{D}_0 = -i\frac{d}{dx}$ commute. Now we write a series which gives the operator $e^{-t(\Delta+2a\mathcal{D}_0)}$

$$e^{-t(\Delta+2a\mathcal{D}_0)} = e^{-t\Delta} + (-1)^n E_n(t) \quad ,$$

where

$$E_n(t) =$$

$$(2a)^n \int_0^t ds_1 \int_0^{s_1} ds_2 ... \int_0^{s_{n-1}} ds_n e^{-s_n\Delta}\mathcal{D}_0 e^{-(s_{n-1}-s_n)\Delta}\mathcal{D}_0 ... \mathcal{D}_0 e^{-(t-s_1)\Delta} =$$

$$(2a)^n (\mathcal{D}_0)^n e^{-t\Delta} \cdot \int_0^t ds_1 \int_0^{s_1} ds_2 ... \int_0^{s_{n-1}} ds_n = (2a)^n \cdot \frac{t^n}{n!} (\mathcal{D}_0)^n e^{-t\Delta} \quad .$$

Theorem 5.1 follows from the fact that $\mathcal{D}_0 = -i\frac{d}{dx}$ has a symmetric spectrum which implies the equality

$$Tr\ \mathcal{D}_0^{2n+1} e^{-t\mathcal{D}^2} = 0 \quad ,$$

for any natural number n .

□

Corollary 5.3 *For any real* a

$$\zeta_{\mathcal{D}_a^2}(0) = 0 \quad . \tag{37}$$

Proof. We have just proved

$$Tr\ e^{-t\mathcal{D}_a^2} - Tr\ e^{-t\mathcal{D}_0^2} < \frac{c}{\sqrt{t}} \quad ,$$

for $0 < t < 1$.

□

6 η-invariant, The Phase of the ζ-determinant.

In this section we study the η-invariant of the operator $\mathcal{D}_a$. The main result is

Theorem 6.1 *The function $\eta_{\mathcal{D}_a}(s)$ is a holomorphic function of s for $Re(s) > -2$.*

Once again we prove this result by studying *Heat Kernels*. We recall the formula

$$\eta_{\mathcal{D}_a}(s) = \frac{1}{\Gamma(\frac{s+1}{2})} \cdot \int_0^\infty t^{\frac{s-1}{2}} Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2} dt \quad .$$

We now decompose $Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2}$ and use the fact that $Tr\ \mathcal{D}_0 e^{-t\mathcal{D}_0^2}$ is equal to 0

$$Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2} = Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2} - Tr\ \mathcal{D}_0 e^{-t\mathcal{D}_0^2} =$$

$$Tr(\mathcal{D}_a - \mathcal{D}_0) e^{-t\mathcal{D}_a^2} - Tr\ \mathcal{D}_0 (e^{-t\mathcal{D}_a^2} - e^{-t\mathcal{D}_0^2}) = a \cdot Tr\ e^{-t\mathcal{D}_a^2} - Tr\ \mathcal{D}_0 (e^{-t\mathcal{D}_a^2} - e^{-t\mathcal{D}_0^2}) \quad .$$

We know the expansion of the first summand on the right side

$$a \cdot Tr\ e^{-t\mathcal{D}_a^2} - a \cdot \sqrt{\frac{\pi}{t}} + c_1 \cdot \sqrt{t} = O(t^{\frac{3}{2}}) \quad . \tag{38}$$

We remember that

$$e^{-t\mathcal{D}_a^2} = e^{-ta^2} e^{-t(\Delta - 2a\mathcal{D}_0)} = e^{-ta^2} (e^{-t(\Delta - 2a\mathcal{D}_0)} - e^{-t\Delta}) + e^{-ta^2} e^{-t\Delta} \quad ,$$

and now we have to study

$$Tr\ \mathcal{D}_0 (e^{-t\mathcal{D}_a^2} - e^{-t\mathcal{D}_0^2}) = \tag{39}$$
$$Tr\ e^{-ta^2} \mathcal{D}_0 (e^{-t(\Delta - 2a\mathcal{D}_0)} - e^{-t\Delta}) + Tr\ (e^{-ta^2} - Id) \mathcal{D}_0 e^{-t\Delta} \quad .$$

The second term on the right side is again equal to 0 and finally we only have to show that $Tr\ e^{-ta^2}\mathcal{D}_0(e^{-t(\Delta-2a\mathcal{D}_0)}-e^{-t\Delta})$ has the correct asymptotic. It was already observed in Section 5 that

$$e^{-t(\Delta-2a\mathcal{D}_0)}-e^{-t\Delta}=\sum_{n=1}^{\infty}\frac{(2at)^n}{n!}\mathcal{D}_0^n e^{-t\Delta} \quad . \tag{40}$$

Hence we do have

$$Tr\ \mathcal{D}_0(e^{-t\mathcal{D}_a^2}-e^{-t\mathcal{D}_0^2})=Tre^{-ta^2}\mathcal{D}_0e^{-t(\Delta-2a\mathcal{D}_0)}=$$

$$Tr\ e^{-ta^2}\sum_{n=1}^{\infty}\frac{(2at)^n}{n!}\mathcal{D}_0^{n+1}e^{-t\Delta}=Tr\ e^{-ta^2}\sum_{k=1}^{\infty}\frac{(2at)^{2k-1}}{(2k-1)!}\mathcal{D}_0^{2k}e^{-t\Delta} \quad .$$

Now let us use an extra symmetry we have in this formula

$$Tr\ \mathcal{D}_0^{2k}e^{-t\Delta}=(-1)^k(\frac{d}{dt})^k(Tr\ e^{-t\mathcal{D}^2})=(-1)^k(\frac{d}{dt})^k(\frac{\sqrt{\pi}}{\sqrt{t}}+O(e^{-\frac{c}{t}})) \quad . \tag{41}$$

We have proved the following result

Proposition 6.2 *There exist a sequence of constants* $\{b_k\}$ *such that*

$$Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2}\sim\frac{b_0}{\sqrt{t}}+b_1\sqrt{t}+\ldots \quad . \tag{42}$$

However, we are not out of trouble if we take into consideration the situation in the neighborhood of $s=0$. This is due to the fact that we have to study

$$\lim_{s\to 0}\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^{\infty}t^{\frac{s-1}{2}}Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2}dt \quad ,$$

and in this situation factor $\frac{1}{\Gamma(s)}$ is replaced by $\frac{1}{\Gamma(\frac{s+1}{2})}$; hence, the singularity which comes from the heat kernel is not cancelled by singularity of $\Gamma(s)$. Now, when we study more precisely formulas (38) and (41) , we come to the conclusion that actually we have

$$b_0 = 0 \ , \tag{43}$$

in (42), but as in these notes the focus is on different methods related to *Heat Kernel* we offer yet another argument. This type of argument is used quite often in different contexts in *Spectral Geometry.* We define a function

$$\mathcal{R}(a) = Res_{s=0}\eta_{\mathcal{D}_a}(0) \quad .$$

This is a smooth function of the parameter a , and we have

Theorem 6.3

$$\frac{d\mathcal{R}}{da} = 0 \tag{44}$$

We need the next Lemma in the proof of Theorem 6.3.

