RENORMALIZED TRACES AS A GEOMETRIC TOOL

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ζ-function regularization techniques are commonly used by physicists in Quantum Field Theory - e.g. to compute partition functions using ζ-function determinants - and by mathematicians in geometry and topology - e.g. to define invariants such as the η invariant which plays a fundamental part in the Atiyah-Patodi-Singer index or the analytic torsion involving again ζ-function regularized determinants. Using ζ-function regularization techniques we define renormalized traces of (possibly non trace-class) classical pseudo-differential operators ; these renormalized traces serve as a tool to generalize some well-known finite dimensional geometric concepts involving traces to a class of infinite dimensional vector bundles and manifolds. We express the obstructions preventing a straightforward extension in terms of Wodzicki residues of classical pseudo-differential operators and describe some cases for which these obstructions vanish.

Contents

Introduction

Traces play an important role in finite dimensional differential geometry.

Given a (finite dimensional) Riemannian manifold (M, g), traces arise in different concepts such as (see e.g. [GHL], [J], [BGV], [CDD], [N]...)

(i) the *divergence* of a one form given by minus the trace of the covariant derivative $\delta\alpha := -tr(\nabla\alpha)$, where ∇ is the Levi-Civita connection, α a one form on the manifold M,

(ii) the *Laplacian* on functions given by minus the trace of a Hessian $\Delta f := -tr(Hessf) = -tr(\nabla\nabla f)$, where f is a C^2 function, ∇ the Levi-Civita connection,

(iii) the *Ricci curvature* given by the trace of an operator involving the *curvature*, $Ricc(U, V) := tr(\Omega(U, \cdot), V)$, where Ω is the curvature of the Levi-Civita connection, U, V two vector fields on the manifold.

In complex geometry, they arise in a fundamental way when building characteristic classes using invariant polynomials (see e.g. [BT], [BGV], [KN], [N]....) For instance the *j-th characteristic class* on a complex vector bundle is defined using the $2j$ form $tr(\Omega^j)$ in terms of any connection ∇ on the bundle, Ω denoting the corresponding curvature tensor. These lead in turn to the *Chern character* $tr(e^{-\Omega})$ and its de Rham cohomology class.

Our goal here is to try to understand how some of these concepts may be extended to an infinite dimensional framework, namely a class of *Fréchet*, resp. *Hilbert* manifolds and vector bundles [L]. Among the manifolds we shall consider are *loop groups* (see e.g. [PS]) which are manifolds of maps on the circle with values in a Lie group. They offer an interesting but yet tractable example of infinite dimensional manifold and some of their geometric properties have been studied extensively in the literature (see e.g. [DL], [F], [Pr], [PS], [SW]).

A first step towards the generalization we are aiming at is to extend the notion of trace on the algebra of matrices to a class of operators acting on a class of infinite dimensional spaces, namely to the algebra of *classical pseudo-differential operators* [Sh], [T] on some given closed manifold. Since we do not want to restrict ourselves to *trace-class operators* for which there is a well-defined notion of trace, we build these "traces" from the *finite part* [H], [Sc] of otherwise divergent expressions. When the operators happen to be trace-class, they coincide with the ordinary trace.

Here we use the terminology "trace" abusively since there will be obstructions to the tracial property $tr[A, B] = 0$ which can be expressed in terms of a *Wodzicki residue* [K], [MN], [O], [W]. To compute this obstruction we use methods developped (independently) by Kontsevich, Vishik [KV1, 2] and Okikiolu [O]. Typically such algebraic obstructions arise as *cocycles* [Mi], [MN] in the Hochschild cohomology on the algebra of classical P.D.O.s.

These extended "traces" are defined with the help of some extra data, an *elliptic operator* and their dependence on the choice of such an operator can be measured (once again) in terms of a Wodzicki residue. This gives rise to the notion of *weighted vector space* $(\Gamma(M,E),Q)$, namely a Fréchet or Hilbert space $\Gamma(M,E)$ of smooth or Sobolev type sections of some (Riemannian/ hermitian) vector bundle E based on a closed Riemannian manifold M, combined with a positive self-adjoint elliptic operator Q acting on $\Gamma(M,E)$ (or possibly only on a dense subspace of that space).

A first rather trivial example is the space $C^\infty(S^1, I\!R^n)$ of smooth maps on the unit circle S^1 with values in $I\!R^n$, in which case the manifold M is the unit circle and the vector bundle E is the trivial bundle $S^1 \times I\!R^n$. A natural weight would be the Laplace-Beltrami operator [R], [G], [BGV] on S^1. A slightly more sophisticated example is that of a space $H^s(M,E), s>0$ (in fact s should be larger than $\frac{dim M}{2}$ if we want continuous sections) of Sobolev type sections of a finite rank vector bundle E based on a closed Riemannian manifold M. If E is equipped with a connection ∇, then there is a natural weight given by the associated generalized Laplacian $\Delta := -tr(\nabla\nabla)$.

Given a weight Q on $\Gamma(M,E)$, we can build *weighted traces* on the algebra $CL(M,E)$ of classicalP.D.O.s acting on $C^\infty(M,E)$. This carries out to families of vectors spaces $\{\Gamma(M_b,E_b), b\in B\}$ in the above class parametrized by a manifold B, if equipped with a family of operators $\{Q_b, b\in B\}$ which are locally invertible admissible (this notion is recalled in section 3) elliptic operator. In that setting we define a notion of weighted trace $\{tr^{Q_b}, b\in B\}$ as a bundle map on the bundle based on B with fibres given by algebras $\{CL(M_b,E_b), b\in B\}$ of classicalP.D.O.s. This construction makes sense provided we require that the transition maps of the infinite rank bundle $\mathcal{E}$ with fibre $\Gamma(M_b,E_b)$ be elliptic operators.

With this at hand, we can introduce the concepts of *weighted Laplacians*, of *weighted Ricci curvature*, *weighted first Chern form...* defined -in a similar way to the finite dimensional case- as (minus) weighted traces of Hessians, weighted traces of operators involving the curvature... We can also extend characteristic forms $tr(\Omega^j)$ to this infinite dimensional setting naively replacing the trace by a weighted trace.

Classical results of Riemannian geometry such as Weitzenböck type formulae or classical constructions in the complex setting such as the Chern-Weil construction do not go through to this infinite dimensional setting in a straight forward manner since obstructions of *geometric type* can arise which again are described by a Wodzicki residue (see [ABP],[Pa], [PR]). However, as we shall show in section 6, these classical results can be generalized up to some extra Wodzicki residue type term involving the commutator $[\nabla, tr^Q]$ where ∇ is the connection and tr^Q the weighted trace; in the finite dimensional case, the connection "goes through" the trace so that such a commutator does not arise. In particular, when defining the *weighted first Chern form* on a weighted complex vector bundle equipped with a connection as the weighted trace of the curvature, unlike in the finite dimensional case, one does not expect it to be closed in general. Only in very special cases, as in the case of loop groups and given a good choice of weight (this corresponds to the two step trace of [F]), can one check that the obstructions vanish [CDMP], [Ma], [Pa].

These obstructions being rather cumbersome, we might want to circumvent them introducing some compensating terms. This can indeed be done when the weight arises as the square of a family of Dirac type operators using tools such as trace forms familiar to non commutative geometers but it would be out of the scope of this article to describe such a procedure in detail (see [PR], [Pa] for further investigations along these lines).

We hope that these lecture notes - through the question we raise of how to extend classical geometric concepts to the infinite dimensional setting - beyond the relevence of this problematic, will offer the reader a different outlook on some classical and therefore maybe well-known to him/her basic concepts of Riemannian and complex geometry. Along the way, our approach to this problem brings in as well as some geometric tools mentioned above, various algebraic tools (such as algebraic cocycles in section 5), analytical tools (such as finite parts and pseudo-functions in section 2) and functional analytical tools (such as renormalization procedures in section 4), which we feel are interesting for their own sake.

There are yet many other ways of approaching the question of how to extend classical geometric concepts to the infinite dimensional setting, one of which is to focus on measures rather than on traces. A natural thing to do from this point of view is to replace the Riemannian volume measure by Gaussian type measures in infinite dimensions, this leading - in the case of path manifolds - to some fascinating constructions on Wiener space. This second approach raises the question of how to construct a Laplacian which is self-adjoint, a problem which in our context does not make sense since there are no measures around and hence no notion of self-adjointness. However, the two approaches do have some fundamental common feature. Namely elliptic operators arise in both constructions; here, the Laplacian that underlies the construction of Brownian motion has been replaced by the weight Q.

1 Traces in riemannian and complex geometry

1.1 Traces on vector bundles

In this section we recall briefly some basic concepts of Riemannian and complex geometry involving traces. Our presentation, which might in some places seem a little unusual to the reader, is organized in view of generalizing the concepts to the infinite dimensional setting in section 6.

A trace on an algebra A over a field K (in practice $K = \mathbb{R}$ or $K = \mathbb{C}$) is a K-linear map $tr : A \to \mathbb{C}$ with the *tracial property*:

$$tr([a,b]) := tr(ab - ba) = 0 \quad \forall a, b \in A. \tag{1.1}$$

A well-known example is the algebra $A = M(K,n)$ of $n \times n$ square matrices with coefficients in K and $tr(a) := \sum_{i=1}^n a_{ii} \quad \forall a = (a_{ij}) \in A$.

As a consequence of the tracial property, given an invertible element c of A, we have $tr(a) = tr(c^{-1}ac) \quad \forall a \in A$.

Let now $\pi : E \to X$ be a (smooth) finite rank K-vector bundle based on a (smooth) finite dimensional manifold X. A vector bundle morphism of $\pi : E \to X$ over the identity of X is a fibre preserving map that is linear on each fibre; i.e a map $u : E \to E$ such that $\pi u = \pi$ and such that for any $x \in X$, $u_x \in Hom(E_x)$ where E_x denotes the fibre above x. Let $Hom(E)$ denote the homomorphism bundle of E with fibre $Hom(E)_x = Hom(E_x)$. Given a local chart (U, ϕ) of X and a local bundle chart

$$\begin{aligned} \Phi : \quad E|_U &\to \phi(U) \times K^d \\ (x, v_x) &\to (\phi(x), \Phi_x(v_x)) \end{aligned}$$

where d is the rank of E and given an element u_x of the fibre $Hom(E_x)$ over $x \in U$, then $\Phi_x^\sharp u_x := \Phi_x u_x \Phi_x^{-1} \in M(K,r)$. The expression $tr(u_x) := tr(\Phi_x^\sharp u_x)$ is independent of the choice of local trivialization; indeed given another local trivialization (ψ, Ψ) above V with $x \in U \cap V$, using the tracial property we find $tr(\Psi_x^\sharp u_x) = tr(\Psi_x \circ \Phi_x^{-1}(\Phi_x^\sharp u_x)\Phi_x \circ \Psi_x^{-1}) = tr(\Phi_x^\sharp u_x)$. The map

$$\begin{aligned} tr : Hom(E) &\to X \times K \\ (x, u_x) &\to (x, tr(u_x)) \end{aligned} \tag{1.2}$$

defines a bundle morphism which we shall refer to as the *trace on the bundle* $Hom(E)$.

1.2 *Variation of traces*

Let $u_t, t \in \mathbb{R}^+ - \{0\}$ be a smooth one parameter family of matrices in $M(K,d)$, then the map $t \to tr(u_t)$ is also smooth and we have $\frac{d}{dt} tr(u_t) = tr(\frac{d}{dt} u_t)$. More generally, if $\pi : E \to X$ is a trivial vector bundle over X, and if $x \to u_x$ is a smooth section of $Hom(E)$, then

$$dtr(u) = tr(du). \tag{1.3}$$

Let ∇ be a connection on a (finite rank) vector bundle $\pi : E \to X$. It induces a connection $\nabla^{Hom} = \nabla^* \otimes 1 + 1 \otimes \nabla$ on the bundle $Hom(E) = E^* \otimes E$ where ∇^* is the unique connection on the dual bundle E^* such that $d\langle \sigma^*, \eta \rangle = \langle \nabla^* \sigma^*, \eta \rangle + \langle \sigma^*, \nabla \eta \rangle$ for any sections σ^* of E^* and η of E, where $\langle \cdot, \cdot \rangle$ stands for the canonical dual pairing.

Exercise: Show that for any section u of $Hom(E)$ and any section σ of E we have $\nabla(u\sigma) = (\nabla^{Hom} u)\sigma + u \nabla \sigma$.

Hint: Given a local chart (e_i) and the corresponding dual chart (e_i^*) (i.e $e_j^*(e_i) = \delta_{ij}$), write $u = \sum_{ij} u_{ij} e_i^* \otimes e_j$ and use the property $(\nabla^* e_i^*)(e_j) = -e_i^*(\nabla e_j)$.

On the grounds of the results of this exercise we shall henceforth use the notation $\nabla^{Hom} u := [\nabla, u] = \nabla u - u \nabla$.

Let (U, ϕ) be a local chart on X and let as before Φ be a bundle chart of E over U. In this local chart the connection reads $\nabla = d + \omega^\Phi$ with $\omega^\Phi \in \Omega^1(\phi(U), gl_d(K)$ and the connection ∇^{Hom} reads $\nabla^{Hom} = d + [\omega^\Phi, \cdot]$. Applying (1.3) to $\Phi^\sharp u : \phi(U) \to M(K,d)$ we can write, locally on U:

$$\begin{aligned} dtr(u) &= dtr(\Phi^\sharp u) \\ &\quad \text{per definition} \\ &= tr(d(\Phi^\sharp u)) \\ &\quad \text{using (1.3)} \\ &= tr(d(\Phi^\sharp u)) + tr([\omega^\Phi, \Phi^\sharp u]) \\ &\quad \text{since the trace is tracial} \\ &= tr(\nabla^{Hom} u). \end{aligned} \tag{1.4}$$

Hence the connection ∇ commutes with the trace, namely

$$[\nabla, tr]u := dtr(u) - tr([\nabla, u]) = 0 \quad \forall u \in C^\infty(Hom(E)) \tag{1.5}$$

where for a bundle F, $C^\infty(F)$ denotes the space of smooth sections of F. We have used two ingredients to prove this elementary result, a purely algebraic property, namely the tracial property of the trace c.f. (1.1) combined with an analytic property, namely the fact that the trace commutes with the ordinary differentiation c.f.(1.3).

1.3 Geometric concepts involving traces

We refer the reader to e.g. [BGV], [GHL], [J], [N] for the concepts in Riemannian geometry needed here. We also refer the reader to the lecture by O.Hijazi in this volume. We shall focus on some basic concepts involving traces.

Let (X,g) be an n dimensional smooth Riemannian manifold equipped with the Levi-Civita connection ∇.

• **The divergence:** Given a one form α on X, the tensor $\nabla\alpha$ is a section of $T^*X \otimes T^*X \simeq Hom(TX)$. Here we have used the Riemannian metric to make the identification $T^*X \simeq TX$ via the (musical) map:

$$\begin{aligned} TX &\to T^*X \\ u &\to u^\flat \end{aligned}$$

where in local coordinates we have $u^\flat_i := g_{ij}u^j$ and the inverse of which is $\alpha \to \alpha^\sharp$ where in local coordinates we have $\left(\alpha^\sharp\right)^i := g^{ij}\alpha_j$. Applying the above constructions to the tangent bundle $E = TX$ of X we can define the divergence of the one form α:

$$\delta\alpha := -tr(\nabla\alpha). \tag{1.6}$$

Exercise: Show that

$$\delta(\alpha) = -\sum_{i=1}^n e_i d\alpha(e_i) + \sum_{i=1}^n \alpha(\nabla_{e_i} e_i)$$

where (e_i) is a local orthonormal frame.