Lemma 6.4

$$\frac{d}{da}(e^{-t\mathcal{D}_a^2}) = -\int_0^t e^{-s\mathcal{D}_a^2}(\dot{\mathcal{D}}_a\mathcal{D}_a + \mathcal{D}_a\dot{\mathcal{D}}_a)e^{-(t-s)\mathcal{D}_a^2}ds \ . \tag{45}$$

Remark 6.5 Of course (45) is the formula which holds for the smooth, 1-parameter family of Dirac operators over a closed manifold. In the particular case of the operator $\mathcal{D}_a = -i\frac{d}{dx} + a$ on S^1 , we have $\dot{\mathcal{D}}_a = \frac{d\mathcal{D}_a}{da} = 1$ and (45) becomes

$$\frac{d}{da}(e^{-t\mathcal{D}_a^2}) = -2\int_0^t \mathcal{D}_a e^{-t\mathcal{D}_a^2}ds = -2t\mathcal{D}_a e^{-t\mathcal{D}_a^2} \quad .$$

Proof. In the Lemma 6.4 we study the variation of the *Heat Kernel* under the smooth change of the Dirac operator

$$\frac{d}{da}\{e^{-t\mathcal{D}_a^2}\} = \lim_{r\to 0}\frac{1}{r}\cdot(e^{-t\mathcal{D}_{a+r}^2} - e^{-t\mathcal{D}_a^2}) = \lim_{r\to 0}\frac{1}{r}\cdot\int_0^t \frac{d}{ds}(e^{-s\mathcal{D}_{a+r}^2}e^{-(t-s)\mathcal{D}_a^2})ds =$$

$$\lim_{r\to 0}\frac{1}{r}\cdot\int_0^t e^{-s\mathcal{D}_{a+r}^2}(\mathcal{D}_a^2 - \mathcal{D}_{a+r}^2)e^{-(t-s)\mathcal{D}_a^2}ds =$$

$$\lim_{r\to 0}\int_0^t e^{-s\mathcal{D}_a^2}\frac{\mathcal{D}_a^2 - \mathcal{D}_{a+r}^2}{r}e^{-(t-s)\mathcal{D}_a^2}ds +$$

$$\lim_{r\to 0}\int_0^t \frac{1}{r}(e^{-s\mathcal{D}_{a+r}^2} - e^{-s\mathcal{D}_a^2})(\mathcal{D}_a^2 - \mathcal{D}_{a+r}^2)e^{-(t-s)\mathcal{D}_a^2}ds =$$

$$-\int_0^t e^{-s\mathcal{D}_a^2}\dot{\mathcal{D}}_a^2 e^{-(t-s)\mathcal{D}_a^2}ds + 0 = -\int_0^t e^{-s\mathcal{D}_a^2}(\dot{\mathcal{D}}_a\mathcal{D}_a + \mathcal{D}_a\dot{\mathcal{D}}_a)e^{-(t-s)\mathcal{D}_a^2}ds \quad .$$

□

Proof. Now we are ready to prove Theorem 6.3.

We have

$$\mathcal{R}(a) = \lim_{s\to 0} s\cdot\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}} Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2}dt \ , \tag{46}$$

and we study the variation of the integral on the right side of (46)

$$\frac{d}{da}\{\int_0^\infty t^{\frac{s-1}{2}} Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2}dt\} = \int_0^\infty t^{\frac{s-1}{2}} Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2}dt \ +$$

$$\int_0^\infty t^{\frac{s-1}{2}} Tr\ \mathcal{D}_a \{e^{-t\mathcal{D}_a^2}\dot{\}}dt = \int_0^\infty t^{\frac{s-1}{2}} Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2}dt + \int_0^\infty t^{\frac{s-1}{2}} Tr\ \mathcal{D}_a \dot{e^{-t\mathcal{D}_a^2}}dt =$$

$$\int_0^\infty t^{\frac{s-1}{2}} Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2}dt - 2\int_0^\infty t^{\frac{s-1}{2}} Tr\ t\dot{\mathcal{D}}_a\mathcal{D}_a^2 e^{-t\mathcal{D}_a^2} = \int_0^\infty t^{\frac{s-1}{2}} Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2}dt +$$

$$2\int_0^\infty t^{\frac{s+1}{2}}\frac{d}{dt}(Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2})dt = \int_0^\infty t^{\frac{s-1}{2}} Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2}dt\ +$$

$$2\lim_{\epsilon\to 0}(t^{\frac{s+1}{2}} Tr\ \dot{\mathcal{D}}_a e^{-\epsilon\mathcal{D}_a^2})|_{\varepsilon}^{\frac{1}{\varepsilon}} - 2\int_0^\infty \frac{d}{dt}(t^{\frac{s+1}{2}}) Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2}dt\ \ .$$

The limit

$$\lim_{\epsilon\to 0}(t^{\frac{s+1}{2}} Tr\ \dot{\mathcal{D}}_a e^{-\epsilon\mathcal{D}_a^2})|_{\varepsilon}^{\frac{1}{\varepsilon}}$$

is equal to 0 for the invertible operator $\mathcal{D}_a$ and $s > 0$ and we obtain a crucial formula

$$\frac{d}{da}\{\int_0^\infty t^{\frac{s-1}{2}} Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2}dt\} = -s\int_0^\infty t^{\frac{s-1}{2}} Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2}dt\ \ . \tag{47}$$

Once again the formula simplifies in the case of $\mathcal{D}_a$ on S^1 as $\dot{\mathcal{D}}_a = 1$. Now we finally have information about the variation of the residue

$$\frac{d\mathcal{R}}{da} = \frac{d}{da}\lim_{s\to 0} s\cdot\eta_{\mathcal{D}_a}(s) = -\lim_{s\to 0}\frac{s}{\Gamma(\frac{s+1}{2})}\frac{d}{da}\{\int_0^\infty t^{\frac{s-1}{2}} Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2}dt\} =$$

$$-\frac{1}{\sqrt{\pi}}\lim_{s\to 0} s^2\int_0^\infty t^{\frac{s-1}{2}} Tr\ e^{-t\mathcal{D}_a^2}dt = 0$$

□

Let us observe that in fact we also obtain the formula for the variation of the η-invariant i.e. the number $\eta_{\mathcal{D}_a}(0)$.

$$\dot{\eta}_{\mathcal{D}_a}(0) = \lim_{s\to 0}\frac{-s}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}} Tr\ e^{-t\mathcal{D}_a^2}dt = -\frac{1}{\sqrt{\pi}}\cdot\lim_{s\to 0} s\cdot\int_0^1 t^{\frac{s-1}{2}} Tr\ e^{-t\mathcal{D}_a^2}dt =$$

$$-\frac{1}{\sqrt{\pi}}\cdot\lim_{s\to 0} s\cdot\int_0^1 t^{\frac{s-1}{2}}(\sqrt{\frac{\pi}{t}} + r_1\sqrt{t} + O(t^{\frac{3}{2}}))dt = -\frac{1}{\sqrt{\pi}}\cdot\lim_{s\to 0} s\cdot\int_0^1 t^{\frac{s-1}{2}}\sqrt{\frac{\pi}{t}}dt =$$

$$-\lim_{s\to 0} s\cdot \int_0^1 t^{\frac{s}{2}-1} dt = -\lim_{s\to 0} s\cdot \frac{1}{\frac{s}{2}} = -2 \quad .$$

Theorem 6.3 implies that $\eta_{\mathcal{D}_a}(s)$ is a holomorphic function of s in the neighborhood of $s=0$, because we know that

$$\mathcal{R}(0) = 0 \quad ,$$

and by Theorem 6.3 , this extends to any real number a . In fact, we have proved more, because equality (42) implies that there exists constant $c>0$ such that for any $0<t\leq 1$ we have

$$|Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2}| < c\sqrt{t} \quad , \tag{48}$$

which gives us this representation

$$Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2} = b_1\sqrt{t} + b_2 t^{\frac{3}{2}} + O(t^{\frac{5}{2}}) \quad .$$

Now, we can discuss the structure of the η-function of $\mathcal{D}_a$

$$\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}} Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2} dt = \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}}(b_1\sqrt{t} + b_2 t^{\frac{3}{2}} + O(t^{\frac{5}{2}}))dt =$$

$$\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^1 t^{\frac{s-1}{2}}(b_1\sqrt{t} + b_2 t^{\frac{3}{2}} + O(t^{\frac{5}{2}}))dt + h(s) =$$

$$\frac{b_1}{\Gamma(\frac{s+1}{2})}\int_0^1 t^{\frac{s}{2}} dt + \frac{b_2}{\Gamma(\frac{s+1}{2})}\int_0^1 t^{\frac{s}{2}+1} + h_1(s) + h(s) \quad ,$$

where $h(s)$ is a holomorphic function on the whole complex plane and $h_1(s)$ is holomorphic for $Re(s) > -6$. This shows that there exists $h_2(s)$ a function holomorphic for $Re(s) > -6$, such that

$$\eta_{\mathcal{D}_a}(s) = \frac{b_1}{\Gamma(\frac{s+1}{2})}\cdot\frac{2}{s+2} + \frac{b_2}{\Gamma(\frac{s+1}{2})}\cdot\frac{2}{s+4} + h_2(s) \quad . \tag{49}$$

In particular Theorem 6.1 is proved.

Corollary 6.6

$$\eta_{\mathcal{D}_a}(0) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2} dt \quad . \tag{50}$$

Proof. Theorem 6.1 implies that we can apply the second formula in (12) for any s with $Re(s) > -2$. □

Corollary 6.7

$$\dot{\eta}_{\mathcal{D}_a}(0) = -\frac{2}{\sqrt{\pi}} \cdot \lim_{\varepsilon \to 0} \sqrt{\varepsilon} \cdot Tr\ \dot{\mathcal{D}}_a e^{-\varepsilon \mathcal{D}_a^2} \quad , \tag{51}$$

which again in the case of operator $\mathcal{D}_a$ on S^1 gives $\dot{\eta}_{\mathcal{D}_a}(0) = -2$.

Proof. We differentiate equation (50)

$$\frac{d}{da}\eta_{\mathcal{D}_a}(0) = \frac{d}{da}\left(\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2} dt\right) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2} dt +$$

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \sqrt{t} \cdot \frac{d}{dt}(Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2}) dt = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2} dt +$$

$$\frac{2}{\sqrt{\pi}} \cdot \lim_{\varepsilon \to 0} (\sqrt{t} \cdot Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2})|_\varepsilon^{\frac{1}{\varepsilon}} - \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} Tr\ \dot{\mathcal{D}}_a e^{-t\mathcal{D}_a^2} dt =$$

$$-\frac{2}{\sqrt{\pi}} \cdot \lim_{\varepsilon \to 0} \sqrt{\varepsilon} \cdot Tr\ \dot{\mathcal{D}}_a e^{-\varepsilon \mathcal{D}_a^2} \quad .$$

In the case of $\mathcal{D}_a$ on S^1 we have

$$-\frac{2}{\sqrt{\pi}} \cdot \lim_{\varepsilon \to 0} \sqrt{\varepsilon} \cdot Tr\ \dot{\mathcal{D}}_a e^{-\varepsilon \mathcal{D}_a^2} = -\frac{2}{\sqrt{\pi}} \cdot \lim_{\varepsilon \to 0} \sqrt{\varepsilon} \cdot \left(\sqrt{\frac{\pi}{\varepsilon}} + O(\sqrt{\varepsilon})\right) = -2 \quad .$$

□

Theorem 6.8

$$\eta_{\mathcal{D}_a}(0) = 1 - 2a \ \ . \tag{52}$$

Proof. We have just proved that

$$\eta_{\mathcal{D}_a}(0) = -2a \ \ mod\ \mathbf{Z}\ .$$

More precisely the continuous part $\tilde{\eta}_{\mathcal{D}_a}(0)$ of $\eta_{\mathcal{D}_a}(0)$ (as a function of a) is equal to

$$\int_0^a \dot{\eta}_{\mathcal{D}_r}(0)dr = -2\cdot \int_0^a dr = -2a \ \ .$$

We have a formula

$$\eta_{\mathcal{D}_a}(s) = \frac{1}{a^s} + \sum_{k \neq 0} sign(k+a)|k+a|^s$$

which gives $\lim_{s\to 0} \eta_{\mathcal{D}_a}(s) = 1 + \tilde{\eta}_{\mathcal{D}_a}(0) = 1 - 2a$. □

It follows now from Theorem 6.8 and Corollary 5.3 that we know the phase of the ζ-determinant of the operator $\mathcal{D}_a$

Corollary 6.9 *The phase of the $det_\zeta \mathcal{D}_a$ is equal to $e^{\frac{i\pi}{2}(2a-1)}$.*

In the next Section we deal with the modulus of the ζ-determinant.

7 Modulus of $det_\zeta \mathcal{D}_a$

There are some reasons beyond the scope of these notes (see for instance [$ScW2$],[$ScW3$], which tell us that we should pick the operator $\mathcal{D}_{\frac{1}{2}} = -i\frac{d}{dx} + \frac{1}{2}$ and assume that its ζ-determinant is equal to 1 and then obtain the value of the ζ-determinant for other $\mathcal{D}_a$ by studying the variation of the determinant with respect to a. Therefore we discuss now the variation of the modulus with respect to the parameter a. We know that absolute value of the ζ-determinant is equal to $-\ half$ of $\zeta'_{\mathcal{D}_a^2}(0) = \frac{d}{ds}\zeta_{\mathcal{D}_a^2}(s)|_{s=0}$. The function $\zeta_{\mathcal{D}_a^2}(s)$ is holomorphic in the neighborhood of $s = 0$ and we can take the derivative with respect to s

$$\lim_{s\to 0} \frac{d}{ds}(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}_a^2} dt) = \lim_{s\to 0}(-\frac{\Gamma'(s)}{\Gamma(s)^2}\int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}_a^2} dt) +$$

$$\lim_{s\to 0}(\frac{1}{\Gamma(s)}\frac{d}{ds}(\int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}_a^2} dt)) = \lim_{s\to 0}(-\frac{\Gamma'(s)}{\Gamma(s)^2}\int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}_a^2} dt) \ .$$

Remark 7.1 The fact that $\lim_{s\to 0}(\frac{1}{\Gamma(s)}\frac{d}{ds}(\int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}_a^2} dt))$ disappears is due to an absence of the singularity of the function $\mathcal{K}(s) = \int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}_a^2} dt$ at $s = 0$ and it is characteristic for dimension 1 , or more generally for odd dimensional manifold M . In the even-dimensional case this part may produce additonal contributions.

We know that $\Gamma'(s) = -\frac{1}{s^2} + h(s)$, where $h(s)$ is a holomorphic function in the neighborhood of $s = 0$, hence we have arrived at the equation

$$\frac{d}{ds}(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}_a^2} dt)|_{s=0} = \lim_{s\to 0}\int_0^\infty t^{s-1} Tr\ e^{-t\mathcal{D}_a^2} dt = \int_0^\infty \frac{1}{t}\cdot Tr\ e^{-t\mathcal{D}_a^2} dt \ . \quad (53)$$

We use Lemma 6.4 to study the variation of the right side of (53) and obtain

$$\frac{d}{da}\int_0^\infty \frac{1}{t}\cdot Tr\ e^{-t\mathcal{D}_a^2}dt = -2\cdot\int_0^\infty Tr\ \dot{\mathcal{D}}_a\mathcal{D}_a e^{-t\mathcal{D}_a^2}dt =$$

$$-2\cdot\int_0^\infty Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2}dt = -2\cdot\eta_{\mathcal{D}_a}(1)\ \ . \tag{54}$$

We work on the expression $\int_0^\infty Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2}dt$ in order to figure out the exact value of the variation . We may assume that $\mathcal{D}_a$ is invertible, hence

$$\int_0^\infty Tr\ \mathcal{D}_a e^{-t\mathcal{D}_a^2}dt = \int_0^\infty Tr\ \mathcal{D}_a^{-1}\mathcal{D}_a^2 e^{-t\mathcal{D}_a^2}dt = -\int_0^\infty \frac{d}{dt}(Tr\ \mathcal{D}_a^{-1}e^{-t\mathcal{D}_a^2})dt =$$

$$-\lim_{\varepsilon\to 0}(Tr\ \mathcal{D}_a^{-1}e^{-t\mathcal{D}_a^2})|_\varepsilon^{\frac{1}{\varepsilon}} = +\lim_{\varepsilon\to 0}Tr\ \mathcal{D}_a^{-1}e^{-\varepsilon\mathcal{D}_a^2}\ \ .$$