The operator δ coincides with the formal adjoint d^* of the exterior differential $d : C^\infty(X) \to C^\infty(T^*X)$ for the scalar product induced by the metric i.e $\langle df, \alpha\rangle = \langle f, d^*\alpha\rangle$ where for two smooth functions $f, \tilde f$ on X the scalar product is given by $\langle f, \tilde f\rangle := \int_X dvol(x) f(x)\tilde f(x)$ and where for two one-forms $\alpha, \tilde\alpha$ on X the scalar product is defined by

$$\langle \alpha, \tilde\alpha\rangle = \int_X dvol(x) g^{ij}(x)\alpha_i(x)\tilde\alpha_j(x) = \sum_{i=1}^n \int_X dvol(x)\alpha(e_i(x))\tilde\alpha(e_i(x))$$

for any local orthonormal chart $(e_i(x))$ and where (g^{ij}) is the inverse matrix of (g_{ij}). Here and thereafter, we shall often make use of the Einstein summation convention omitting to write the summation symbol which is subintended on repeated indices.

• **The Laplacian:** Given a smooth function f, and taking $\alpha = df$ yields the Laplacian

$$\Delta f := \delta df = d^* df = -tr(\nabla df) \tag{1.7}$$

It therefore corresponds to minus the trace of the Hessian $Hess f := \nabla df$ of f.
Let us recall that for a k-covariant tensor T we have:

$$(\nabla_U T)(V_1, \cdots, V_k) = \nabla_U(T(V_1, \cdots, V_k)) - \sum_{i=1}^{k} T(V_1, \cdots, V_{i-1}, \nabla_U V_i, \cdots, V_k) \tag{1.8}$$

for any vector fields $U, V_1, \cdots, V_k$. In particular for any two vector fields U, V and any smooth function f we have

$$(\nabla^2 f)(U \otimes V) := (\nabla_U \nabla f)(V) = \nabla_U \nabla_V f - \nabla_{\nabla_U V} f.$$

Hence we can write:

$$-\Delta f = \sum_{i=1}^{n} \nabla^2 f(e_i \otimes e_i) = \sum_{i=1}^{n} \nabla_{e_i} \nabla_{e_i} f - \nabla_{\nabla_{e_i} e_i} f. \tag{1.9}$$

For two smooth functions f_1 and f_2 on X we have:

$$\Delta(f_1 f_2) = (\Delta f_1) f_2 + f_1 \Delta f_2 - 2tr(df_1 \otimes df_2).$$

The notion of Laplacian extends to vector bundles as follows. Let E be a vector bundle based on a Riemannian manifold equipped with a connection ∇^E. The Levi-Civita connection ∇ on X combined with thes connection ∇^E yields a connection on $\nabla^{T^*X \otimes E}$ on the tensor product $T^*X \otimes E$ so that we obtain an operator $\nabla^{T^*X \otimes E} \nabla^E : C^\infty(X, E) \to C^\infty(X, T^*X \otimes T^*X \otimes E)$. In other words $\nabla^{T^*X \otimes E} \nabla^E \in C^\infty(X, T^*X \otimes T^*X \otimes Hom(E))$. Identifying TX with T^*X via the musical maps, given a section σ of E we can therefore view $\nabla^{E \otimes T^*X} \nabla^E \sigma$ as a map from TX to $TX \otimes E$ and we can define:

$$\begin{aligned} \Delta^E \sigma &:= -tr(\nabla^{T^*X \otimes E} \nabla^E \sigma) \\ &= -\sum_{i=1}^{n} (\nabla^{T^*X \otimes E} \nabla^E \sigma)(e_i \otimes e_i) = -\sum_{i=1}^{n} (\nabla_{e_i}^{T^*X \otimes E} \nabla_{e_i}^E \sigma) + \nabla_{\nabla_{e_i} e_i}^E \sigma \end{aligned}$$

for any local orthonormal frame $(e_i)_{i_1, \cdots, n}$ of TX.

Remark: The sign convention sometimes vary from one text book to the other; the one we choose here leads to a positive operator **(exercise)**.

• **The Ricci and scalar curvature:** From the curvature Ω associated to the Levi-Civita connection ∇, one can build the curvature tensor $R \in C^\infty(T^*X^{\otimes 4})$ defined by $R(u,v,w,t) = \langle \Omega(u,v)w, t\rangle$ where $\langle \cdot, \cdot \rangle$ is the scalar product induced by the Riemannian metric. For fixed vector fields u, v, $R(u,\cdot,v,\cdot)$ defines a symmetric bilinear form on each tangent space and the Ricci curvature is the symmetric 2-tensor

$$Ricc(u,v) := (tr_{2,4}R)(u,v) = \sum_{i=1}^n R(u,e_i,v,e_i). \tag{1.10}$$

The scalar curvature is the trace of this two tensor:

$$Scal = tr(Ricc). \tag{1.11}$$

The Bochner-Weitzenböck formula relates the Laplacian on functions to the Ricci curvature on a Riemannian manifold:

A Bochner-Weitzenböck formula:

$$([\Delta, \nabla]f)(V) = Ricc(V, \nabla f) \quad \forall f \in C^\infty(X, \mathbb{C}), \forall V \in C^\infty(TX) \tag{1.12}$$

Proof. Let us first recall that for any C^2 function f on a manifold X equipped with a connection ∇ without torsion, for any point $x \in X$, the operator $Hessf(x)$ is a bounded symmetric (with respect to the scalar product induced by the Riemannian structure) operator acting on the tangent space T_xX at point x. This follows from the fact that

$$\begin{aligned}
\nabla^2 f(U \otimes V - V \otimes U) &= \nabla_U \nabla_V f - \nabla_{\nabla_U V} f - \nabla_V \nabla_U f + \nabla_{\nabla_V U} f \\
&= [\nabla_U, \nabla_V] f - \nabla_{\nabla_U V - \nabla_V U} f \\
&= [\nabla_U, \nabla_V] f - \nabla_{[U,V]} f = 0
\end{aligned}$$

where we have used the fact that $[U,V] = \nabla_U V - \nabla_V U$ because ∇ has no torsion.

Let $V \in T_xX$ and let $(e_i, i = 1, \cdots, dimX)$ be a local orthonormal frame at point x.

$$\begin{aligned}
([-\Delta, \nabla]f)(V) &= (-\Delta \nabla f)(V) + \nabla_V \Delta f \\
&= \sum_i \nabla^3 f(e_i \otimes e_i \otimes V) - \nabla_V tr(Hessf)
\end{aligned}$$

$$= \sum_i \nabla^3 f(e_i \otimes V \otimes e_i) - tr(\nabla_V \nabla^2 f)$$

where we have used the fact that $Hess f$ is symmetric

$$= \sum_i \big(\nabla^3 f(e_i \otimes V \otimes e_i) - \nabla^3 f(V \otimes e_i \otimes e_i)\big)$$

$$= \sum_i \nabla^2 \nabla f((e_i \wedge V) \otimes e_i)$$

$$= \sum_i (\nabla^2_{e_i \wedge V} \nabla f) e_i$$

$$= \sum_i \langle \Omega(e_i, V) \nabla f, e_i \rangle$$

$$= -Ricc(V, \nabla f). \quad \bullet$$

First Chern form on a vector bundle: Let E be a smooth complex vector bundle of rank d based on a Riemannian manifold X equipped with a connection ∇^E. The curvature Ω^E lies in $C^\infty(T^*X \otimes T^*X \otimes Hom(E))$, i.e for any two vector fields U, V on X, $\Omega^E(U, V) \in Hom(E)$. The first Chern form is the two form given by:

$$r_1^E(U, V) := tr(\Omega^E(U, V)) \quad \forall \quad U, V \in C^\infty(TX) \tag{1.14}$$

where the trace is taken on the local matrix representation of the curvature tensor. This definition is of course independent of the choice of local chart. Equivalently, if E is hermitian we can write:

$$r_1^E(U, V) = \sum_{i=1}^n \langle \Omega^E(U, V) e_i, e_i \rangle$$

where (e_i) is a local orthonormal frame, i.e. a local section of $O(E)$.

It follows from the Bianchi identity $[\nabla^E, \Omega^E] = 0$ and from (1.4) that the first Chern form is closed. Indeed we have:

$$\begin{aligned} dr_1^E &= dtr(\Omega^E) \\ &= tr([\nabla^E, \Omega^E]) \\ &= 0. \end{aligned} \tag{1.15}$$

Thus we can define a de Rham cohomology class:

$$c_1(E) := [r_1^E] \in H^2(M)$$

called the *first Chern class*. It has the following properties:

(i) It does not depend on the choice of the connection and is a topological characteristic of the bundle

(ii) Naturality. Given a bundle morphism f,

$$c_1(f^*E) = f^*c_1(E)$$

where f^*E is the pull-back of the bundle E by f.

(iii) $c_1(E) = c_1(E')$ if $E \simeq E'$.

Let us now generalize this construction to other invariants involving the curvature for which we shall give a proof of the independence of the choice of the connection for the corresponding cohomology classes, thus proving (i) as a particular case.

• **Chern-Weil invariants:** It also follows from the above considerations that any polynomial P of degree k on $gl_n(K)$ gives rise to a well-defined form $tr(P(\Omega^E))$ for we have (using again the cyclicity of the trace)

$$tr(P(\Psi^\sharp\Omega^E)) = tr\left(C\left(\Phi^\sharp\Omega^E\right)C^{-1}\right) = tr(P(\Phi^\sharp\Omega^E))$$

with $C := \Psi^{-1}\circ\Phi \in GL_n(K)$. Let us consider the case of a monomial $P(M) := M^k, k \in I\!N$, since the case of a general polynomial then easily follows taking linear combinations. Since $tr(P(C^{-1}MC)) = tr(P(M))$ for all $C \in GL_n(K)$, setting $C := e^{tB}, B \in gl_n(K)$ and differentiating w.r.to t we find

$$\sum_{i=1}^{k} tr\ (\tilde{P}(A_1,\cdots,A_{i-1},[B,A_i],A_{i+1},\cdots,A_k)) = 0$$
$$\forall B \in gl_n(K), A_j \in gl_n(K), j = 1,\cdots,k. \tag{1.16}$$

Here $\tilde{P}(A_1,\cdots,A_k) := A_1\cdots A_k$ is the polarized k-form corresponding to P.

Let E be a vector bundle of rank d equipped with a connection ∇^E. Let Ω^E be the corresponding curvature and let P be a polynomial on the algebra of $d\times d$ matrices. Then the following classical result holds:

$$dtr(P(\Omega^E)) = 0 \tag{1.17}$$

and the de Rham class of $P(\Omega^E)$ is independent of the choice of the connection ∇^E.

Proof. As before, we can restrict ourselves to a monomial $P(A) := A^k$. Let $\alpha_i,\cdots,\alpha_k$ be $Hom(E)$-valued forms on X and let $\nabla^E = d+\theta^E$ be the local representation of a connection ∇^E in some given local trivialization (U, Φ, ϕ). Applying (1.16) to $B := \theta^E$, $A_i = \Phi^\sharp \alpha_i$ we can write:

$$\begin{aligned}
dtr\left(\tilde{P}(\Phi^\sharp\alpha_1, \cdots, \Phi^\sharp\alpha_k)\right) &= \sum_{i=1}^k tr(\tilde{P}(\Phi^\sharp\alpha_1, \cdots, d\left(\Phi^\sharp\alpha_i\right), \cdots, \Phi^\sharp\alpha_k)) \\
&= \sum_{i=1}^k tr(\tilde{P}(\Phi^\sharp\alpha_1, \cdots, d(\Phi^\sharp\alpha_i), \cdots, \Phi^\sharp\alpha_k)) \\
&+ \sum_{i=1}^k tr(\tilde{P}(\Phi^\sharp\alpha_1, \cdots, [\theta^E, \Phi^\sharp\alpha_i], \cdots, \Phi^\sharp\alpha_k)) \\
&= \sum_{i=1}^k tr(\tilde{P}(\alpha_1, \cdots, [\nabla^E, \alpha_i], \cdots, \alpha_k)). \qquad (*)
\end{aligned}$$

Applying (*) to $\alpha_i = \Omega^E \quad \forall i =, 1 \cdots, k$ and using the Bianchi identity $[\nabla^E, \Omega^E] = 0$ yields that $dtr(P(\Omega^E)) = 0$ and hence that $tr(P(\Omega^E))$ is closed.

Let us now check the independence of the corresponding de Rham class of the choice of the connection. Let ∇^E and $\tilde{\nabla}^E$ be two connections and let $\eta := \tilde{\nabla}^E - \nabla^E$, $\nabla_t^E := t\tilde{\nabla}^E + (1-t)\nabla^E = \nabla^E + t\eta$. It is easy to check that $\Omega_t^E := \nabla_t^E \nabla_t^E = \Omega^E + t[\nabla^E, \eta] + t^2 \eta \wedge \eta$ and from this to deduce that $\frac{\partial}{\partial t}\Omega_t^E = [\nabla_t^E, \eta]$. The following holds:

$$\begin{aligned}
\frac{\partial}{\partial t} tr(P(\Omega_t^E)) &= \frac{\partial}{\partial t} tr(\tilde{P}(\Omega_t^E, \cdots, \Omega_t^E)) \\
&= \sum_{j=1}^k tr(\tilde{P}(\Omega_t^E, \cdots, \frac{\partial}{\partial t}\Omega_t^E, \cdots, \Omega_t^E)) \\
&= \sum_{j=1}^k tr(\tilde{P}(\Omega_t^E, \cdots, [\nabla_t^E, \eta], \Omega_t^E, \cdots, \Omega_t^E)) \\
&\quad \text{where} \quad [\nabla_t^E, \eta] \quad \text{is at place j} \\
&= \sum_{j=1}^k \Big(tr(\tilde{P}(\Omega_t^E, \cdots, [\nabla_t^E, \eta], \Omega_t^E, \cdots, \Omega_t^E)) \\
&+ \sum_{i \neq j} tr(\tilde{P}(\Omega_t^E, \cdots, [\nabla_t^E, \Omega_t^E], \cdots, \eta, \cdots, \Omega_t^E, \cdots, \Omega_t^E)) \Big) \\
&\quad \text{where} \quad [\nabla_t^E, \eta], [\nabla_t^E, \eta] \text{ are at place j}
\end{aligned}$$

$$= \sum_{j=1}^{k} dtr(\tilde{P}(\Omega_t^E, \cdots, \eta, \Omega_t^E, \cdots, \Omega_t^E))$$

where η is at place j.

To establish the last equality we applied (*) (replacing ∇^E by ∇_t^E) to $\alpha_i = \Omega_t^E \quad \forall i \neq j$ and $\alpha_j = \eta$. Hence $tr(P(\Omega_1^E)) - tr(P(\Omega_0^E)) = d\int_0^1 \sum_{j=1}^k dtr(\tilde{P}(\Omega_t^E, \cdots, \eta, \Omega_t^E, \cdots, \Omega_t^E))dt$ is an exact form, which ends the proof.

Using these polynomial invariants, we can also define the *Chern character*

$$Ch(\nabla^E) = tr(e^{-\Omega^E}) := \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k}{k!} tr\left(\left(\Omega^E\right)^k\right)$$

(where n is the dimension of the underlying manifold) which is again a closed form, the cohomology class of which is independent of the choice of the connection. We recover the first Chern form introduced above as minus the term of degree 2 of the Chern character of the bundle:

$$r_1^E = tr(\Omega^E) = tr((\nabla^E)^2) = -\left[tr(e^{-\Omega^E})\right]_{[2]} = -Ch(\nabla^E)_{[2]}. \tag{1.18}$$

We refer the reader to [BGV],[CDD], [KN], [N] ...for a more detailed account of these concepts.