We have just proved

Proposition 7.2

$$\frac{d}{da}\zeta'_{\mathcal{D}_a^2}(0) = -2\cdot\lim_{\varepsilon\to 0}Tr\ \mathcal{D}_a^{-1}e^{-\varepsilon\mathcal{D}_a^2}\ \ . \tag{55}$$

We need a formula for the operator $\mathcal{D}_a^{-1}$ in order to get the right side of (55) . The point is that we have an explicit formula for $k_a(x,y)$ the kernel of the operator $\mathcal{D}_a^{-1}$

$$k_a(x,y) := \begin{cases} -\frac{ie^{-i(x-y)}}{1-e^{2\pi i a}} \text{ for } x<y \\ \\ +\frac{ie^{-i(x-y)}}{1-e^{-2\pi i a}} \text{ for } x>y \end{cases} \tag{56}$$

Now, we have

$$\frac{d}{da}\zeta'_{\mathcal{D}_a^2}(0) = -2\cdot\lim_{\varepsilon\to 0}\int_{S^1}dx\int_{S^1}dy\ k_a(x,y)e_{\mathcal{D}_a^2}(t;y,x) =$$

$$-2\cdot\lim_{\varepsilon\to 0}\int_{S^1} dx \int_{|x-y|<\delta} dy\ k_a(x,y) e_{\mathcal{D}_a^2}(t;y,x) \quad .$$

The last equality follows from, earlier established, fact that the *Heat Kernel* $e(t;x,y)$ is exponentially decaying when the distance between x and y is bounded away from 0 and time is going to 0 . In other words

$$\lim_{\varepsilon\to 0}\ k_a(x,y) e_{\mathcal{D}_a^2}(t;y,x) = 0 \quad for \quad |x-y| > \delta \quad .$$

Therefore, we have a sequence of equalities

$$\frac{d}{da}\zeta'_{\mathcal{D}_a^2}(0) =$$

$$-2\cdot\lim_{\varepsilon\to 0}\int_{S^1} dx \int_{r<\delta} dr\{k_a(x;x+r) e_{\mathcal{D}_a^2}(\varepsilon;x+r,x) + k_a(x;x-r) e_{\mathcal{D}_a^2}(\varepsilon;x-r,x)\} =$$

$$-2\cdot\lim_{\varepsilon\to 0}\int_{S^1} dx \int_{r<\delta} dr e^{-ira}\{-\frac{i}{1-e^{2\pi ia}} + \frac{i}{1-e^{-2\pi ia}}\}\frac{1}{\sqrt{4\pi\varepsilon}} e^{-\frac{r^2}{4\varepsilon}} =$$

$$-4\pi\cdot\frac{sin\ 2\pi a}{1-cos\ 2\pi a}\cdot\lim_{\varepsilon\to 0}\int_{r<\delta} dr e^{-ira}\frac{1}{\sqrt{4\pi\varepsilon}} e^{-\frac{r^2}{4\varepsilon}} = -2\pi\cdot\frac{sin\ 2\pi a}{1-cos\ 2\pi a} \quad .$$

and we have proved

Proposition 7.3

$$\frac{d}{da}\zeta'_{\mathcal{D}_a^2}(0) = -2\pi\cdot\frac{sin\ 2\pi a}{1-cos\ 2\pi a} \quad . \tag{57}$$

The variation of the modulus of the determinant is given respectively by

$$\pi\cdot\frac{sin\ 2\pi a}{1-cos\ 2\pi a} \quad .$$

Theorem 7.4 *The following equality holds for any* $0 < b < 1$

$$\frac{det_\zeta \mathcal{D}_b^2}{det_\zeta \mathcal{D}_{\frac{1}{2}}^2} = sin\ \pi b \ . \tag{58}$$

Proof. We have

$$-\frac{1}{2}\zeta'_{\mathcal{D}_b^2}(0) - (-\frac{1}{2}\zeta'_{\mathcal{D}_{\frac{1}{2}}^2}(0)) = \pi\cdot \int_{\frac{1}{2}}^{b} \frac{sin\ 2\pi a}{1 - cos\ 2\pi a} da = \frac{1}{2}\cdot \int_{\frac{1}{2}}^{b} \frac{sin\ 2\pi a}{1 - cos\ 2\pi a} 2\pi da =$$

$$\frac{1}{2}\cdot \int_{2}^{1-cos\ 2\pi b} \frac{du}{u} = \frac{1}{2}\cdot ln\ \frac{1 - cos\ 2\pi b}{2} = \frac{1}{2}\cdot ln\ sin^2 \pi b = ln\ sin\ \pi b \ .$$

□

Corollary 7.5

$$det_\zeta \mathcal{D}_a = e^{\frac{i\pi}{2}(2a-1)} \cdot sin\ \pi a \cdot det_\zeta \mathcal{D}_{\frac{1}{2}} \ , \tag{59}$$

for any $0 < a < 1$.

It is quite natural to assume now that $det_\zeta \mathcal{D}_{\frac{1}{2}} = 1$ and take

$$e^{\frac{i\pi}{2}(2a-1)} \cdot sin\ \pi a$$

as the ζ-determinant of the operator $\mathcal{D}_a$. This is even more sound if you notice that

$$e^{\frac{i\pi}{2}(2a-1)} \cdot sin\ \pi a = \frac{1 - e^{2\pi i a}}{2} \ ,$$

which actually plays the role of the determinant. We can generalize the whole story as follows: let us observe that the operator $\mathcal{D}_a$ is from the spectral point of view, equivalent to the operator $-i\frac{d}{dx}$ on the interval $[0, 2\pi]$ subject to the boundary condition

$$f(2\pi) = e^{2\pi i a} f(0) \ , \tag{60}$$

This means that we consider the operator $\mathcal{D}_a = -i\frac{d}{dx}$ on the subspace of $C^\infty([0,2\pi])$, which consists of only functions which satisfies condition (60). This becomes a nice, self-adjoint operator on $L^2([0,2\pi])$ with a spectrum equal to the spectrum of the operator $\mathcal{D}_a$ on S^1 . The last can be easily seen by the straightforward solution of the eigenvalue problem

$$\begin{cases} -i\frac{d}{dx}\phi(x) = \lambda\phi(x) \text{ for } 0 \leq x \leq 2\pi \\ \phi(2\pi) = e^{2\pi a}\phi(0) \ . \end{cases} \tag{61}$$

which gives

$$\phi(x) = e^{i\lambda x} \quad where \quad \lambda 2\pi = 2\pi a + 2\pi k \ ,$$

for any integer k . More generally we can assume that $-i\frac{d}{dx}$ acts on $\mathbf{C}^n$ valued functions $f(x)$ and that the boundary conditions is given by a unitary matrix $T : \mathbf{C}^n \to \mathbf{C}^n$. We denote the resulting operator by $(-i\frac{d}{dx})_T$. Then using the fact that we can diagonalize the operator we can apply our method to prove

$$det_\zeta(-i\frac{d}{dx})_T = det\ \frac{Id_{\mathbf{C}^n} - T}{2} \ , \tag{62}$$

where once again we use the *"relative ζ-determinant"*, i.e. we assumed that

$$det_\zeta(-i\frac{d}{dx})_{-Id} = 1 \ .$$

Actually, we can compute the value of $det_\zeta(-i\frac{d}{dx})_{-Id}$ using *Elementary Number Theory* and it appears to be equal to 2^n . We refer to paper $[BSW]$ for details. It is much more amazing that most of this generalizes to the general situation in which we deal with elliptic boundary problems for the Dirac operatos (see $[Sc1]$,$[Sc2]$,$[ScW1]$,$[ScW2]$,$[ScW3]$,[?]).