2 Renormalized limits

In this section, preparing for the notion of renormalized traces to be described in section 3, we recall the notion of finite part of an otherwise diverging function. In contrast to other sections, this one might seem rather technical and the reader might like to skip it at first reading. I however strongly encourage her/him to come back to it later since renormalized limits are an essential tool in various approaches to Quantum Field Theories. The main reference used in this section is [Sc] but other more recent references on related issues are e.g. [EF], [EK] and some mentioned therein. I also refer the reader to the lectures by K. Wojciechowski and E. Langmann in this volume where similar renormalization techniques are used.

Let me finally address my thanks again to Marc Arnaudon at this point since it was some informal discussions with him that served as the initial motivation to write up this section.

2.1 Finite parts

In this section we consider a (complex valued) measurable function g on $]0,1[$ such that $g \in L^1_{loc}(]0,1])$. Hence $g \in L^1(]\varepsilon,1[)$ for any $\varepsilon > 0$ but g might not belong to $L^1(]0,1[)$. We make the following

Assumption: There is a finite subset J of $\mathbb{N}$, complex numbers b, a_j, and λ_j indexed by J and there is a function $h \in L^1(]0,1[)$ such that:

$$g(t) = \sum_{j \in J} a_j t^{-\lambda_j} + b t^{-1} + h(t) \tag{2.1}$$

with $Re(\lambda_j) > 1, \lambda_j \neq 1 \quad \forall j \in J$.

• **Finite part of an integral:** The notion of finite part was first introduced by Hadamard (see [H], [Sc] for the history of the concept) and then further investigated by Schwartz [Sc]. Under assumption (2.1), for any $\varepsilon > 0$ we can write:

$$\begin{aligned}
\int_\varepsilon^1 g(t)dt &= \sum_{j \in J} a_j \left[\frac{1}{-\lambda_j + 1} t^{-\lambda_j + 1} \right]_\varepsilon^1 - b \log \varepsilon + \int_\varepsilon^1 h(t)dt \\
&= - \sum_{j \in J} \frac{a_j}{-\lambda_j + 1} \varepsilon^{-\lambda_j + 1} - b \log \varepsilon + \sum_{j \in J} \frac{a_j}{-\lambda_j + 1} + \int_\varepsilon^1 h(t)dt \\
&= I(\varepsilon) + F(\varepsilon)
\end{aligned}$$

where $I(\varepsilon) := -\sum_{j\in J} \frac{a_j}{-\lambda_j+1}\varepsilon^{-\lambda_j+1} - b\log\varepsilon$ and $F(\varepsilon) := \sum_{j\in J} \frac{a_j}{-\lambda_j+1} + \int_\varepsilon^1 h(t)dt$. When ε tends to zero, $F(\varepsilon)$ converges to a finite expression:

$$\begin{aligned} f.p. \textstyle\int_0^1 g(t)dt &:= \sum_{j\in J} \frac{a_j}{-\lambda_j+1} + \int_0^1 h(t)dt \\ &= \lim_{\varepsilon\to 0}\left(\int_\varepsilon^1 g(t)dt + \sum_{j\in J}\frac{a_j}{-\lambda_j+1}\varepsilon^{-\lambda_j+1} + b\log\varepsilon\right) \end{aligned} \tag{2.2}$$

called the *finite part of the integral* $\int_0^1 g(t)dt$ whereas $I(\varepsilon)$ diverges.

Notice that whenever $g \in L^1(]0,1[)$, then $f.p. \int_0^1 g(t)dt = \int_0^1 g(t)dt$. We leave to the reader the easy **exercise** to prove that $f.p. \int_0^1 : L^1_{loc}(]0,1]) \to \mathbb{C}$ defines a linear map.

• **Behaviour under a change of variable:** The finite part of the integral of a function satisfying assumption (2.1) is *not* in general invariant under a change of variable in the integral. Let $t \to u(t)$ be a smooth map from $[0,1]$ to $[u(0)=0, u(1)]$ which is a diffeomorphism on $]0,1[$. Let $u \to t(u)$ denote its inverse; we want to check that $\tilde{g} : u \mapsto g(t(u))t'(u)$ satisfies condition (2.1).

Let k_0 be the first integer for which $t^{(k_0)}(0) \neq 0$, then $t(\varepsilon) = \sum_{k=k_0}^K \frac{\varepsilon^k}{k!}t^{(k)}(0) + o(\varepsilon^K) = \frac{t^{(k_0)}(0)\varepsilon^{k_0}}{k_0!}(1 + \sum_{k=1}^{K-k_0} \varepsilon^k \frac{k_0!}{(k+k_0)!}\frac{t^{(k+k_0)}(0)}{t^{(k_0)}(0)} + o(\varepsilon^{K-k_0}))$ and

$$\begin{aligned} \tilde{g}(u) = \sum_{j\in J} a_j &\left(\frac{t^{(k_0)}(0)}{k_0!}\right)^{-\lambda_j+1} u^{-k_0\cdot\lambda_j}\left(\sum_{k=0}^{K-k_0} u^k \frac{k_0!\, t^{(k+k_0)}(0)}{(k+k_0)!\, t^{(k_0)}(0)} + o(u^{K-k_0})\right)^{-\lambda_j} \cdot \\ &\cdot \left[\frac{t^{(k_0)}(0)u^{k_0-1}}{(k_0-1)!}\left(\sum_{k=0}^{K-k_0} u^k \frac{k_0!}{(k+k_0)!}\frac{t^{(k+k_0)}(0)}{t^{(k_0)}(0)} + o(u^{K-k_0})\right) + \right. \\ &+ \left.\frac{t^{(k_0)}(0)u^{k_0}}{k_0!}\left(\sum_{k=1}^{K-k_0} k u^{k-1}\frac{k_0!}{(k+k_0)!}\frac{t^{(k+k_0)}(0)}{t^{(k_0)}(0)} + o(u^{K-k_0-1})\right)\right] + \\ &+ \left(\frac{k_0 b}{t^{(k_0)}(0)}u^{-k_0} + \tilde{h}(u)\right)\cdot\left(\sum_{k=0}^{K-k_0} u^k \frac{k_0!}{(k+k_0)!}\frac{t^{(k+k_0)}(0)}{t^{(k_0)}(0)} + o(u^{K-k_0})\right)^{-\lambda_j} \cdot \\ &\cdot \left[\frac{t^{(k_0)}(0)u^{k_0-1}}{(k_0-1)!}\left(\sum_{k=0}^{K-k_0} u^k \frac{k_0!}{(k+k_0)!}\frac{t^{(k+k_0)}(0)}{t^{(k_0)}(0)} + o(u^{K-k_0})\right) + \right. \end{aligned}$$

$$+ \frac{t^{(k_0)}(0)u^{k_0}}{k_0!}\left(\sum_{k=1}^{K-k_0} k u^{k-1}\frac{k_0!}{(k+k_0)!}\frac{t^{(k+k_0)}(0)}{t^{(k_0)}(0)} + o(u^{K-k_0-1})\right)\Bigg]$$

for some $\tilde{h} \in L^1(]0,1[)$. ¿From this (rather complicated !) expression, one can see that the term $u^{-k_0\cdot\lambda_j}$ gets multiplied by terms of the form $u^i, i \in I\!N$ this giving rise to terms of the type $u^{-k_0\cdot\lambda_j+i}, i \in I\!N$ which are compatible with condition (2.1) provided $-k_0\cdot\lambda_j - i = -1$ or $k_0\cdot Re(\lambda_j) - i > 1$. This singles out the following cases:

Case $k_0 = 1: \quad i)\lambda_j \in I\!N, \ \ ii)Re(\lambda_j) \notin I\!N, \ \ iii)Re(\lambda_j) \in I\!N \quad$ and $\ Im\lambda_j \neq 0.$

Case $k_0 \neq 1: \quad i)\lambda_j \in \mathbb{Q}, \ \ ii)Re(\lambda_j) \notin \mathbb{Q}, \ \ iii)Re(\lambda_j) \in \mathbb{Q} \quad$ and $\ Im\lambda_j \neq 0.$

In cases i) and ii) the map $\tilde{g}$ satisfies condition (2.1) but not in case iii) which we shall therefore leave aside.

It follows from (2.1) that

$$\begin{aligned}\int_\varepsilon^{u(1)} g(t(u))t'(u)du &= \sum_{j\in J}\int_\varepsilon^{u(1)} a_j(t(u))^{-\lambda_j}t'(u)du \\ &\quad + b\int_\varepsilon^{u(1)} t'(u)t(u)^{-1}du + \int_\varepsilon^{u(1)} h(t(u))t'(u)du \\ &= \sum_{j\in J}\frac{a_j}{-\lambda_j+1}\left[t^{-\lambda_j+1}\right]_{t(\varepsilon)}^1 + b[\log t]_{t(\varepsilon)}^1 + \int_{t(\varepsilon)}^1 h(t)dt.\end{aligned}$$

we have:

$$\int_\varepsilon^{u(1)} g(t(u))t'(u)du = I_1(\varepsilon) + F_1(\varepsilon)$$

where

$$\begin{aligned}I_1(\varepsilon) &:= -\sum_{j\in J}\frac{a_j}{-\lambda_j+1}\varepsilon^{k_0(-\lambda_j+1)}\left(\frac{t^{(k_0)}(0)}{k_0!}\right)^{-\lambda_j+1}\cdot \\ &\quad \left(1+\sum_{k=1}^{K-k_0}\frac{k_0!}{(k+k_0)!}\frac{t^{(k+k_0)}(0)}{t^{(k_0)}(0)}\varepsilon^k + o(\varepsilon^{K-k_0})\right)^{-\lambda_j+1} - b\cdot\log(t(\varepsilon))\end{aligned}$$

and where

$$F_1(\varepsilon) := \sum_{j\in J}\frac{a_j}{-\lambda_j+1} + \int_{t(\varepsilon)}^1 h(t)dt.$$

Let us first assume that $t'(0) \neq 0$ so that $k_0 = 1$.

(i) If $Re(\lambda_j) \notin I\!N$ for any $j \in J$ and $b = 0$ then $I_1(\varepsilon)$ leads to a divergent expression when $\varepsilon \to 0$ and we have:

$$
\begin{aligned}
f.p. \left(\int_0^{u(1)} g(t(u))t'(u)du \right) &:= \sum_{j \in J} \frac{a_j}{-\lambda_j + 1} + \int_0^1 h(t)dt \\
&= \lim_{\varepsilon \to 0} \left(\int_\varepsilon^{u(1)} g(t(u))t'(u)du + \right. \\
&\qquad \left. \sum_{j \in J} \frac{a_j}{-\lambda_j + 1} \varepsilon^{-\lambda_j + 1} \left(t'(0)\right)^{-\lambda_j + 1} \left(1 + \alpha(\varepsilon)\right)^{-\lambda_j + 1} \right) \\
&= f.p. \int_0^1 f(t)dt \qquad (2.3)
\end{aligned}
$$

where we have set $\alpha(\varepsilon) := \sum_{k=2}^{K-1} \frac{\varepsilon^{k-1}}{k!} \frac{t^{(k)}(0)}{t'(0)} + o(\varepsilon^{K-1})$.

(ii) If $Re(\lambda_j) \notin I\!N$ for any $j \in J$ and $b \neq 0$ then the expression $I_1(\varepsilon)$ gives rise to an extra finite term in the limit $\varepsilon \to 0$:

$$
\begin{aligned}
f.p. \left(\int_0^{u(1)} g(t(u))t'(u)du \right) &:= \sum_{j \in J} \frac{a_j}{-\lambda_j + 1} + \int_0^1 h(t)dt - b \log(t'(0)) \\
= \lim_{\varepsilon \to 0} & \left(\int_\varepsilon^{u(1)} g(t(u))t'(u)du + \sum_{j \in J} \frac{a_j}{-\lambda_j + 1} \varepsilon^{-\lambda_j + 1} \left(t'(0)\right)^{-\lambda_j + 1} \cdot \right. \\
&\qquad \left. \cdot \left(1 + \alpha(\varepsilon)\right)^{-\lambda_j + 1} + b \log \varepsilon \right) \qquad (2.4)
\end{aligned}
$$

Comparing (2.4) with (2.2) we see that

$$
f.p. \left(\int_{u(0)}^{u(1)} g(t(u))t'(u)du \right) - f.p. \left(\int_0^1 g(t)dt \right) = -b \log(t'(0)).
$$

(iii) If there exists $j_0 \in I\!N$ such that $\lambda_{j_0} \in I\!N$, say $\lambda_{j_0} = n_0 \in I\!N$ then the divergent term in ε^{-n_0+1} multiplied with the term in ε^{n_0-1} arising in the expansion $(1+\alpha(\varepsilon))^{-n_0+1} = 1 + (1-n_0)\alpha(\varepsilon) + \frac{(n_0-1)n_0}{2}\alpha(\varepsilon)^2 + \cdots$ gives rise to an extra finite term in the limit $\varepsilon \to 0$, thus adding an extra term to the finite part.

Let us now consider the case when $t'(0) = 0$ i.e $k_0 > 1$. We also distinguish three cases.

(i) If $Re(\lambda_j) \notin \mathbb{Q}$ for any $j \in J$ and $b = 0$ then $I_1(\varepsilon)$ leads to a purely divergent expression when $\varepsilon \to 0$ and we have:

$$f.p. \left(\int_0^{u(1)} g(t(u))t'(u)du \right) = f.p. \int_0^1 f(t)dt.$$

(ii) If $Re(\lambda_j) \notin \mathbb{Q}$ for any $j \in J$ and $b \neq 0$ then the expression $I_1(\varepsilon)$ gives rise to an extra finite term in the limit $\varepsilon \to 0$ and we have:

$$f.p. \left(\int_0^{u(1)} g(t(u))t'(u)du \right) - f.p. \int_0^1 f(t)dt = -b \log(t^{(k_0)}(0)) + b \log(k_0!).$$

(iii) If for some $j_0 \in J$, $\lambda_{j_0} \in \mathbb{Q}$, then for any k_0 such that $(-\lambda_{j_0} + 1)k_0 \in \mathbb{Z}$, a compensation arises between the divergent term $\varepsilon^{(-\lambda_j+1)k_0}$ and a term in the expansion of of $(1 + \alpha(\varepsilon))^{-\lambda_j+1}$ thus giving rise to an extra finite term.

Hence only if $Re(\lambda_j) \notin \mathbb{Q}$ for any $j \in J$ and $b = 0$ (none of the powers in the asymptotic expansion (2.1) are rationals) is the finite part invariant under a change of variable:

$$f.p. \left(\int_{u(0)}^{u(1)} g(t(u))t'(u)du \right) = f.p. \int_0^1 g(t)dt.$$

Our conclusion therefore differs from [Sc] where it was asserted that the invariance under a change of variable held true for all g of type (2.1).

• **A meromorphic extension:** For $z \in \mathbb{C}$ with $Re(z) > 0$ and large enough, the function $G(z) := \int_0^1 g(t)t^z dt$ is well defined. Indeed, let $Re(z) > sup(Re(\lambda_j)) - 1$ or $Re(z) > 0$ if J is void, then:

$$\begin{aligned} G(z) &= \int_0^1 g(t)t^z dt \\ &= \sum_{j \in J} a_j \int_0^1 t^{z-\lambda_j} dt + b \int_0^1 t^{z-1} dt + \int_0^1 h(t)t^z dt \\ &= \sum_{j \in J} \frac{a_j}{z - \lambda_j + 1} + \frac{b}{z} + \int_0^1 h(t)t^z dt \end{aligned}$$

is well defined. It extends to a meromorphic function on the upper half complex plane $Re(z) > 0$ with a finite number of poles $\{0, \lambda_j - 1, j \in J\}$. Let us focus on the pole at zero.