8 Heat Equation and Index

So far, we stayed on the elementary level and a smart undergraduate student having $[RS1]$,$[RS2]$ in sight, should be able to go through this text alone without help from more advanced colleague. In the last part of my lecture I move to a more complicated set-up. I discuss the index of a single Dirac operator and its counterpart in the self-adjoint case, the spectral flow of the 1-parameter family of Dirac operators. This gives me an opportunity to discuss a new application of *Duhamel's Principle*, which is a technical tool used in these lectures and arguably the main hero of the story. Still, I think that a smart undergrad can deal with this presentation by himself/herself looking some things up in the first Chapter of $[G]$.

Let $A : C^\infty(M;E) \to C^\infty(M;F)$ denote an elliptic differential operator acting on sections of smooth vector bundles E and F over a smooth, closed manifold M . We assume that we have given Riemannian structure on M and Hermitian structures on E and F . The spaces *kernel of* A and *cokernel of* A are defined as

$$ker\ A = \{s \in C^\infty(M;E); As = 0\} \quad ,$$

and

$$coker\ A = ker A^* = \{s \in \{s \in C^\infty(M;F); A^*s = 0\} \quad .$$

They are finite dimensional. The number

$$index\ A = dim\ ker\ A - dim\ coker\ A$$

is a homotopy invariant of the operator A , which means that it does not change under smooth deformation of the operator A .

The model here is the Dirac operator acting on sections of S , a bundle of Clifford modules over a closed manifold. This is a self-adjoint operator hence

$$ker\ \mathcal{D} = coker\ \mathcal{D} \quad ,$$

and the *index* is trivial in this case. However, we know that in the case $dim\ M$ is an even number then spinor bundle splits $S = S^+ \oplus S^-$ onto the subbundle of spinors of positive and negative chirality. This gives a decomposition

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix} : C^\infty(M;S^+) \oplus C^\infty(M;S^-) \to C^\infty(M;S^+) \oplus C^\infty(M;S^-) \ , \tag{63}$$

where the operators $\mathcal{D}^\pm : C^\infty(M;S^\pm) \to C^\infty(M;S^\mp)$ now, in general, have nontrivial indices. Let us observe that

$$\mathcal{D}^- = (\mathcal{D}^+)^* \ \ hence \ \ index\ \mathcal{D}^- = -index\ \mathcal{D}^+ \ \ .$$

There is a formula which expresses the *index* in terms of topological invariants determined by manifold M , bundles E and F , and coefficients of the operator $\mathcal{D}$. This formula is a content of the famous *Atiyah-Singer Index Theorem* proved by Atiyah and Singer in early 60th and we refer reader to $[G, ?]$ for formulation, proof, and extensive bibliography related to this result.

One can use the *Heat Equation Formula* in order to prove the *Atiyah-Singer Index Theorem.* Let us consider the operator A from the beginning of the Section. The starting point of the proof is the following formula due to Bott, which holds for any $t > 0$:

$$index\ A = Tr\ e^{-tA^*A} - Tr\ e^{-tAA^*} \ \ . \tag{64}$$

Proof. The operators A^*A and AA^* are self-adjoint elliptic operators, hence, they have non-negative spectrum. Formula (64) follows from two facts. First, it is easy to observe that the 0-eigenspace of A^*A is equal to the kernel of the operator A and the 0-eigenspace of A^* is the cokernel of A . Now let λ denote a positive eigenvalue of the operator A^*A corresponding to the eigensection ϕ_λ. Then, we have

$$AA^*(A\phi_\lambda) = A(A^*A)\phi_\lambda = \lambda(A\phi_\lambda) \ \ .$$

We use this information to compute the right side of (64). Let $E_\lambda(A^*A)$ and $E_\lambda(AA^*)$ denote the eigenspaces of A^*A and AA^* corresponding to λ . We have just seen that those are isomorphic spaces and in particular $dim\ E_\lambda(A^*A) = dim\ E_\lambda(AA^*)$. Now we compute

$$Tr\ e^{-tA^*A} - Tr\ e^{-tAA^*} =$$

$$\sum_{\lambda \in spec(A^*A)} e^{-t\lambda} dim\ E_\lambda(A^*A)\ - \sum_{\lambda \in spec(AA^*)} e^{-t\lambda} dim\ E_\lambda(AA^*) =$$

$$dim\ ker\ A - dim\ coker\ A^* + \sum_{\lambda \neq 0} e^{-t\lambda}(dim\ E_\lambda(A^*A) - dim\ E_\lambda(AA^*)) = index\ A\ .$$

□

The *Index Theorem* follows from the analysis of the limit

$$\lim_{t \to 0}(Tr\ e^{-tA^*A} - Tr\ e^{-tAA^*})\ .$$

Using the construction of the *Heat Kernel* we can show that at least for the Dirac operators we have a *"local"* density $\alpha(x)$ such that

$$index\ A = \lim_{t \to 0}(Tr\ e^{-tA^*A} - Tr\ e^{-tAA^*}) = \int_M \alpha(x)dx\ , \tag{65}$$

where *"local"* means that $\alpha(x)$ is constructed (in a complicated way) from the coefficients of the operator A at the point x .

9 Spectral Flow η-invariant and ξ-invariant

The index of a self-adjoint elliptic operator is equal to 0 . However, there is an invariant which corresponds to the index in the self-adjoint case. Let us consider $\{B_u\}_{0\leq u\leq 1}$ a 1-parameter, smooth family of self-adjoint elliptic operators $B_u : C^\infty(M;E) \to C^\infty(M;E)$. The space $ker\ B_u = coker\ B_u$ has finite dimension for any u , which implies that only finitely many eigenvalues change sign when u travels from 0 to 1 .

Definition 9.1 The number of the eigenvalues which change sign from $-$ to $+$ *minus* the number of eigenvalues which change sign from $+$ to $-$, when u varies from 0 to 1 is called spectral flow of the family $\{B_u\}$ and is denoted by $sf\{B_u\}$.

Spectral flow was introduced in paper $[APS2]$ in order to study a non-continous part of the η-invariant. To explain the situation let us think about the family $\{\mathcal{D}_a\}_{-\frac{1}{2}\leq a\leq \frac{1}{2}}$. We have the equality

$$\mathcal{D}_{\frac{1}{2}} = e^{-ix}\mathcal{D}_{-\frac{1}{2}}e^{ix} \quad ,$$

hence $\mathcal{D}_{\frac{1}{2}}$ is unitary equivalent to the operator $\mathcal{D}_{-\frac{1}{2}}$ and

$$\eta_{\mathcal{D}_{\frac{1}{2}}}(0) = \eta_{\mathcal{D}_{-\frac{1}{2}}}(0) \quad ,$$

equal to 0 . On the other hand we have computed the variation of the η-invariant (see Corollary 6.7)

$$\dot{\eta_{\mathcal{D}_a}} = -2 \quad .$$

Hence, at least formally we expect

$$\eta_{\mathcal{D}_{\frac{1}{2}}}(0) = \eta_{\mathcal{D}_{-\frac{1}{2}}}(0) - 2 \quad .$$

However, this is compensated by the appearence of the spectral flow of the family $\{\mathcal{D}_a\}_{-\frac{1}{2}\leq a\leq\frac{1}{2}}$. As parameter a crosses 0 the largest negative eigenvalue becomes the smallest positive eigenvalue and the value of the η-invariant jumps by $+2$.

To get to the spectral flow of the family of the Dirac operators $\{\mathcal{D}_u\}_{0\leq u\leq 1}$ we introduce ξ-invariant of the Dirac operator $\mathcal{D}$ as

$$\xi_{\mathcal{D}} = \frac{dim\ ker\ \mathcal{D} + \eta_{\mathcal{D}}(0)}{2} \quad . \tag{66}$$

Invariant $\xi_{\mathcal{D}}$ is a well-defined function on a space of Dirac operators. Once again this function has a well-defined continous part $\tilde{\xi_{\mathcal{D}}}$ and integer value component, which this time is equal to the *Spectral Flow.* To be more precise we have the following result

Theorem 9.2 *Let* $\{\mathcal{D}_u\}_{0\leq u\leq 1}$ *denote a smooth family of Dirac operators, then*

$$\xi_{\mathcal{D}_1} - \xi_{\mathcal{D}_0} = \tilde{\xi_{\mathcal{D}_1}} - \tilde{\xi_{\mathcal{D}_0}} + sf\{\mathcal{D}_u\}_{0\leq u\leq 1} \quad , \tag{67}$$

and

$$\tilde{\xi_{\mathcal{D}_1}} - \tilde{\xi_{\mathcal{D}_0}} = \int_0^1 \frac{d}{du}\xi_{\mathcal{D}_u} du \quad .$$

Instead of offering the proof of the Theorem for a general case we allow the reader to figure out this result in our simple scenario on S^1 .

10 Index and spectral Flow

The Theorem which we prove in this Section is due to Atiyah, Patodi and Singer (see [$APS2$]). The novelty here is the proof, which is due to the author of these notes (see [$W1$]). In the unpublished paper I studied a slight generalization of the famous *Witten's Holonomy Theorem* and the proof of the equality of the *Index* and the *Spectral Flow* offered in [$W1$] was a simple variant of a general method used to prove *Holonomy Theorem*.

Let for any $0 \leq u \leq 1$ $B_u : C^\infty(M;S) \to C^\infty(M;S)$ denote a compatible Dirac operator acting on sections of a bundle of Clifford modules over closed, odd-dimensional manifold M . We assume that $\{B_u\}$ is a smooth family and we also make a technical assumption which makes the pasting of the operators easy

$$B_u = B_0 \quad for\ 0 \leq u \leq \varepsilon \quad and \quad B_u = B_1 \quad for\ 1-\varepsilon \leq u \leq 1 \ .$$

Moreover, we assume that there exists a unitary bundle automorphism $g : S \to S$, such that

$$B_1 = gBg^{-1} \ . \tag{68}$$

The equality (68) lets us think about $\{B_u\}$ as the family over S^1 , as B_1 is now unitary equivalent to the operator B_0 . On the other hand, we can construct a corresponding object, an elliptic operator A on a manifold $S^1 \times M$. We introduce S_g a smooth vector bundle over $S^1 \times M$ as follows: we take bundle $I \times S$ over $I \times M$ and now we identify

$$(1,y;s) \sim (0,y;g(s)) \ , \tag{69}$$

to obtain a bundle over $S^1 \times M$, where $S^1 = \mathbf{R}/_{\mathbf{Z}}$. Let u denote a coordinate on S^1 , then we define operator $A : C^\infty(S^1 \times M; S_g) \to C^\infty(S^1 \times M; S_g)$ by the formula

$$A = \partial_u + B_u \ .$$

Though it is not essential, here is the point when we use the assumption that family $\{B_u\}$ is constant near 0 and 1 in order to have a well-defined operator

A . To see that A is indeed an operator acting on sections of a twisted bundle let us take section $s \in C^\infty(S^1 \times M; S_g)$. We have

$$s(1,y) = g(y)s(0,y) \quad for\ any\ y \in M \ ,$$

and now we apply operator A

$$As(1,y) = (\partial_u + B_1)s(1,y) =$$

$$(\partial_u + gBg^{-1})g(y)s(0,y) = g(\partial_u + B_0)s(0,y) = gAs(0,y) \quad ,$$

which shows that As is again a section of bundle S_g

Theorem 10.1

$$index\ A = sf\{B_u\} \ . \tag{70}$$

We use (64) in order to prove Theorem 10.1. One would expect that we need to construct exact *Heat Kernels* for the operators A^*A and AA^*. However, the information we need is not hidden deep in those kernels. We show that it is enough to know an appriopriate approximations of those kernels. To get rid of the unnecessary terms we use the *Adiabatic Argument* . We introduce an auxiliary parameter ε and define the operator A_ε as

$$A_\varepsilon = \varepsilon\partial_u + B_u \quad .$$

This procedure corresponds to replacing the original S^1 of the length 1 by the circle of the length $\frac{1}{\varepsilon}$, as basically we replace normal coordinate u by $v = \frac{u}{\varepsilon}$. Index of the operator is unchanged under the smooth deformation hence we have

$$index\ A = index\ A_\varepsilon = Tr\ e^{-tA_\varepsilon^* A_\varepsilon} - Tr\ e^{-tA_\varepsilon A_\varepsilon^*} \ , \tag{71}$$

for any positive number ε . We use *Duhamel's Principle* to approximate the operators $e^{-tA_\varepsilon^* A_\varepsilon}$ and $e^{-tA_\varepsilon A_\varepsilon^*}$. This time; however, we do not start with the solution of the *Heat Equation.* Instead of that we use an operator close enough to the *Heat Operator* to be a good starting point for the whole procedure described earlier in these notes. Such an operator is usually called a *Parametrix* in the literature, though this may lead to a misunderstanding as this term is also used in some other parts of the theory of partial differential equations. We define $Q_\varepsilon(t)$ as the operator which serves as the parametrix for both $e^{-tA_\varepsilon^* A_\varepsilon}$ and $e^{-tA_\varepsilon A_\varepsilon^*}$ as follows

$$Q_\varepsilon(t) = e^{-tB^2} e^{\varepsilon^2 t \partial_u^2} \quad .$$

The operator $Q_\varepsilon(t)$ is a composition of the operator e^{-tB^2} , or more precisely at each $u \in S^1$ we have the operator

$$Q_\varepsilon(t;u) = e^{\varepsilon^2 t \partial_u^2} e^{-tB_u^2} \quad .$$

The reason that it is not the *Heat Operator* define by $A_\varepsilon^* A_\varepsilon$ (or $A_\varepsilon A_\varepsilon^*$) is the fact that operators ∂_u and B_u do not commute. We have

$$A_\varepsilon^* A_\varepsilon = -\varepsilon^2 \partial_u^2 + B_u^2 - \varepsilon \dot{B}_u \quad and \quad A_\varepsilon A_\varepsilon^* = -\varepsilon^2 \partial_u^2 + B_u^2 + \varepsilon \dot{B}_u \quad . \tag{72}$$

Now we use *Duhamel's Principle*

$$e^{-tA_\varepsilon^* A_\varepsilon} = Q_\varepsilon(t) + \int_0^t \frac{d}{ds}(e^{-sA_\varepsilon^* A_\varepsilon} Q_\varepsilon(t-s)ds =$$

$$Q_\varepsilon(t) + \int_0^t e^{-sA_\varepsilon^* A_\varepsilon}(-A_\varepsilon^* A_\varepsilon + \frac{d}{ds})Q_\varepsilon(t-s)ds =$$

$$Q_\varepsilon(t) + \int_0^t Q_\varepsilon(s)(-A_\varepsilon^* A_\varepsilon + \frac{d}{ds})Q_\varepsilon(t-s)ds +$$

$$\int_0^t ds \int_0^s dr e^{-rA_\varepsilon^* A_\varepsilon}(-A_\varepsilon^* A_\varepsilon + \frac{d}{dr})Q_\varepsilon(s-r)(-A_\varepsilon^* A_\varepsilon + \frac{d}{ds})Q_\varepsilon(t-s) \quad .$$