The case $b=0$*:* since $\lambda_j \neq 1$, the function $z \to G(z)$ extends continuously to 0 and we have:

$$G(0) = \sum_{j \in J} \frac{a_j}{-\lambda_j + 1} + \int_0^1 h(t)dt = f.p. \int_0^1 g(t)dt. \qquad (2.5.a)$$

In other words, the finite part arises as an analytic extension of an ordinary integral.

The case $b \neq 0$ *:* we first need to extract a (simple) pole at zero before taking the limit at $z=0$:

$$\lim_{z\to 0}\left(G(z) - \frac{b}{z}\right) = \sum_{j \in J} \frac{a_j}{-\lambda_j + 1} + \int_0^1 h(t)dt = f.p. \int_0^1 g(t)dt. \qquad (2.5.b)$$

• **Pseudo-functions:** We now assume that $b=0$ and that none of the $\lambda_j, j \in J$ are integers. Then for any $n \in I\!N$, in a similar way as in the case $n=0$, we can define

$$G(n) := \sum_{j \in J} \frac{a_j}{n - \lambda_j + 1} + \int_0^1 h(t)t^n dt = f.p. \int_0^1 g(t)t^n dt. \qquad (2.6)$$

Let ϕ be a smooth function on $]0,1[$. The Taylor-Lagrange formula at order K yields:

$$\phi(t) = \sum_{k=0}^{K} \frac{\phi^{(k)}(0)}{k!} t^k + \frac{t^{K+1}}{K!} \int_0^1 (1-u)^K \phi^{(K+1)}(tu)du$$

so that choosing $K = [sup_{j \in J}(Re(\lambda_j))]$, the integral $\int_0^1 g(t)\frac{t^{K+1}}{K!}\int_0^1 (1-u)^K \phi^{(K+1)}(tu)du$ is well-defined. Moreover the map $t \to g(t)\phi(t)$ satisfies condition (2.1) and hence it follows from the linearity of the finite part of an integral that:

$$\begin{aligned} f.p. \int_0^1 g(t)\phi(t)dt \; &= \sum_{k=0}^{K} \frac{\phi^{(k)}(0)}{k!} f.p. \left(\int_0^1 g(t)t^k dt\right) + \\ & \int_0^1 g(t)\frac{t^{K+1}}{K!}\left(\int_0^1 (1-u)^K \phi^{(K+1)}(tu)du\right). \qquad (2.7) \end{aligned}$$

¿From the above discussion, we know that provided $Re(\lambda_j) \notin I\!\!Q$ and $b=0$, $f.p. \int_0^1 g(t)t^k dt$ is independent of the choice of parametrization inside the integral so that the integral symbol on the l.h.s. of (2.7) makes sense. Moreover the map $\phi \to f.p. \int_0^1 g(t)\phi(t)dt$ is linear and continuous on $C^\infty([0,1])$ for the topology of uniform convergence of all derivatives. Furthermore for some large enough integer K there is a constant $C > 0$ such that

$$|f.p. \int_0^1 g(t)\phi(t)dt| \leq C\|\phi\|_{C^K([0,1])}$$

where $C^K([0,1])$ is the sapce of C^k functions on $[0,1]$ and $\|\cdot\|_{C^K([0,1])}$ the norm of uniform convergence in all j-th derivatives up to $j = K$. Thus if $b = 0$ and $Re(\lambda_j) \notin \mathbb{Q}$, $\phi \to f.p. \int_0^1 g(t)\phi(t)dt$ is a linear map on $C^\infty([0,1])$ which is continuous on $C^K([0,1])$ for some K large enough and hence it defines a distribution of finite order called *pseudo-function.*

2.2 Mellin transform [BGV]

Recall that $\Gamma(z) = \int_0^\infty e^{-t}t^{-z}dt$ is a meromorphic function with poles at the negative integers. Moreover $\Gamma(z)^{-1} = z + \gamma z^2 + O(z^3)$ where γ is the Euler constant $\gamma = \lim_{n\to\infty}(log\, n - \sum_{k=1}^{n+1} \frac{1}{k})$.

Unlike in the previous section where we considered a finite interval, here we will be working on $]0, +\infty[$. We shall therefore need some condition on the functions at infinity in order to have well-defined integrals. Let us consider functions $f \in C^\infty(]0,\infty[)$ such that:

(i) $\exists l > 0$, $\exists C > 0$, such that $|f(t)| \leq Ce^{-\lambda t}$ for large enough t

(ii) There are positive integers $N, q \neq 0$ and real numbers a, α_j,β_j, γ_j with $j \in \mathbb{N}$ such that

$$f(\varepsilon) \sim_0 \sum_{j=0}^{\infty} \alpha_j \varepsilon^{\frac{j-a-N}{q}} + \sum_{j=0,\, \frac{j-a-N}{q} \in \mathbb{Z}}^{\infty} \beta_j \varepsilon^{\frac{j-a-N}{q}} \log\varepsilon + \sum_{j=0}^{\infty} \gamma_j \varepsilon^j. \quad (2.8)$$

Note that when a is an integer, there is a redundancy in this expression since constant terms can arise as γ_0 or as α_{a+N}. By the symbol $\sim$ we mean that for any $J \in \mathbb{N}$, $\exists K_J := [a] + qJ + N \in \mathbb{N}$ such that

$$f(\varepsilon) = \sum_{j=0}^{K_J} \alpha_j \varepsilon^{\frac{j-a-N}{q}} + \sum_{j=0,\, \frac{j-a-N}{q} \in \mathbb{Z}}^{K_J} \beta_j \varepsilon^{\frac{j-a-N}{q}} \log\varepsilon + \sum_{j=0}^{K_J} \gamma_j \varepsilon^j + o(\varepsilon^J).$$

For such functions the *Mellin transform* (see e.g. [BGV])

$$M(f)(z) := \frac{1}{\Gamma(z)} \int_0^\infty f(t)t^{z-1}dt \quad (2.9)$$

is well defined for $Re(z)$ large enough.

It is useful to notice that for $\lambda > 0$ we have:

$$\lambda^{-z} = M(t \to e^{-t\lambda})(z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-t\lambda} dt.$$

Let us show that $M(f)$ extends to a meromorphic function with poles in $\{\frac{N+a-j}{q}, j \in I\!N\}$. Provided $Re(z)$ is chosen large enough we can write:

$$\begin{aligned}
&\Gamma(z)M(f)(z) = \int_0^1 f(t)t^{z-1}dt + \int_1^\infty f(t)t^{z-1}dt \\
&= \sum_{j=0}^{K_J} \alpha_j \int_0^1 t^{\frac{j-a-N}{q}+z-1}dt + \sum_{j=0,\frac{j-a-N}{q}\in \mathbb{Z}}^{K_J} \beta_j \int_0^1 t^{\frac{j-a-N}{q}+z-1} \log t \, dt \\
&\quad + \sum_{j=0}^{J} \gamma_j \int_0^1 t^{j+z-1}dt + \int_0^1 o(t^J)t^{z-1}dt + \int_1^\infty f(t)t^{z-1}dt \qquad (2.10) \\
&= \sum_{j=0}^{K_J} \frac{\alpha_j}{\frac{j-a-N}{q}+z} - \sum_{j=0,\frac{j-a-N}{q}\in \mathbb{Z}}^{J} \frac{\beta_j}{(\frac{j-a-N}{q}+z)^2} + \sum_{j=0}^{J} \frac{\gamma_j}{j+z} + \\
&\quad R(z) + \int_1^\infty f(t)t^{z-1}dt
\end{aligned}$$

(using an integration by parts to compute the integral involving the logarithmic terms) where $z \to R(z)$ is holomorphic for $Re(z) > \frac{-K_J+1}{q}$. The map $z \to \int_1^\infty f(t)t^{z-1}dt$ is analytic in z since $f(t)$ decreases exponentially at infinity. Hence the map $z \to \Gamma(z)M(f)(z)$ is meromorphic with poles in the set $\{\frac{N+a-j}{q}, j \in I\!N\}$. Since Γ^{-1} is analytic, the map $z \to M(f)(z)$ is also meromorphic with poles in the set $\{\frac{N+a-j}{q}, j \in I\!N\}$.

For our purposes, it is useful to know how the Mellin transform behaves at zero. It only has a simple pole at zero and:

$$\begin{aligned}
&M(f)(z) + \beta_0 z^{-1} = \alpha_{a+N} + \gamma_0 - \gamma\beta_{a+N} + O(z) \\
&\quad = \text{ constant term } \quad - \gamma \cdot (\text{coefficient of the logarithmic term}) \qquad (2.11)
\end{aligned}$$

This follows from (2.10) combined with the fact that $\Gamma(z)^{-1} = z + \gamma z^2 + O(z^3)$ where γ is the Euler constant.

2.3 Renormalized limits [BGV]

• **Renormalizable functions:** We shall call a smooth (complex valued) function f on $C^\infty(]0,\infty[)$ *renormalizable* whenever it satisfies conditions (i) and (ii) of (2.8).

Since the poles of the function $M(f)$ lie in $\{\frac{N+a-j}{q}, j \in I\!N\}$, if *a is not an integer*, then 0 is not a pole and we can define $M(f)(0)$. However for a renormalizable function f of the type (2.8) with $a \in Z\!\!\!Z$, one expects a pole in zero and we need to "extract divergences" before taking the limit. On the grounds of (2.11), for $\mu \in I\!R$ we define the *μ-renormalized limit* of f at 0:

$$\begin{aligned} LIM^\mu_{t\to 0} f(t) &:= \alpha_{a+N} + \gamma_0 - \mu\beta_{a+N} \\ &= \text{constant term} \quad - \mu \cdot (\text{coefficient of the logarithmic term}) \end{aligned} \tag{2.12}$$

From (2.11) it follows that:

$$\lim_{z\to 0}(M(f)(z) + \beta_0 z^{-1}) = \alpha_{a+N} + \gamma_0 - \gamma\beta_{a+N} = LIM^\gamma_{t\to 0} f(t). \tag{2.13}$$

Notice that *if a is not an integer*, then $\beta_0 = 0$ and

$$LIM^\mu_{t\to 0} f(t) = lim_{z\to 0} M(f)(z) = M(f)(0).$$

• **A class of renormalizable functions:** Let g be of the form (2.8) with $\beta_j = 0$ for all $j \in I\!N$. We set $\lambda_j := \frac{a+N-j}{q}$. Then $f(\varepsilon) := \int_\varepsilon^\infty g(t)dt$ satisfies assumptions (2.8) and $f(\varepsilon) \sim_0 A_0 - \sum_{\lambda_j\neq 1} \frac{\alpha_j}{-\lambda_j+1}\varepsilon^{-\lambda_j+1} - \alpha_{q(a+N)} \log\varepsilon$ for some constant $A_0 = \sum_{\lambda_j\neq 1} \frac{\alpha_j}{-\lambda_j+1} + \int_0^1 \left(g(t) - \sum_{\lambda_j\neq 1}\alpha_j t^{-\lambda_j} dt\right) + \int_1^\infty g(t)dt$. Since $g \in L^1(]\varepsilon, +\infty[)$ for any $\varepsilon > 0$, we can extend the notion of finite part defined in (2.2) for the integral $\int_0^1 g(t)dt$ on $]0,1[$ in a straightforward way to the integral $\int_0^\infty g(t)dt$ on $]0,\infty[$:

$$\begin{aligned} f.p.\int_0^\infty g(t)dt &:= f.p.\int_0^1 g(t)dt + \int_1^\infty g(t)dt \\ &= \sum_{\lambda_j\neq 1} \frac{\alpha_j}{-\lambda_j+1} + \int_0^1 (g(t) - \sum_{\lambda_j\neq 1}\alpha_j t^{-\lambda_j})dt + \int_1^\infty g(t)dt \end{aligned}$$

and we have $A_0 = f.p.\int_0^\infty g(t)dt$. As before $f.p.\int_0^\infty g(t)dt$ is invariant under a change of variable if $Re(\lambda_j) \notin I\!\!\!Q$; this condition holds in particular if $a \notin Z\!\!\!Z$. Hence, by (2.12)

$$LIM^\mu_{\varepsilon\to 0} f(\varepsilon) = f.p.\int_0^\infty g(t)dt + \mu\alpha_{(a+N)q}.$$

In particular, taking $\mu = \gamma$ and using (2.13) yields:

$$\lim_{z\to 0}\left(M(f)(z) - \alpha_{(a+N)q} z^{-1}\right) = f.p. \int_0^\infty g(t)dt + \gamma \alpha_{(a+N)q}.$$

Exercise: Let $g(t) := -\frac{e^{-t\lambda}}{t}$, $\lambda > 0$ and $f_\varepsilon(t) := -\int_\varepsilon^\infty \frac{e^{-\lambda t}}{t} dt$. Check that $LIM^\gamma(f) = log\lambda = -\zeta'_\lambda(0)$ where we have set $\zeta_\lambda = M(t \to e^{-t\lambda})$.

Let us summarize where we now stand.

Proposition: *For g satisfying (2.8) (i) and (ii) with $a \notin \mathbb{Z}$ then:*

(i) the finite part $f.p.\left(\int_0^\infty g(t)dt\right)$ is invariant under a change of variable,

(ii) $\phi \to \int_0^\infty g(t)\phi(t)dt$ defines a distribution of finite order,

(iii) $z \to M(f)(z)$ has no poles.

Proof. This is a direct consequence of the above discussion since $a \notin \mathbb{Z} \Rightarrow \beta_j = 0$ and $Re(\lambda_j) = \frac{-j+a+N}{q} \notin \mathbb{Q} \quad \forall j \in \mathbb{N}$.

3 Pseudo-differential operators

This section gives a brief presentation of the basic tools in our framework, namely classical pseudo-differential operators and particularly elliptic ones, their logarithms and their complex powers. Classical references are [G], [LM], [Se], [Sh]. We also refer to [D] for a detailed presentation of the tools briefly described in this section.

An *ordinary partial differential operator* of order $m \in \mathbb{N}$ is a polynomial expression

$$P = \sum_{|\alpha| \leq m} p_\alpha(x) D_x^\alpha$$

where $p_\alpha \in C^\infty(\mathbb{R}^d)$ and $D_x^\alpha := (-i)^{|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1} \cdots + \frac{\partial^{\alpha_d}}{\partial x_d}$. Here $|\alpha| = \alpha_1 + \cdots \alpha_d$. It can also be written:

$$Pu(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma(x,\xi) \hat{u}(\xi) d\xi \quad \text{where} \quad \sigma(x,\xi) = \sum_{|\alpha| \leq m} p_\alpha(x) \xi^\alpha$$

and where $\cdot$ denotes the canonical scalar product in $\mathbb{R}^d$. Allowing more general expressions for $\sigma(x,\xi)$ leads to the notion of pseudo-differential operator.

3.1 Pseudo-differential operators on an open set of $\mathbb{R}^d$

•**The symbol set:** Let U be an open subset of $\mathbb{R}^d$. Given $\alpha \in \mathbb{R}$, let us denote by $S^\alpha(U)$ the set of real valued smooth functions

$$\begin{aligned} U \times \mathbb{R}^d &\to \mathbb{R} \\ (x,\xi) &\to \sigma(x,\xi) \end{aligned}$$

with compact support in x and satisfying the following property. Given any two multiindices $\gamma = (\gamma_1, \cdots, \gamma_d)$ and $\delta = (\delta_1, \cdots, \delta_d)$ in $\mathbb{N}^d$, there exists a constant $C_{\gamma,\delta}$ such that

$$|D_x^\gamma D_\xi^\delta \sigma(x,\xi)| \leq C_{\gamma,\delta}(1 + \|\xi\|)^{\alpha - |\delta|} \quad \forall x \in U, \forall \xi \in \mathbb{R}^d \tag{3.1}$$

where we use the symbol $\|\cdot\|$ to denote the norm on $\mathbb{R}^d$. An element of $S^\alpha(U)$ is called a *symbol of order* α. Notice that the above condition is a requirement on the behaviour when $\|\xi\|$ goes to infinity of the expression $(1 + \|\xi\|)^{|\delta| - \alpha} |D_x^\gamma D_\xi^\delta \sigma(x,\xi)|$.