This equality may be used in order to construct $e^{-tA_\varepsilon^* A_\varepsilon}$. Calculations are longer and more complicated then before, but they are not really more difficult. The new element here is due to the fact that the operator $-\varepsilon^2\partial_u^2$ in $A_\varepsilon^* A_\varepsilon$ does not commute with $e^{-tB_u^2}$ in $Q_\varepsilon(t)$ and we have to write the first term in the expansion of $e^{-tA_\varepsilon^* A_\varepsilon}$ with respect to $Q_\varepsilon(t)$ as follows

$$\int_0^t Q_\varepsilon(s)(-A_\varepsilon^* A_\varepsilon + \frac{d}{ds})Q_\varepsilon(t-s)ds = \varepsilon\cdot \int_0^t Q_\varepsilon(s)\dot{B}_u Q_\varepsilon(t-s)ds +$$

$$\varepsilon^2 \int_0^t Q_\varepsilon(s)[e^{-(t-s)B_u^2};\partial_u^2]e^{\varepsilon^2(t-s)\ partial_u^2}ds \quad .$$

Now, the careful reader observe that all other terms in the series which gives $e^{-tA_\varepsilon^* A_\varepsilon}$ are of order at least ε^2 with respect to the parameter ε . Similarly we see that

$$e^{-tA_\varepsilon A_\varepsilon^*} - Q_\varepsilon(t) + \varepsilon\cdot \int_0^t Q_\varepsilon(s)\dot{B}_u Q_\varepsilon(t-s)ds = O(\varepsilon^2) \quad .$$

We reached the main technical result of this Section

Theorem 10.2 *The index of the operator A is given by the follwing formula which holds for any $t > 0$*

$$index\ A = index\ A_\varepsilon = \frac{\sqrt{t}}{\sqrt{\pi}} \int_{S^1} Tr\ \dot{B}_u e^{-tB_u^2} du \quad . \tag{73}$$

Proof. We have

$$index\ A = \lim_{\varepsilon\to 0}(Tr\ e^{-tA_\varepsilon^* A_\varepsilon} - Tr\ e^{-tA_\varepsilon A_\varepsilon^*}) =$$

$$2\cdot \lim_{\varepsilon\to 0} \varepsilon \int_0^t Tr(Q_\varepsilon(s)\dot{B}_u Q_\varepsilon(t-s))ds = 2\cdot \lim_{\varepsilon\to 0} \varepsilon Tr\ \dot{B}_u \int_0^t Q_\varepsilon(t)ds =$$

$$2\cdot \lim_{\varepsilon\to 0}(\varepsilon t) Tr\ \dot{B}_u Q_\varepsilon(t) = 2\cdot \lim_{\varepsilon\to 0}(\varepsilon t)\cdot \frac{1}{\sqrt{4\pi\varepsilon^2 t}}\cdot \int_{S^1} Tr\ \dot{B}_u e^{-tB_u^2} du \quad ,$$

where in the last line we do have trace on $S^1 \times M$ (or more precisely of the operator acting on $L^2(S^1 \times M; S_g)$) on the left side and the trace on M on the right side. The Theorem is proved. □

Now we use Corollary 6.7 (see (51)) and Theorem 10.2 to show the equality

$$\begin{gathered} index\ A = \frac{\sqrt{t}}{\sqrt{\pi}} \int_{S^1} Tr\ \dot{B}_u e^{-tB_u^2} du = \\ \lim_{t\to 0} \frac{\sqrt{t}}{\sqrt{\pi}} \int_{S^1} Tr\ \dot{B}_u e^{-tB_u^2} du = -\frac{1}{2}\int_{S^1} \eta_{\dot{B}_u}(0) du \quad . \end{gathered} \tag{74}$$

We use Theorem 9.2 to make the final step in the proof of the Theorem 10.1

$$index\ A = -\frac{1}{2}\int_{S^1} \eta_{\dot{B}_u}(0) du = -\int_{S^1} \xi_{\dot{B}_u} du = sf\{B_u\} \quad .$$

11 A Simple Example

We refer the reader to Section 26 of $[BW]$ for the extended analysis of the two-dimensional case of the *Index Theorem.* Now you can read this Section without looking anywhere else in this book, which is devoted to the study of *Elliptic Boundary Value Problems.* However, for your convenience I decided to extract simplest possible example of the equality of the index and the spectral flow. We already know that spectral flow of the family $\{-i\frac{d}{dx} + u\}_{0\leq u\leq 1}$ is equal to 1 . Now, how to get to the index from this point. Let us consider the operator $\overline{\partial}$ acting on smooth, complex-valued functions on two dimensional torus $T^2 = S^1 \times S^1$. The operator

$$\overline{\partial} : C^{\infty}(T^2) \to C^{\infty}(T^2)$$

can be realized as the operator $\overline{\partial} = \partial_u + i\partial_x$ acting on smooth periodic functions in $\mathbf{R}^2$. We choose periods to be 1 and 2π respectively, which means that we consider functions which satisfy the following equality

$$f(u+k, x+2\pi l) = f(u,x) \ , \tag{75}$$

for each couple of integers $k, l \in \mathbf{Z}$. It is not difficult to see that both operators $\overline{\partial}$ and $\partial = \overline{\partial}^*$ have trivial index. You can use method presented below, in the non-trivial case, to prove that. We also have a constant family of the operators in the normal direction $\{B_u = i\frac{d}{dx}\}$, which produces no spectral flow. To change that we replace smooth functions by sections of non-trivial line bundle over T^2 . We replace condition (75) by

$$f(u+k, x+2\pi l) = e^{ikx} f(u,x) \ , \tag{76}$$

All smooth $f(u,x)$ satisfying (76) are sections of the bundle S_g over T^2 , with $g(x) = e^{ix}$ (see (69)). The corresponding operator $\overline{\partial}_g : C^{\infty}(T^2; S_g) \to C^{\infty}(T^2; S_g)$ is equal to

$$\overline{\partial}_g(u,x) = \partial_u + i\partial_x + u \ , \tag{77}$$

as follows from the following identifications

$$(\partial_u + i\partial_x + u + 1) f(u+1,x) = (\partial_u + i\partial_x + u + 1)(e^{ix} f(u,x)) =$$

$$e^{ix}\partial_u f(u,x) + e^{ix}\partial_x f(u,x) - e^{ix} f(u,x) + (u+1) f(u,x) = e^{ix}(\partial_u + i\partial_x + u) f(u,x) \ .$$

We calculate the *index* of the operator $\overline{\partial}_g$. We start with the *kernel*. We have to find all smooth solutions of the operator $(\partial_u + i\partial_x + u)$ on $\mathbf{R}^2$, which satisfies pewriodicity condition (76). Any smooth f satisfying (76) is of the form

$$f(u,x) = \sum_{k\in\mathbf{Z}} f_k(u) e^{ikx} \quad ,$$

where coefficients $f_k(u)$ fulfill condition

$$f_k(u) = f_{k+1}(u+1) \ . \tag{78}$$

Now we see that

$$(\partial_u + i\partial_x + u + 1) f_k(x) e^{ikx} = (f_k'(x) + (u-k) f_k(x)) e^{ikx} \quad ,$$

hence

$$f_k(u) = c_k e^{-\frac{1}{2}(u-k)^2} \quad .$$

This implies that solution of the equation $\overline{\partial}_g f = 0$ has the form

$$f(u,x) = \sum_{k\in\mathbf{Z}} c_k e^{-\frac{1}{2}(u-k)^2} e^{ikx} \quad .$$

Condition (78) implies that $c_k = c_{k+1}$ for any $k \in \mathbf{Z}$ and we proved

Proposition 11.1 *Kernel of the operator $\overline{\partial}_g$ is the one-dimensional subspace of $C^\infty(T^2; S_g)$ spanned by* $\sum_{k\in\mathbf{Z}} e^{-\frac{1}{2}(u-k)^2} e^{ikx}$.

We can compute kernel of the operator $\overline{\partial}_g^*$ in the same way. This time the solution is

$$c \cdot \sum_{k \in \mathbf{Z}} e^{+\frac{1}{2}(u-k)^2} e^{ikx} \quad .$$

The coefficients of this series explode as $k \to \pm\infty$, hence it is a formal solution only, and we have

$$ker\ \overline{\partial}_g^* = coker\ \overline{\partial}_g = \{0\} \quad .$$

We have just proved the main result of this Section.