The notion of symbol of order α extends in an obvious manner to complex valued symbols but also to matrix valued symbols

$$\begin{array}{ll} U \times \mathbb{R}^d & \to \mathcal{M}_{r_1,r_2}(\mathbb{R}) \\ (x,\xi) & \to \sigma(x,\xi), \end{array}$$

the components of which are symbols of order α as defined above.

Exercise: Let $\Delta := -\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$. Show that $\sigma_\Delta(x,\xi) := \|\xi\|^2$ on $\mathbb{R}^d$ is a symbol of order 2 and that $\sigma_{(1+\Delta)^{-k}}(x,\xi) := (1+\|\xi\|^2)^{-k}$ is a symbol of order $-2k$ with $k \in \mathbb{N}$.

The *principal part* of the symbol $\sigma \in S^\alpha(U)$ (*or leading symbol*) is defined as follows:

$$\sigma_L(x,\xi) = \lim_{t\to+\infty} \frac{\sigma(x,t\xi)}{t^\alpha}. \tag{3.2}$$

A *smoothing symbol* is a symbol in

$$S^{-\infty}(U) := \bigcap_{\alpha\in\mathbb{R}} S^\alpha(U\ )$$

and the relation

$$\sigma \simeq \tilde{\sigma} \Leftrightarrow \sigma - \tilde{\sigma} \in S^{-\infty}(U)$$

defines an equivalence relation on $S(U) := \bigcup_\alpha S^\alpha(U)$.

For a real valued function σ on $U \times \mathbb{R}^d$ with the property that there exist $\sigma_j \in S^{m_j}(U), j \in \mathbb{N}$ with $(m_j)_{j\in\mathbb{N}}$ a decreasing sequence in $\mathbb{Z}$ verifying the following condition:

$$\forall N \in \mathbb{N}, \exists J_N \in \mathbb{N}, \quad \sigma(x,\xi) - \sum_{j=0}^J \sigma_j(x,\xi) \in S^{-N}(U) \quad \forall \quad J \geq J_N$$

we shall write:

$$\sigma(x,\xi) \sim \sum_{j=0}^\infty \sigma_j(x,\xi).$$

A symbol of order α is called *classical* if there exist $\sigma_{\alpha-j} \in S^{\alpha-j}(U)$, $j \in \mathbb{N}$ such that:

$$\sigma(x,\xi) \sim \sum_{j=0}^\infty \sigma_{\alpha-j}(x,\xi) \tag{3.3}$$

which are positively homogeneous, i.e

$$\sigma_{\alpha-j}(x, t\xi) = t^{\alpha-j}\sigma_{\alpha-j}(x,\xi) \quad \forall t \in I\!R^+, \forall \xi \in I\!R^d. \tag{3.4}$$

Let $\Psi \in C^\infty(I\!R^d)$ such that Ψ vanishes for for $\|\xi\| < \frac{1}{4}$ and $\Psi(\xi) = 1$ for $\|\xi\| \geq \frac{1}{2}$. Then $\sigma_{\alpha-j}(x,\xi) - \Psi(\xi)\sigma_{\alpha-j}(x,\xi)$ belongs to $S^{-\infty}(U)$. Hence for a symbol to be classical, it is sufficient to require that

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \Psi(\xi)\sigma_{\alpha-j}(x,\xi)$$

with $\sigma_{\alpha-j}(x,\xi) \in S^{\alpha-j}(U)$ and positively homogeneous in ξ.
Following Kontsevich and Vishik [KV1], we shall say that a classical symbol lies in the *odd-class* if it has integer order and the positively homogeneous components $\sigma_{\alpha-j}$ are moreover homogeneous i.e:

$$\sigma_{\alpha-j}(x, t\xi) = t^{\alpha-j}\sigma_{\alpha-j}(x,\xi) \quad \forall t \in I\!R \quad \forall \xi \in I\!R^d. \tag{3.5}$$

Polynomial symbols provide examples of symbols in the odd class.

A symbol σ in $\mathcal{S}^\alpha(U)$ is called *elliptic* if its principal part $\sigma_L(x,\xi)$ seen as an endomorphism of the fibre E_x is invertible for any $(x, \xi \neq 0) \in T^*U$. Adding a smoothing symbol to an elliptic symbol therefore does not affect the ellipticity property. An equivalent condition (see e.g. [G]) for the ellipticity of $\sigma \in \mathcal{S}^\alpha(U)$ is the existence of a symbol $\tilde{\sigma} \in \mathcal{S}^{-\alpha}(U)$ and of a smooth plateau function Φ which is constant equal to 1 on $\bar{V}$ with V a non empty open subset of U such that:

$$\Phi(\sigma\tilde{\sigma} - I) \in \mathcal{S}^{-\infty}(U) \quad \text{and} \quad \Phi(\tilde{\sigma}\sigma - I) \in \mathcal{S}^{-\infty}(U). \tag{3.6}$$

All the above mentioned notions extend in an obvious manner to complex or matrix valued symbols.

• **From symbols to pseudo-differential operators:** Following the prescription described in the introduction for partial differential operators, to a symbol $\sigma \in S^\alpha(U)$ we can associate the *pseudo-differential operator (P.D.O.) with symbol* σ defined by:

$$\begin{aligned} A : C_c^\infty(U) &\longrightarrow C^\infty(I\!R^d) \\ u &\longrightarrow (x \to Au(x)) \end{aligned}$$

where

$$Au(x) = \int_{I\!R^d \times U} e^{i\xi\cdot x}\sigma(x,\xi)\hat{u}(\xi)d\xi = \int_{I\!R^d \times U} e^{i\xi\cdot(x-y)}\sigma(x,\xi)u(y)dyd\xi. \tag{3.7}$$

Here $C_c^\infty(U)$ denotes the space of complex valued smooth functions with compact support in U. The *leading symbol* of A is given by the leading part of its symbol. If the symbol is classical, we shall call the corresponding P.D.O. *classical* and the set of classical P.D.O.s of order α is denoted by $CL^\alpha(U)$. The Laplacian $\Delta := -\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is a partial differential operator of order 2 with symbol σ_Δ and $(\Delta+1)^{-k}, k \in I\!N$ a P.D.O. of order $-2k$ with symbol $\sigma_{(\Delta+1)^{-k}}$ given previously.

A *smoothing P.D.O.* is an operator in

$$CL^{-\infty}(U) := \bigcap_{\alpha \in I\!R} CL^\alpha(U).$$

Such an operator sends any function in a Sobolev class H^s to a smooth function and can be represented by a smooth kernel, i.e

$$K \in CL^{-\infty}(U) \Leftrightarrow \exists k \in C^\infty(U \times U),$$

$$\text{such that} \quad Ku(x) = \int_U k(x,y)u(y)dy \quad \forall u \in C_c^\infty(U), \quad \forall x \in U.$$

Two P.D.O.s P and Q are called *equivalent* (and we note $P \simeq Q$) if their difference $P - Q$ is a smoothing operator; one often works with P.D.O.s "up to equivalence".

A partial differential operator is *local* i.e. $(u = 0 \Rightarrow Au = 0)$. But a pseudo-differential operator, because of the smearing produced by the Fourier transform, is not local. However, it has some *pseudo-locality* properties [G], [LM]:

(i) For any $\varepsilon > 0$, there is an ε-local P.D.O. P_ε to which P is equivalent i.e $P \simeq P_\varepsilon$ and such that the support of P_ε is contained in the ε - neighborhood $\{x \in I\!R^d, d(x,U) \leq \varepsilon\}$ of U,

(ii) If u is smooth on an open set U then Pu is also smooth on this open subset.

Given two partial differential operators $P = \sum_\alpha \sigma_\alpha(x) D_x^\alpha$ and $\tilde{P} = \sum_\beta \tilde{\sigma}_\beta(x) D_x^\beta$, a direct computation shows that the symbol of the product $P\tilde{P}$ reads

$$\sigma(P\tilde{P}) = \sum_\alpha D_\xi^\alpha \sigma \cdot D_x^\alpha \tilde{\sigma}.$$

A similar type of relation holds "up to equivalence" for products of pseudo-differential operators, replacing the symbol "=" by the symbol "$\simeq$":

$$\sigma(P\tilde{P}) \simeq \sum_\alpha D_\xi^\alpha \sigma \cdot D_x^\alpha \tilde{\sigma}. \tag{3.8}$$

The various classes of symbols introduced previously induce corresponding classes of pseudo-differential operators. An *odd-class classical P.D.O.* is a classical P.D.O. with integer order with symbol in the odd class, an *elliptic operator* is a P.D.O. with elliptic symbol.

Ordinary partial differential operators with integer order provide examples of (classical) P.D.O.s in the odd class.

From the characterization of ellipticity of the symbols of pseudo-differential operators by the invertibility of their leading symbol, it follows that a pseudo-differential operator A is elliptic whenever its leading symbol $\sigma_L(A)(x,\xi)$ (which is defined globally) is invertible for any $(x, \xi \neq 0) \in T^*M$. From the characterization of the ellipticity of the symbols by "invertibility up to a smoothing operator" described above, it follows that the ellipticity of a P.D.O. P can also be characterized by the existence of a P.D.O. $\tilde{P}$ such that both $\Phi(P\tilde{P} - I)$ and $\Phi(\tilde{P}P - I)$ are infinitely smoothing. We shall refer to this by saying that its localized version is invertible modulo a smoothing operator. For a partial differential operator, there is no need for a plateau function Φ and it is elliptic if and only if it is invertible modulo a smoothing operator.

From the product rule for symbols of ordinary pseudo-differential operators P and $\tilde{P}$, it follows that

$$\sigma_L(P\tilde{P}) = \sigma_L(P)\sigma_L(\tilde{P}). \tag{3.9}$$

This combined with the characterization of ellipticity by the invertibility of the leading symbol shows that that the product of two elliptic pseudo-differential operators is elliptic.

3.2 Pseudo-differential operators acting on sections of vector bundles

The notion of pseudo-differential operator can be carried out to operators acting on sections of vector bundles. Let $\pi_1 : E_1 \to M$, $\pi_2 : E_2 \to M$ be two smooth vector bundles of rank r_1 and r_2 respectively over a smooth *compact* and boundaryless (i.e *closed*) Riemannian manifold M of dimension d. A linear operator

$$A : C^\infty(M, E_1) \to C^\infty(M, E_2)$$

acting from the space $C^\infty(M, E_1)$ of smooth sections of a vector bundle E_1 to the space $C^\infty(M, E_2)$ of smooth sections of a vector bundle E_2 is called a *pseudo-differential operator of order* α if it is locally given by a $(r_2 \times r_1)$ matrix of pseudo-differential operators of order α. Let us make this last requirement more precise.

Given a neighborhood U of any point $m \in M$ and any two smooth functions $\chi, \tilde{\chi}$ with compact support in U, there are local trivializations:

$$\Phi_i : E_{i_{|U}} \to \phi(U) \times \mathbb{C}^{r_i}$$

where r_i is the rank of E_i and where (U, ϕ) is a local chart on M (so that $\phi(U)$ is an open subset of $\mathbb{R}^d$), such that the induced linear map

$$\begin{array}{ll} C_c^\infty(\phi(U), \mathbb{C}^{r_1}) & \to C^\infty(\phi(U), \mathbb{C}^{r_2}) \\ u & \to pr_2 \circ \Phi_2(\tilde{\chi} A \chi) \Phi_1^{-1} u \end{array}$$

is a P.D.O. of order α. In other words, its symbol which takes values in $(r_2 \times r_1)$ matrices has components in $S^\alpha(\phi(U))$.

This definition involves a choice of trivialization but it can be shown that a change of local trivialization $(U, \phi, \Phi_i) \to (\tilde{U}, \tilde{\phi}, \tilde{\Phi}_i)$ transforms a P.D.O. on $\phi(U \cap \tilde{U})$ of order α into another P.D.O. of order α on $\tilde{\phi}(U \cap \tilde{U})$. They only differ by a smoothing operator and the notion of P.D.O. of order α acting on sections of a bundle is actually independent of the choice of trivialization.

The notion of *classical P.D.O.* and *elliptic P.D.O.* - which we recall were both unaffected by adding a smoothing operator - carry out in a similar manner to operators acting on sections of vector bundles. An ordinary partial differential operator in one local trivialization remains an ordinary partial differential operator in another one so that it makes sense to speak of partial differential operators acting on sections of vector bundles.

We denote by $CL^\alpha(M, E_1, E_2)$ the space of all classical P.D.O.s of order α acting from sections of E_1 to sections of E_2. When $E_1 = E_2 = E$, we shall denote this space by $CL^\alpha(M, E)$ and when E is the trivial bundle $M \times \mathbb{C}$ by $CL^\alpha(M)$. We set $CL(M, E) := \bigcup_\alpha CL^\alpha(M, E)$. A *smoothing operator* acting on sections of E is an element of $CL^{-\infty}(M, E) := \bigcap_\alpha CL^\alpha(M, E)$. Since a P.D.O. of order α sends a Sobolev space of sections $H^s(M, E)$ to $H^{s-|\alpha|}(M, E)$, using Sobolev embedding theorems (see [G], [LM], [N] for a review of what we need here and references therein) combined with the compactness of M, yields that a smoothing operator sends $H^s(M, E)$ to $C^\infty(M, E)$. Such an operator "smoothens" Sobolev type distribution sections into smooth ones.

The *symbol* of a P.D.O. acting on sections of vector bundles is only defined locally, namely, on the local chart (U, ϕ) and given the local trivialization $(U, \phi, \Phi_1, \Phi_2)$ it is given by the symbol of the operator $u \to \Phi_2(\tilde{\chi} A \chi) \Phi_1^{-1} u$. However its leading part transforms under a change of trivialization as a section of $\otimes_s^d T^* M \otimes Hom(E_1, E_2)$, where $\otimes_s^d T^* M$ is the symmetrized d-th tensor power of the cotangent bundle T^*M. We shall denote this section called the *leading symbol* of the P.D.O. by $\sigma_L(A)$. The ellipticity of A is characterized by the invertibility of its leading symbol $\sigma_L(A)$.

•**Elliptic (admissible) pseudo-differential operators:** Let $E \to M$ be a finite rank vector bundle based on a closed Riemannian manifold M. We shall denote by $Ell^*(M,E)$ the set of invertible classical elliptic operators, by $Ell^*_{ord>0}(M,E)$ the subset of invertible classical elliptic operators that have strictly positive order. If E is hermitian, we denote by $Ell^{*+}_{ord>0}(M,E)$ the subset of operators in $Ell^*_{ord>0}(M,E)$ with positive leading symbol. Using the fact that $\sigma_L(A)^*\sigma_L(A) = \sigma_L(A^*A)$, one can check that any positive self-adjoint elliptic operator has positive leading symbol.

Warning: We postpone until the next section the proof of some of the properties of elliptic operators acting on sections of E, which we shall take for granted for now.

Because M is compact the spectrum $spec(A)$ of an operator in $Ell^*_{ord>0}(M,E)$ consists of isolated eigenvalues with finite multiplicity (see [Sh] for differential operators, see also [LM]). There is therefore a disc D_R of radius $R > 0$ around the origin which does not contain any point of the spectrum.

We shall call an invertible elliptic P.D.O. A *admissible* if its leading symbol $\sigma_L(A)$ has no eigenvalue in some open cone with vertex 0 in the complex plane and we shall denote by $Ell^{*adm}_{ord>0}(M,E)$ the set of such admissible operators which of course contains $Ell^{*+}_{ord>0}(M,E)$. In that case at most a finite number of eigenvalues of the operator are contained in the cone [Sh] and we shall call *spectral cut* any half line $L_\theta := \{re^{i\theta}, r \geq 0\}$ for some $\theta \in \mathbb{R}$ avoiding the spectrum of A.

• **Complex powers of elliptic operators [Se]:** Let $A \in Ell^{*adm}_{ord>0}(M,E)$ with spectral cut L_θ. As we shall see in the next section, for $Rez << 0$, the complex power A^z_θ of A is a bounded operator on any space $H^s(M,E)$ of sections of E of Sobolev class H^s defined by the contour integral:

$$A^z_\theta = \frac{i}{2\pi}\int_{\Gamma_\theta} \lambda^z (A - \lambda I)^{-1} d\lambda \tag{3.10}$$

where $\Gamma_{1,\theta} = \{\lambda = re^{i\theta}, r \geq R\}$,$\Gamma_{2,\theta} = \{\lambda = Re^{i\phi}, \theta \geq \phi \geq -\theta\}$, $\Gamma_{3,\theta} = \{\lambda = re^{i(\theta-2\pi)}, r \geq R\}$ and $\Gamma_\theta = \Gamma_{1,\theta} \cup \Gamma_{2,\theta} \cup \Gamma_{3,\theta}$. Here $\lambda^z = exp(zlog\lambda)$ where $log\lambda = log|\lambda| + i\theta$ on $\Gamma_{1,\theta}$ and $log\lambda = log|\lambda| + i(\theta - 2\pi)$ on $\Gamma_{3,\theta}$.

This definition is independent of the choice of R but depends on the choice of θ and yields for any $z \in \mathbb{C}$ an elliptic operator A^z_θ of order $z \cdot ord(A)$. When $z = -k, k \in \mathbb{N}$, then A^z coincides with A^{-k} of order $-k \cdot ord(A)$.

It can be extended to all complex powers of A by the semi-group property $A^z = A^{z-k} A^k$ for k large enough. When M is Riemannian, E is hermitian and A is essentially self-adjoint, then A_θ^z is independent of the choice of θ and coincides with the complex powers defined using spectral representation. For $\theta = -\pi$, we shall drop the explicit mention of the angle θ writing simply A^z.

• **Logarithms of elliptic operators:**

For an admissible invertible elliptic operator of order 0, the spectrum is bounded (because the operator is bounded) and does not cross 0 so that it is contained in $D(0,R)/D(0,\varepsilon)$ for some $\varepsilon > 0$ small enough and some $R > 0$ large enough, where $D(0,R)$ denotes the disc centered in 0 with radius R. Let L_θ be a spectral cut and define the logarithm of A by:

$$log_\theta A := \frac{i}{2\pi} \int_{\Gamma_{R,\varepsilon,\theta}} log_\theta \lambda (A - \lambda)^{-1} d\lambda \tag{3.11}$$

where $\Gamma_{R,\varepsilon,\theta} = \bigcup_{i=1}^4 \Gamma_{i,R,\varepsilon,\theta}$ and $\Gamma_{1,R,\varepsilon,\theta} = \{\lambda = re^{i\theta}, \varepsilon \leq r \leq R\}$,$\Gamma_{2,R,\varepsilon,\theta} = \{\lambda = re^{i(\theta-2\pi)}, \varepsilon \leq r \leq R\}$, $\Gamma_{3,R,\varepsilon,\theta} = \{\lambda = Re^{i\phi}, \theta - 2\pi \leq \phi \leq \theta\}$, $\Gamma_{4,R,\varepsilon,\theta} = \{\lambda = Re^{i\phi}, \theta - 2\pi \leq \phi \leq \theta\}$. This definition in fact only depends on θ and not on R and ε.

Given $A \in Ell_{ord>0}^{*adm}(M,E)$, for arbitrary $k \in \mathbb{Z}$, the map $z \to A_\theta^z$ defines a holomorphic function from $\{z \in \mathbb{C}, Rez < k\}$ to the space of bounded linear maps $\mathcal{L}(H^s(M,E), H^{s-k\cdot ordA}(M,E))$ for any $s \in \mathbb{R}$ and we can set:

$$log_\theta A := \left[\frac{\partial}{\partial z} A_\theta^z\right]_{z=0}. \tag{3.12}$$

This defines a (non classical) P.D.O. operator of zero order and hence a bounded operator from $H^s(M,E)$ to $H^{s-\varepsilon}(M,E)$ for any $\varepsilon > 0$ and any $s \in \mathbb{R}$. For $\theta = -\pi$, we shall drop the explicit mention of the angle θ writing simply $logA$.

In local coordinates (x,ξ) on T^*M, the symbol of $log_\theta A$ reads:

$$\sigma_{log_\theta A}(x,\xi) = ord(A) log|\xi| Id + \text{ a classical P.D.O. symbol of order 0.}$$

Although the logarithm of an admissible invertible elliptic classical pseudo-differential operator with spectral cut L_θ is not itself a classical pseudo-differential operator, for two operators $A \in Ell_{ord>0}^{*adm}(M,E)$, $B \in Ell_{ord>0}^{*adm}(M,E)$ admitting spectral cuts L_θ and L_ϕ:

$$\frac{log_\theta A}{ordA} - \frac{log_\phi B}{ordB} \in CL^0(M,E) \tag{3.13}$$

where $CL^0(M,E)$ denotes the algebra of zero order classical pseudo-differential operators acting on sections of E. As a consequence, given $Q \in Ell^{*+}(M,E)$ with strictly positive order, we have:

$$log_\theta(A) = \frac{ordA}{ordQ} logQ + \text{ a classical P.D.O. of order 0.} \tag{3.14}$$

4 Traces on operator algebras

In this section, we consider a particular class of bounded operators, namely trace-class operators which will play an important role in the sequel. They do not form a closed ideal for the topology of bounded operators so that there is no reason for the trace of a family of trace-class operators to converge when the operators converge in the operator norm. However, when restricting ourselves to another class of operators, classical pseudo-differential operators, we can circumvent this problem extracting a finite part of the otherwise diverging limit of traces of certain families of pseudo-differential operators.

Let H be a separable Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{B}(H)$ denote the $*$-algebra of bounded linear operators on H. We briefly recall (without proofs) some basic facts about trace-class and Hilbert-Schmidt operators. We refer the reader to [RS].

4.1 Some remarkable ideals of the algebra of bounded operators

• **Compact operators:** An operator $A \in \mathcal{B}(H)$ is called *compact* if it sends the unit ball (and hence any ball) of H into a set with compact closure in H. Alternatively, A is compact if given any bounded sequence (u_n) in H, one can extract a convergent subsequence $(Au_{\phi(n)})$ of its image (Au_n).

Finite rank operators, i.e operators with finite dimensional range are compact operators. In fact they form a dense subset of the closed subset of compact operators for the operator norm convergence in $\mathcal{B}(H)$. We shall denote by $\mathcal{K}(H)$ the set of compact operators which form a two-sided $*$-ideal in the algebra $\mathcal{B}(H)$, i.e the sum of two compact operators is compact, the adjoint of a compact operator is compact and the product of a compact operator with a bounded operator is compact.

Smoothing operators defined in the previous section provide examples of compact operators on $H = L^2(M, E)$ the space of L^2-sections of a hermitian vector bundle E based on a closed Riemannian manifold M. Here the L^2-hermitian product is built using the Riemannian volume form on the base manifold and the hermitian product on the fibre:

$$\langle \sigma, \rho \rangle := \int_M \langle \sigma(x), \rho(x) \rangle_x dvol(x) \tag{4.0}$$

where $\langle \cdot, \cdot \rangle_x$ is the hermitian product on the fibre above x, $dvol(x)$ the volume measure on M at point x and ρ, σ are any two smooth sections of E.

Let us recall that the non-zero spectrum of a compact operator coincides wth the set of its non-zero eigenvalues and that when the operator is self-adjoint, its spectrum is either reduced to zero, or finite, or a real sequence converging to 0 (see e.g [G]).

•Trace-class operators

A compact linear operator A on H is called *trace-class* if there is a complete orthonormal system (C.O.N.S) $(e_n, n \in I\!N)$ of H such that $\sum_{n=1}^{\infty}\langle |A|e_n, e_n\rangle < \infty$ where $|A| := \sqrt{A^*A}$ (the square root is defined using the spectral representation of A^*A,see e.g. [RS]). If this condition is fulfilled for one C.O.N.S. it is also fulfilled for any other C.O.N.S. and the expression:

$$tr(A) := \sum_{n=1}^{\infty}\langle Ae_n, e_n\rangle \tag{4.1}$$

is well-defined and independent of the choice of C.O.N.S $(e_n, n \in I\!N)$. It is called the *trace* of A. The set of trace-class operators on H build a two sided ideal $\mathcal{I}_1(H)$ in $\mathcal{B}(H)$ i.e.

$$\mathcal{I}_1(H) + \mathcal{I}_1(H) \subset \mathcal{I}_1(H), \quad \mathcal{I}_1(H)\mathcal{B}(H) \subset \mathcal{I}_1(H), \quad \mathcal{B}(H)\mathcal{I}_1(H) \subset \mathcal{I}_1(H).$$

Moreover it is stable under the $*$-operation and $tr(A^*) = \overline{tr(A)}$.

Exercise: Check that $(\Delta + 1)^{-z}$ is trace-class in $L^2(S^1, I\!R)$ for, $Re(z) > 1$ where $\Delta := -\frac{\partial^2}{\partial t^2}$.
Hint: Use a Fourier expansion to investigate the behaviour of the eigenvalues $(\lambda_n, n \in I\!N)$ of $(\Delta + 1)^{-1}$.
In a similar way, one can check that $e^{-t\Delta}$ is also trace-class for $t > 0$. It follows immediately that $e^{-t\Delta}$ lies in all Schatten classes, i.e in every $\mathcal{I}_p(H) := \{A \in \mathcal{K}(H), tr(|A|^p) < \infty\}, p \in I\!N - \{0\}$ where $|A|^p$ is defined via the spectral representation of A^*A.

•Hilbert-Schmidt operators: A compact operator is called *Hilbert-Schmidt* whenever there is a C.O.N.S $(e_n, n \in I\!N)$ such that $\sum_{n=1}^{\infty}\langle Ae_n, Ae_n\rangle < \infty$. In other words a bounded operator A is Hilbert-Schmidt whenever A^*A is trace-class. The *Hilbert-Schmidt norm* of such an operator is defined by

$$\|A\|_{HS}^2 := tr(A^*A). \tag{4.2}$$

Hilbert-Schmidt operators on H also build a two-sided ideal $\mathcal{I}_2(H)$ of $\mathcal{B}(H)$. The adjoint of a Hilbert-Schmidt operator is also Hilbert-Schmidt so that like $\mathcal{I}_1(H)$, the set $\mathcal{I}_2(H)$ is a $*$-ideal in $\mathcal{B}(H)$. For $A \in \mathcal{I}_2(H)$ we have $\|A\|_{HS} = \|A^*\|_{HS}$. The ideal of trace-class operators is contained in the ideal of Hilbert-Schmidt operators $\mathcal{I}_1(H) \subset \mathcal{I}_2(H)$.

The product of two Hilbert-Schmidt operators is a trace-class operator **(exercise)** and we have:

$$tr(AB) = tr(BA) \quad \forall A, B \in \mathcal{I}_2(H). \tag{4.3}$$

Furthermore the Hilbert-Schmidt norm derives from a hermitian product on $\mathcal{I}_2(H)$:

$$\langle A, B\rangle := tr(B^* A) \quad \forall A, B \in \mathcal{I}_2(H). \tag{4.4}$$

It will be useful for our purposes to keep in mind that for any $Q \in Ell^{*+}_{ord>0}(M, E)$ the smoothing operator e^{-tQ} lies in $\mathcal{I}_2(H)$.

Exercise: Let $Q \in Ell^{+*}_{ord>0}(M, E)$ of order q.

(i) Show that the operator Q^{-z} is Hilbert-Schmidt for $Re(z) > \frac{dim M}{2q}$.

Hint: Use the asymptotic behaviour $\lambda_n \sim C n^{\frac{q}{dim M}}$ of the eigenvalues $(\lambda_n, n \in I\!N)$ of Q (which has purely discrete spectrum, see e.g. [G]).

(ii) Show that for any $A \in CL(M, E)$ and any $t > 0$ the operator Ae^{-tQ} is trace-class (it is in fact infinitely smoothing) and that $tr(Ae^{-tQ}) = tr(e^{-tQ}A) = tr(e^{-\frac{t}{2}Q}Ae^{-\frac{t}{2}Q})$.

•**HS-pseudo-traces:** Given $K \in \mathcal{I}_2(H)$, from the properties of Hilbert-Schmidt operators we recalled above, it follows that for any $A \in \mathcal{B}(H)$ the operator K^*AK lies in $\mathcal{I}_1(H)$ **(exercise)** and hence that:

$$\begin{aligned} \mathcal{B}(H) &\to \mathbb{C} \\ A \quad &\to tr^K(A) := tr(K^*AK) \end{aligned} \tag{4.5}$$

is a well-defined linear operator which we will refer to as a *HS-pseudo-trace* ("HS" standing for Hilbert-Schmidt). Moreover from the above properties it also follows that **(exercise)**:

$$tr^K(A) = tr(AKK^*) = tr(KK^*A). \tag{4.6}$$

In general tr^K does not define a trace, namely $tr^K(AB) \neq tr^K(BA)$ and we have **(exercise)**:

$$tr^K([A, B]) = tr([K^*K, A]B) = tr([B, K^*K]A). \tag{4.7}$$

If H is finite dimensional, we can choose $K = Id$ in which case this obstruction vanishes and we are back with the usual trace.

• **Fredholm operators:** A Fredholm operator $A : H \to H_1$ where H_1 is another separable Hilbert space, is a bounded operator $A \in \mathcal{B}(H, H_1)$ such that there exist operators $B \in \mathcal{B}(H, H_1)$ and $C \in \mathcal{B}(H_1, H)$ with the property that both $BA - I$ and $AC - I$ be compact. In other words, A is invertible up to a compact operator. Any operator of the type $A = I + K$ where K is compact, is a Fredholm operator. A Fredholm operator can be characterized by the fact that it is bounded, both the kernel $KerA$ of A and cokernel $KerA^*$ of A^* are finite dimensional and both the the range $R(A)$ of A and the range $R(A^*)$ of A^* are closed.

Let us assume for a moment that $H \subset H_1$ and that the inclusion is compact. If a Fredholm operator $A : H \to H_1$ is self-adjoint, then $A : H \cap Ker(A)^\perp \to H_1 \cap Ker(A)^\perp$ is invertible and one can define its inverse $A^{-1} : H_1 \to H$ extending it by zero on the kernel. Then $A^{-1} : H_1 \to H_1$ is compact as product of a bounded and a compact operator given by the inclusion $H \subset H_1$. It therefore has purely discrete spectrum and hence so has its inverse A. Thus a self-adjoint Fredholm operator $A : H \to H_1$ with $H \subset H_1$ has purely discrete spectrum when the inclusion is compact.