Theorem 11.2

$$index\ \overline{\partial}_g = sf\{i\frac{d}{dx} + u\}_{0 \le u \le 1} = 1 \quad . \tag{79}$$

Students who already took differential geometry should notice that number 1 is nothing else but integral $\int_{T^2} ch(S_g)$, where $ch(S_g)$ denotes *Chern Character* of S_g (see [G]). More generally, let us introduce a unitary transformation

$$g_k(x) = e^{ikx} \quad ,$$

which determines bundle S_k over T^2 . The corresponding operator has the form

$$\overline{\partial}_k = \partial_u + i\partial_x + ku \quad ,$$

and we have formula

$$index\ \overline{\partial}_k = sf\{i\frac{d}{dx} + ku\}_{0 \le u \le 1} = \int_{T^2} ch(S_k) = k \quad . \tag{80}$$

It is more difficult to show that we can replace g_k by any map $h : S^1 \to U(1)$ of degree k and the index of the resulting operator

$$\overline{\partial}_h = \partial_u + i\partial_x + uh^{-1}(x)\frac{dh}{dx}(x) \quad ,$$

is still equal to $k = deg\ h$.

12 Bibliographical Remarks

At the end let me discuss a little history of a subject. My comments are far away from being complete and the reader is again referred to different places for more accurate information. The *Atiyah-Singer Index Theorem* is a breakthrough result, which shows how to use analysis of elliptic operators to obtain a topological information and also how the topology and geometry determines the analysis on a manifold. The first proof using *Cobordism Approach* was published in $[AS]$ (see $[P]$ for the detailed exposition). Soon after the first proof of the *Index Theorem* had been published, Bott observed that we can use *Heat Kernel*, or equivalently ζ-function of an elliptic operator to obtain an index formula. The ζ-function entered global analysis a long time ago, but the serious systematic study of the relation between analysis and geometry based on ζ-function and *Heat Kernel* started with the famous work of Seeley $[Se1]$. Singer and McKean used *Heat Kernel* to study geometrical information contained in the Laplace operator (see $[MS]$). The proof of the *Index Theorem* based on the *Heat Equation* method was found by Patodi and independently by Gilkey. We refer to $[G]$ for bibliographical remarks on this subject. The original *Heat Equation* proof was greatly simplified in 80th. The detail of the modern approach are presented for instance in $[T2]$.

The ζ-determinant of the positive elliptic operator was introduced by Ray and Singer in $[RaS]$. This notion of the determinant was first used by physicists in the middle of 70th (see $[H]$) and has become an established mathematical tool of *Quantum Field Theory* since then. The η-function was introduced by Atiyah, Patodi and Singer in $[APS1]$. Singer observed the relation between η-invariant and the phase of the determinant and he first used (6) as the definition of the ζ-determinant of the Dirac operator (see $[Si]$). The corresponding theory of the ζ-determinant of the elliptic boundary problems for the Dirac operators was studied by author and his collaborators (see $[Sc1]$,$[Sc2]$, $[ScW1]$,$[ScW2]$,$[ScW3]$,$[W2]$).

We also refer to $[BW]$ for the presentation of the theory of elliptic boundary problems for the Dirac operators and to $[M]$ for a different approach to the analysis of the Dirac operators on a manifold with boundary.

References

[APS1] Atiyah, M.F., Patodi, V.K. and Singer, I.M.: 1975, 'Spectral asymmetry and Riemannian geometry. I', *Math. Proc. Cambridge Phil. Soc.* **77**, 43–69.

[APS2] Atiyah, M.F., Patodi, V.K. and Singer, I.M.: 1976, 'Spectral asymmetry and Riemannian geometry. I', *Math. Proc. Cambridge Phil. Soc.* bf 79, 71–99.

[AS] Atiyah, M.F. and Singer, I.M.: 1963, *Bull. Amer. Math. Soc.* **69**, 422–433.

[BSW] Booß–Bavnbek, B., Scott, S. G. and Wojciechowski, K. P.: 1998, 'The ζ-determinant and $\mathcal{C}$-determinant on the Grassmannian in dimension one', *Letters in Math. Phys.* **45**, 353–362.

[BW] Booß–Bavnbek, B. and Wojciechowski, K.P.: 1993, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, Boston.

[G] Gilkey, P. B.: 1995, *Invariance Theory, Heat Equation and the Atiyah-Singer Index Theorem*, CRC Press, Boca Raton.

[H] Hawking, S. W.: 1977, 'Zeta function regularization of path integrals in curved spacetime', *Comm. Math. Phys.* **55**, 133-148.

[MS] McKean, H. H. P. and Singer I. M.: 1967, 'Curvature and the eigenvalues of the Laplacian', *J. Diff. Geom.* **1**, 43–69.

[M] Melrose, R.: 1993, *The Atiyah-Patodi-Singer Index Theorem*, A. K. Peters, Boston.

[P] Palais, R., ed.: 1963, *Seminar on the Atiyah-Singer Index Theorem*, Princeton University Press, Princeton.

[RaS] Ray, D. and Singer, I.M.: 1971, 'R–torsion and the Laplacian on Riemannian manifolds', *Adv. Math.* **7**, 145–210.

[RS1] Reed, M. and Simon, B.: 1972, *Methods of Modern Mathematical Physics 1: Functional Analysis*, Academic Press, New York.

[RS2] Reed, M. and Simon, B.: 1975, *Methods of Modern Mathematical Physics 2: Fourier Analysis, Self-Adjointness*, Academic Press, New York.

[R] Rosenberg, S.: 1997, *The Laplacian on a Riemannian Manifold*, Cambridge University Press, Cambridge.

[Sc1] Scott, S.G.: 1995, 'Determinants of Dirac boundary value problems over odd–dimensional manifolds', *Comm. Math. Phys.* **173**, 43–76.

[Sc2] Scott, S.G.: 1998, 'Splitting the curvature of the determinant line bundle', *Proc. Am. Math. Soc.*, to appear.

[ScW1] Scott, S.G. and Wojciechowski, K.P.: 1997, 'Determinants, Grassmannians and elliptic boundary value problems for the Dirac operator', *Letters in Math. Phys.* **40**, 135–145.

[ScW2] Scott, S.G. and Wojciechowski, K.P.: 1999, 'ζ-determinant and the Quillen determinant on the Grassmannian of elliptic self-adjoint boundary conditions', *C. R. Acad. Sci., Serie I,* **328**, 139–144.

[ScW3] Scott, S.G. and Wojciechowski, K.P.: 1999, 'The ζ-determinant and Quillen determinant for a Dirac operator on a manifold with boundary', *GAFA*, to appear.

[Se1] Seeley, R. T.: 1967, 'Complex powers of an elliptic operator'. AMS *Proc. Symp. Pure Math. X.* AMS Providence, 288–307.

[Se2] Seeley, R. T.: 1969, *'Topics in pseudodifferential operators'.* In: *CIME Conference on Pseudo-Differential operators (Stresa 1968),* pp. 167–305. Cremonese 1969.

[Si] Singer, I.M.: 1985, 'Families of Dirac operators with applications to physics', *Asterisque, hors série*', 323–340.

[T1] Taylor, M. E.: 1996, *Partial Differential Equations. Basic Theory.*, Springer, New York.

[T2] Taylor, M. E.: 1996, *Partial Differential Equations. Qualitative Studies of Linear Equations.*, Springer, New York.

[W1] Wojciechowski,K. P: 1993, *'Witten's Holonomy Theorem in the Non-Loop Case'*, Preprint.

[W2] Wojciechowski, K.P.: 1999, 'The ζ-determinant and the additivity of the η-invariant on the smooth, self-adjoint Grassmannian', *Comm. Math. Phys.* **201**, 423–444.