Let us now take $H = H^s(M, E)$, $s > 0$, $H_1 = L^2(M, E)$ where $\pi : E \to M$ is a finite rank vector bundle based on a closed Riemannian manifold as before. A self-adjoint elliptic operator $A \in Ell(M, E)$ of order s induces a bounded operator $A : H^s(M, E) \to L^2(M, E)$. Being "invertible up to a smoothing operator" (see above discussion) and a smoothing operator being compact, it is Fredholm.

The manifold M being compact, by Sobolev embedding theorems, we know that the inclusion $H^s(M, E) \to L^2(M, E)$ is compact for $s > \frac{dimM}{2}$. It follows that the Fredholm operator $A : H^s(M, E) \to L^2(M, E)$ has purely discrete spectrum. Since its inverse (extended to zero outside the kernel) is compact and has spectrum tending to zero, if A is positive, its eigenvalues tend to $+\infty$. A more detailed investigation shows that the asymptotic behaviour of a partial differential elliptic operator of order s is given by $\lambda_n \sim C n^{\frac{s}{dimM}}$ [G].

4.2 Renormalization procedures

A *renormalization procedure* (R.P. for short) on a Hilbert space H is a one parameter family $\{K_\lambda, \lambda \in \Lambda\}$ where Λ is an open subset of $\mathbb{R}^+ := \{\lambda \in \mathbb{R}, \lambda > 0\}$ or of $\mathbb{C}^+ := \{z \in \mathbb{C}, Re(z) > 0\}$ such that $0 \in \bar{\Lambda}$ with the following properties:

(i) $K_\lambda \in \mathcal{B}(H)\, \forall \lambda \in \Lambda$ and $\lim_{\lambda \to 0} K_\lambda u = u \quad \forall u \in H$

(ii) If $\Lambda = \mathbb{R}^+$ then $K_\lambda \in \mathcal{I}_2(H)$

If $\Lambda = \mathbb{C}^+$ then $\exists \lambda_0$ s.t. $K_\lambda \in \mathcal{I}_2(H)$ if $Re(\lambda) > \lambda_0$.

In general

$$tr^\lambda(A) := tr(K_\lambda^* A K_\lambda) \tag{4.8}$$

does not converge when $\lambda \to 0$ and we shall need to *renormalize* the limit taking -whenever possible- the finite part of this divergent limit. Because of the obstruction (4.7) arising for fixed λ, one expects to see an obstruction of similar type in the renormalized limit, an obstruction we will be investigating in section 5.

•**Heat-operator renormalization procedures:** The spaces we shall be considering are spaces of sections of some hermitian finite rank vector bundle $\pi : E \to M$ on a closed Riemannian manifold. Typically, $H = L^2(M, E)$ where the L^2-scalar product is defined by (4.0). To simplify, let us first consider the case when $E = M \times \mathbb{C}^d$ is a trivial vector bundle and let us consider the operator defined componentwise by the Laplace Beltrami operator Δ on M.

For $\lambda \in \Lambda := \mathbb{R}^+$, the operators $K_\lambda := e^{-\frac{\lambda}{2}\Delta}, \lambda \in \Lambda$ are infinitely smoothing and hence Hilbert-Schmidt. It is easy to check that it defines an R.P. on H.

This generalizes to a general vector bundle E based on M replacing Δ by any operator $Q \in Ell^{*+}_{ord>0}(M, E)$ (see section 3) acting on smooth sections of E and considering the family

$$K_\lambda^Q := e^{\frac{-\lambda}{2} Q}, \lambda \in \Lambda \tag{4.9a}$$

which also defines an R.P. on $L^2(M, E)$.

• **ζ-function renormalization procedure:** Let us come back to the trivial case $E = M \times \mathbb{C}^d$ and let us consider once more the operator defined componentwise by the Laplace Beltrami operator Δ on M. Since the spectrum of Δ is purely discrete and since its eigenvalues behave asymptotically like $Cn^{\frac{2}{dim M}}$ (see [G]) where $C > 0$ (2 corresponds here to the order of the Laplace operator), provided $Re(z) > \frac{dim M}{2}$, $\Delta^{-\frac{z}{2}}$ is Hilbert-Schmidt since the trace $tr(\Delta^{-\frac{z}{2}}\Delta^{-\frac{z}{2}}) = tr(\Delta^{-z})$ is a series with positive general term behaving asymptotically like $n^{\frac{-2z}{dim M}}$.

On the grounds of these results, we can build an R.P. with the help of the Laplace operator. Recall that the kernel of an elliptic operator on a closed manifold is finite dimensional as a result of the Fredholm property of the operator (see e.g.[G]) thus we can consider the orthogonal projection P_Δ onto the kernel $Ker(\Delta)$ of Δ. Let us take $\Lambda = \mathbb{C}^+$, then for any $\lambda \in \Lambda$, it is easy to check that on the grounds of the above remarks, the operators $\{(\Delta + P_\Delta)^{-\frac{\lambda}{2}}, \lambda \in \Lambda\}$ define an R.P. on H.

This generalizes to a vector bundle $\pi : E \to M$ replacing Δ by any element of $Ell^{*adm}_{ord>0}(M,E)$ and considering the family

$$K_\lambda^Q := Q^{-\frac{\lambda}{2}}, \lambda \in \Lambda \tag{4.9.b}$$

(such complex powers were defined in section 3) which also defines an R.P. on $L^2(M,E)$.

• **Renormalizable families of traces:** Let $\pi : E \to M$ be a finite rank vector bundle on a closed Riemannian manifold M of dimension N. Given an element $A \in CL(M,E)$ or order a and $Q \in Ell^{+*}(M,E)$ of order $q > 0$ then the map $t \to tr(Ae^{-tQ})$ defines a renormalizable functions ([GS], [Le]) and

$$tr(Ae^{-tQ}) \sim \sum_{j=0}^\infty \alpha_j t^{\frac{j-a-N}{q}} + \sum_{j=0, \frac{j-a-N}{q} \in \mathbb{Z}}^\infty \beta_j t^{\frac{j-a-N}{q}} \log t + \sum_{j=0}^\infty \gamma_j t^j \tag{4.10}$$

for some complex coefficicents $\alpha_j, \beta_j, \gamma_j, j \in \mathbb{N}$. When $Q \in Ell^{*+}_{ord>0}(M,E)$, from the fact that $\lambda^{-z} = M(t \to e^{-t\lambda})(z)$ it follows $tr(AQ^{-z}) = M(f_{A,Q})(z)$ with $f_{A,Q}(t) := tr(e^{-tQ})$ and hence, using (2.11) that

$$M(f_{A,Q})(z) + \beta_0 z^{-1} = \alpha_0 + \gamma_0 - \gamma\beta_0 + O(z). \tag{4.11}$$

4.3 Traces on algebras of pseudo-differential operators

Let as before $A \in CL(M,E)$, $Q \in Ell^{*adm}_{ord>0}(M,E)$.

It follows from the discussion in section 2 that the map $z \to M(f_{A,Q})(z)$ has at most a simple pole at zero and with the notations of (2.12) we define a linear map:

$$tr^{Q,\mu} : CL(M,E) \to C^\infty(M,\mathbb{C})$$

$$A \mapsto tr^{Q,\mu}(A) := LIM^\mu_{z\to 0} tr(AQ^{-z}). \tag{4.12}$$

and the *Wodzicki residue*

$$res(A) := ord(A) Res_{z=0} tr(AQ^{-z}) \tag{4.13}$$

where $Res_{z=z_0}$ stands for the complex residue at the point z_0. When $\mu = \gamma$ the Euler constant, then we shall simply write $tr^Q(A)$ and refer to this expression as the *Q-weighted trace of A*. We have:

$$\begin{aligned} tr^Q(A) &= LIM_{z\to 0}\left(tr(AQ^{-z})\right) \\ &:= \lim_{z\to 0}\left(tr(AQ^{-z}) - \frac{1}{z} res_{z=0} tr(AQ^{-z})\right) \\ &= \lim_{z\to 0}\left(tr(AQ^{-z}) - \frac{1}{z\cdot ordQ} res(A)\right). \end{aligned}$$

In general the weighted trace depends on the spectral cut chosen to define the complex power Q^{-z}; if $Q \in Ell^{*+}_{ord>0}(M,E)$ we take $L_\pi := \{re^{i\pi}, r \geq 0\}$ as common spectral cut. Whenever the order a of A is such that $a < -dim M$ then A is trace-class and $tr^Q(A) = tr(A)$ coincides with the usual trace of A so that the Q dependence disapears alltogether.

As we shall see shortly, tr^Q does not define a trace on the algebra of classical pseudo-differential operators. In fact the Wodzicki residue is (up to a multiplicative constant) the only trace on the algebra of classical P.D.O.s [W], see also [K], [Le].

• **The Kontsevich-Vishik trace [KV], [Le]:** The linear functional tr^Q restricts to the *canonical trace* introduced by Kontsevich and Vishik trace [KV] which we denote by TR on classical pseudo-differential operators with non-integer order.

If the order a of A is not an integer then it follows from the results of section 2.2 that zero is not a pole of $M(f_{A,Q})$ so that we can take the (ordinary) limit $tr^Q(A) := tr(AQ^{-z})_{z=0}$. The linear map tr^Q has the tracial property on non-integer order operators:

Proposition [Le]: *Given any $Q \in Ell^{*+}_{ord>0}(M,E)$*

(i) $TR(A) := tr^Q(A)$ is independent of the choice of Q.

(ii) $TR([A,B]) = 0$ for $A, B \in CL(M,E)$ of orders a and b such that $a+b$ is not an integer.

Idea of proof. It is based on the following formula:

$$e^{-tQ}A = \sum_{j=0}^{N-1} \frac{(-t)^j}{j!} \nabla_Q^j A e^{-tQ} + R_N(A,Q,t)$$

where for N large enough $R_N(t) = O(t)$ and where $\nabla_Q^0 A = A, \nabla_Q^{j+1} A := [Q, \nabla_Q^j A]$.

(i) Given a one parameter family $\{Q_u, u \in I\!R^+\} \in Ell^{*,adm}_{ord>0}(M,E)$, and applying the above formula to $A = \frac{d}{du}Q_u$, the Duhamel formula (see e.g [BGV]) yields:

$$\begin{aligned}\frac{d}{du}tr(Ae^{-tQ_u}) &= -\int_0^t tr\left[Ae^{-(t-t_1)Q_u}\left(-\frac{d}{du}Q_u\right)e^{-t_1Q_u}dt_1\right] \\ &= \sum_{j=0}^{N-1} \frac{(-1)^{j+1}}{(j+1)!} tr(A\nabla_Q^j \frac{d}{du}Q_u e^{-tQ_u}) + O(t).\end{aligned}$$

As can easily be seen from definition (4.12), we can assume without restriction that Q has integer order since $tr^Q(A) = tr^{Q^s}(A)$ for any $s > 0$. Thus A having non-integer order, so has $A\nabla_Q^j \frac{dQ_u}{du}$ non-integer order and hence the asymptotic expansion of $tr(A\nabla_Q^j \frac{dQ_u}{du} e^{-tQ_u})dt$ around $t = 0$ has neither terms of type t^{-j-1} nor terms of type $t^{-j-1}\log t$. Thus the constant term as well as the logarithmic term vanish in the asymptotic expansion of $\frac{d}{du}(tr(Ae^{-tQ_u})$ around $t = 0$ from which it follows that $\frac{d}{du}(tr(AQ^{-z})) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \frac{d}{dt} tr(Ae^{-tQ})$ has no pole and has limit zero when $z \to 0$.

(ii) Let A, B be two P.D.O.s then

$$\begin{aligned} tr([A,B]e^{-tQ}) &= tr(ABe^{-tQ}) - tr(Ae^{-tQ}B) \\ &\sim \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} t^j tr(A(\nabla_Q^j B)e^{-tQ}) \end{aligned}$$

and the rest of the proof goes as above.

5 Algebraic obstructions

The weighted trace tr^Q defined above is not a trace and this fact leads to obstructions. In order to investigate these obstructions, it is useful to establish some properties of the Kontsevich and Vishik trace defined prviously on classical pseudo-differential operators of non-integer order.

5.1 A fundamental property

Kontsevich and Vishik introduced a functional $A \to TR(A)$ on classical pseudo-differential operators with non-integer order for which they proved a property which plays a fundamental part for our needs. Let us first describe the general lines of the construction of TR in a heuristic way. For a (classical) pseudo-differential operator $A \in CL(M, E)$ (see notations of section 3) locally given by:

$$Au(x) := \int_{\mathbb{R}^n} a(x,\xi)\hat{u}(\xi)d\xi \tag{5.1}$$

where $a(x,\xi)$ is the (locally defined) total symbol, we would like to define

$$"TR(A) := \int_M \int_{\mathbb{R}^n} tr_x a(x,\xi) d\xi d\mathrm{vol}(x)" \tag{5.2}$$

- here tr_x denotes the trace on the fibre E_x above x of the vector bundle E - which in general does not make sense since $a(x,\xi)$ typically has components of degree $\geq -dimM$. However, it does make sense for an operator of order $< -dimM$ and yields the ordinary trace. We should therefore find a way of picking up only the finite term in such an expression. We shall come across similar problems to the ones described in section 2, namely that of defining a finite part invariant under a change of variable, which means here defining a covariant finite part.

To go from the approach we are about to describe back to the zeta-function approach to traces described in section 4 (see e.g. (4.12)), one way is to find an asymptotic expansion of $tr(A(Q-\lambda I)^{-1})$ as $|\lambda| \to \infty$ and then using a contour integral from which one deduces the asymptotics of $tr(Ae^{-tQ})$ when $t \to 0$ from which follow in turn the asymptotics of $tr(AQ^{-z})$ around $z = 0$ [GS].

Let us first recall some basic facts about homogeneous distributions.

•**Homogeneous distributions ([H] chap. 3.2):** An element u of $L^1_{loc}(\mathbb{R}^n - \{0\})$ - which in particular is a distribution on $\mathbb{R}^n - \{0\}$ - is called *homogeneous* of degree z when

$$u(tx) = t^z u(x) \quad \forall t > 0, x \neq 0. \tag{5.3}$$

If $Re(z) < -n$ then for a smooth function Ψ on $\mathbb{R}^n$ which is zero in $B(0, \frac{1}{4})$ and equal to 1 on $\mathbb{R}^n - B(0, \frac{1}{2})$ we have $\Psi u \in L^1(\mathbb{R}^n)$. Indeed writing $x = r\xi$ with $\|\xi\| = 1$ we have $|\Psi(x)u(x)| = r^{Re(z)}|\Psi(\xi)u(\xi)|$ so that for any $R > 0$ the integral :

$$\int_{B(0,R)} \Psi(x)u(x)dx = \int_0^R \int_{\|\xi\|=1} \Psi(r\xi)u(\xi)r^{n-1+Re(z)}drd\xi$$

converges. However problems might arise for negative integer values of z because of divergences coming into play.

Take $z = -n$ in (5.3). Then in a similar way to what was done in section 2, we have:

$$\begin{aligned}\int_{B(0,R)} \Psi(x)u(x)dx &= \int_{B(0,1)} \Psi(x)u(x)dx + \int_1^R \int_{\|\xi\|=1} u(\xi)r^{-1}drd\xi \\ &= \int_{B(0,1)} \Psi(x)u(x)dx + \log R \int_{\|\xi\|=1} u(\xi)d\xi\end{aligned}$$

which diverges when $R \to \infty$. Following methods presented in section 2, we can define the finite part

$$\begin{aligned}f.p.\Big(\int_{\mathbb{R}^n} \Psi(x)u(x)dx\Big) &:= lim_{R\to+\infty} \left(\int_{B(0,R)} \Psi(x)u(x) - \log R \int_{\|\xi\|=1} u(\xi)d\xi\right) \\ &= \int_{B(0,1)} \Psi(x)u(x)dx\end{aligned}$$

but this is not invariant under a change of variable anymore (we refer the reader to the discussion in section 2).

Take $z = -n + k, k > 0$ for some $k \in \mathbb{N} - \{0\}$. A similar computation yields:

$$\begin{aligned}\int_{B(0,R)} \Psi(x)u(x)dx &= \int_{B(0,1)} \Psi(x)u(x)dx + \int_1^R \int_{\|\xi\|=1} u(\xi)r^{-1+k}drd\xi \\ &= \int_{B(0,1)} \Psi(x)u(x)dx + \frac{R^k}{k} \int_{\|\xi\|=1} u(\xi)d\xi - \frac{1}{k} \int_{\|\xi\|=1} u(\xi)d\xi.\end{aligned}$$

Here again we can define the finite part:

$$f.p.(\textstyle\int_{\mathbb{R}^n} \Psi(x)u(x)dx) := lim_{R\to+\infty} \left(\displaystyle\int_{B(0,R)} \Psi(x)u(x) - \frac{1}{k}R^k \int_{\|\xi\|=1} u(\xi)d\xi\right)$$

$$= \int_{B(0,1)} \Psi(x)u(x)dx - \frac{1}{k}\int_{\|\xi\|=1} u(\xi)d\xi.$$

•**The Kontsevich-Vishik canonical density:** Let us consider a classical P.D.O. A of order α with symbol $a = \sum_{j=0}^{N} \Psi a_{\alpha-j} + a_{(N)}$ where $a_{\alpha-j} \in S^{\alpha-j}(\mathbb{R}^n)$, $a_{(N)} \in S^{\alpha-N-1}(\mathbb{R}^n)$ and where as before Ψ is a smooth function on $\mathbb{R}^n$ which is zero in $B(0,\frac{1}{4})$ and equal to 1 on $\mathbb{R}^n - B(0,\frac{1}{2})$. Since $a_{(N)}(x,\xi) = O(|\xi|^{\alpha-N})$ we have $\int_{B(0,R)} a_{(N)}(\xi)d\xi = \int_{\mathbb{R}^n} a_{(N)}(x,\xi)d\xi + O(R^{\alpha+n-N})$. On the other hand, splitting the integrals

$$\int_{B(0,R)} \Psi a_{\alpha-j}(x,\xi)d\xi = \int_{B(0,1)} \Psi a_{\alpha-j}(x,\xi)d\xi + \int_{B(0,R)-B(0,1)} a_{\alpha-j}(x,\xi)d\xi,$$

we use the fact that $a_{\alpha-j}$ is homogeneous of order $\alpha - j$ to express the last integral. If α is an integer then there is an integer j_0 such that $\alpha - j + n = 0$ and we have for $N > j_0$:

$$\sum_{j=0}^{N} \int_{B(0,R)/B(0,1)} \Psi a_{\alpha-j}(x,\xi)d\xi = \sum_{j=0}^{N} \int_1^R \int_{|\xi|=1} r^{\alpha-j+n-1} a_{\alpha-j}(x,\xi)d\xi dr$$

$$\sim_{R\to\infty} \sum_{j=0,n+\alpha-j\neq 0}^{N} \frac{1}{\alpha+n-j} R^{\alpha-j+n} \int_{|\xi|=1} a_{\alpha-j}(x,\xi)d\xi +$$

$$+ \log R \int_{|\xi|=1} a_{-n}(x,\xi)d\xi + \text{ constant term.}$$

Finally this yields the existence of an asymptotic expansion in $R \to +\infty$ and of a constant $C(a(x,\cdot))$ dependent of R such that

$$\int_{B(0,R)} a(x,\xi)d\xi \sim_{R\to\infty} \sum_{j=0,n+\alpha-j\neq 0}^{N} \frac{1}{\alpha+n-j} R^{\alpha-j+n} \int_{|\xi|=1} a_{\alpha-j}(x,\xi)d\xi +$$
$$+ \log R \int_{|\xi|=1} a_{-n}(x,\xi)d\xi + C(a(x,\cdot)). \tag{5.4}$$

Because of the logarithmic term in R, one does not expects the finite part

$$f.p.(\int_{\mathbb{R}^n} a(x,\xi)d\xi) = LIM_{R\to\infty} \int_{B(0,R)} a(x,\xi)d\xi \tag{5.5}$$

-defined taking the limit when $R \to \infty$ of the r.h.s of (5.4) after substracting out divergences- to be invariant under a change of variable of $\mathbb{R}^n$. However if the order α is not an integer, there is no logarithmic divergence and $f.p.\left(\int_{\mathbb{R}^n} a(x,\xi)d\xi\right)$ is independent of the local representation $a(x,\xi)$ of A:

Proposition [KV] (see also [Le] in Prop. 5.2): *Provided $A \in CL(M,E)$ has non-integer order:*

$$TR(A) := \int_M tr_{E_x}\left(f.p.(\int_{\mathbb{R}^n} a(x,\xi)d\xi)\right) dvol(x) \tag{5.6}$$

and coincides with the previous definition of section 4.3.

• **The Wodzicki residue:** When A has integer order, there is a pole at zero arising in $tr(AQ^{-z})$ which by the Mellin transform corresponded to a logarithmic divergence in the asymptotic expansion of $tr(Ae^{-\varepsilon Q})$. This in turn gives rise to a logarithmic divergence in $\int_{B(0,R)} a(x,\xi)d\xi$ when $R \to \infty$ where a was the symbol of A. We have:

Proposition [W] (see also [Le]): *For any $A \in CL(M,E)$, $Q \in Ell^{*+}_{ord>0}(M,E)$,*

$$\begin{aligned} res(A) &:= ord(A) Res_{z=0} tr(AQ^{-z}) \\ &= -ord(A)\, \big(\textit{coefficient of } \log t \\ &\quad \textit{in the asymptotic expansion of } tr(Ae^{-tQ}) \ \textit{ as } \ t \to 0\big) \\ &= \int_M \left(\int_{\|\xi\|=1} tr_x(a_{-n}(x,\xi))d\xi\right) dvol(x) \\ &= \textit{coefficient of } \log R \ \textit{ as } \ R \to \infty \\ &\quad \textit{in the asymptotic expansion of } \int_{B(1,R)} tr_x a(x,\xi) d\xi \end{aligned} \tag{5.7}$$

(where tr_x denotes as before the trace on the fibre E_x above x) defines a trace on the algebra $CL(M,E)$ i.e. we have

$$res(AB) = res(BA) \quad \forall A, B \in CL(M,E).$$

It follows from the definition that the Wodzicki trace vanishes on trace-class operators. This is the reason why we shall not be using it as a tool to extend the finite dimensional trace, and why we consider weighted traces. However it plays an important role since it is - up to a multiplicative constant factor - the only trace on the algebra of classical P.D.O.s [W]. The notion of Wodzicki residue also generalizes to a class of non classical operators (which includes the classical ones) as described in [Le].

The third indentity in (5.7) also shows (**exercise**) that the residue vanishes when the underlying manifold is odd dimensional and the operator in the odd class (a concept introduced in section 3).

• **Smooth and holomorphic families of P.D.O.s.**

The algebra $CL(M,E)$ inherits a Fréchet structure equipped with a topology induced by the natural Fréchet topology on symbols given by the following family of semi-norms indexed by multiindices $\gamma \in \mathbb{N}^d$, $\delta \in \mathbb{N}^d$ and $i \in \{1,\cdots,N\}$, $k \in \mathbb{N}$:

$$\|\sigma\|_{\gamma,\delta,k} := max_i\Big(sup_{y\in\overline{V_i},\xi\in\mathbb{R}^d,\|\xi\|=1}\|D_y^\gamma D_\xi^\delta \sigma_k(y,\xi)\|\Big)$$

where σ_k is the homogeneous component of order k, $\{V_i\}_{i=1,\cdots,N}$ is a finite open cover (with $\bar{V}_i$ compact) of M associated to a partition of unity $\{V_i,\xi_i\}_{i=1,\cdots,N}$ on M subordinated to some finite open covering $\{U_i\}_{i=1,\cdots,N}$.

This topology combined with natural semi-norms on smooth kernels of compact operators, induces a Fréchet structure on the space of classical pseudo-differential operators $CL(M,E)$ via the identification of operators with their symbol up to a smoothing operator (see e.g [KV1] section 3 and [D] for further details).

This topology leads to the following definitions. Given a domain $D \subset \mathbb{C}$ (resp. an open subset $U \subset \mathbb{R}^n$), we shall call a family of PDOs $A(z) \in CL(M,E)$ of order $\alpha(z)$ with $z \in D$ *holomorphic* (resp. $A(x)$ of order $\alpha(x)$ with $x \in U$ smooth) if:

(i) $z \to \alpha(z) \in \mathbb{C}$ is holomorphic (resp. $x \to \alpha(x)$ is smooth)

(ii) the homogeneous components $z \to a_{\alpha(z)-j}$ (resp. $x \to a_{\alpha(x)-j}$) of the symbol of $A(z)$ (resp. $A(x)$) are holomorphic (resp. smooth) in z (resp. in x) as functions on the cotangent unit sphere S^*M restricted to some local chart,

(iii) the kernel $K_{A(z)}(u,v)$ (resp. $K_{A(x)}(u,v)$) of the family $A(z)$ (resp. $A(x)$) is holomorphic (resp. smooth) for u,v in disjoint local charts,

(iv) there is a local chart V such that on $V\times V$ and for all $N \in \mathbb{N}$ there exists an integer M_N such that the map

$$(u,v) \to \ K_{A(z)}(u,v) - \sum_{j=0}^{M_N}\int \Psi(\|\xi\|)\|\xi\|^{\alpha(z)-j}a_{\alpha(z)-j}(z)\left(x,\frac{\xi}{\|\xi\|}\right)exp(i\langle u-v,\xi\rangle)d\xi$$

(resp. the same expression with z replaced by x) is a C^{M_N} smooth (local) kernel on this neighborhood of the diagonal which is holomorphic in z (smooth in x) together with its partial derivatives in u, v up to order M_N. Here Ψ is a cut-off function already introduced previously (see below formula (5.3)).

•**Fundamental property [KV] (see also [Le] Prop.5.5):** *Let $A(z) \in CL(M,E)$ be a holomorphic family of operators on a domain $D \subset \mathbb{C}$ such that $\alpha'(z_0) \neq 0$ for any $z_0 \in \mathbb{C}$ such that $\alpha(z_0) \in \{-n, -n+1, \cdots\}$. Then the function $z \to TR(A(z))$ defines a meromorphic function on D with poles in $\alpha^{-1}(\{-n, -n+1, \cdots\})$ of order 1. Moreover:*

$$Res_{z=z_0} TR(A(z)) = -\frac{1}{\alpha'(z_0)} res(A(z_0)) \tag{5.8}$$

when $\alpha(z_0) \in \{-n, -n+1, \cdots\}$.

Proof. We shall only prove the case $\alpha(z) = z$, leaving the general case as an **exercise** for the reader. Let n denote the dimension of M and let $a(z)(x,\xi)$ denote the symbol of $A(z)$. We have:

$$a(z) = \sum_{j=0}^{N} \Psi(\xi) a_{z-j}(z) + g_{(N)}(z)$$

where $g_{(N)}(z)(x,\xi) = O(|\xi|^{z-N})$ as $\|\xi\| \to \infty$. Notice that for $Re(z) < N - n$ the map $\xi \to g_{(N)}(z)(x,\xi)$ lies in $L^1(\mathbb{R}^n)$. As before and with the same notations, for $Re(z)$ negative enough we can write:

$$\int_{\mathbb{R}^n} a_{z-j}(\xi) = lim_{R\to\infty} \left[\sum_{j=0}^{N} \int_{B(0,R)/B(0,1)} \Psi(\xi) a_{z-j}(z)(x,\xi) d\xi + \right.$$

$$\left. \sum_{j=0}^{N} \left(\int_1^R t^{z-j+n-1} dt \right) \cdot \left(\int_{S^{n-1}} a_{z-j}(z)(x,\xi) d\xi \right) + \int_{R^n} g_{(N)}(z)(\xi) d\xi \right]$$

$$= -\sum_{j=0}^{N} \frac{1}{z-j+n} \left(\int_{S^{n-1}} a_{z-j}(z)(x,\xi) d\xi \right) + \sum_{j=0}^{N} \int_{\mathbb{R}^n/B(0,1)} \Psi(\xi) a_{z-j}(z)(x,\xi) d\xi$$

$$+ \int_{R^n} g_{(N)}(z)(\xi) d\xi.$$

Hence the map $z \to \int_{\mathbb{R}^n} \Psi(\xi) a_{z-j}(\xi)$ extends to a meromorphic map on the complex plane with simple poles at $-dimM, -dimM+1, \cdots$ and residue at $k \in \mathbb{Z}$ given by $res_{z=k} tr(A(z)) = -\int_{S^{n-1}} a_{-dimM}(k)(x,\xi) d\xi = -res(A(k))$.

5.2 Obstructions in terms of the Wodzicki residue

We saw at the end of section 4 that the Q-weighted trace tr^Q is tracial on the set of pseudo-differential operators with non-integer order (where it is denoted by TR) and that it is independent of Q. These properties fail to hold on the whole algebra of pseudo-differential operators and we shall express the corresponding obstructions in terms of a Wodzicki residue.

•**The weighted Radul cocycle (see also the lecture by E.Langmann in this volume):** We first need the notion of cocycle in the cohomology of Lie algebras (see e.g. [Mi]). Let L be a Lie algebra, then a *cochain of degree* n (or n-cochain) is an antisymmetric multilinear map: $C : L \times \cdots \times L \to \mathbb{C}$ where the product is taken n times. The set of cochains on L is denoted by $C^n(L)$. The *coboundary operator* operator

$$\begin{aligned} \delta : C^n(L) &\to C^{n+1}(L) \\ c &\to \delta c \end{aligned}$$

defined by

$$(\delta c)(x_1, \cdots, x_{n+1}) = \sum_{i<j} (-1)^{i+j+1} c([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j \cdots, x_{n+1})$$

has the property $\delta^2 = 0$. An n-cochain c is called a *cocycle* if $\delta c = 0$ and cocycles of the form δc are called *coboundaries.* Two n-cocycles c, c' are *equivalent* whenever their difference is a coboundary i.e if there is an $(n-1)$-cochain c'' such that $c - c' = \delta c''$. A two-cocycle c on L gives rise to a central extension of L namely $\tilde{L} := L \oplus C$ with bracket $[\widetilde{(a,\alpha),(b,\beta)}] := ([a,b], c(a,b))$. The fact that c is a cocycle is equivalent to the Jacobi identity for the bracket on $\tilde{L}$.

Proposition-Definition : *Let* $A, B \in CL(M,E)$, $Q \in Ell^{*adm}_{ord>0}(M,E)$. *Then*

$$c_R^Q(A,B) := tr^Q([A,B]) = \lim_{z\to 0} tr([A,B]Q^{-z}) = -\frac{1}{ordQ} res([\log Q, A]B) \tag{5.9}$$

is called the Q*-weighted Radul cocycle.*

This was proved independently in [MN] and [CDMP].

Proof. Let us first note that for z with large enough real part, we have $tr([A,B]Q^{-z}) = tr([Q^{-z}, A]B)$ and let us set $C(z) := [Q^{-z}, A]B$ which is a P.D.O. of order $\alpha(z) = a - z\cdot q + b$, $a = ord(A), b = ord(B), c = ord(C)$. $z \to$

$C(z)$ is a holomorphic family and we have $C(0)=0$. Hence $res_{z=0}TR(C(z)) = -res(C(0)) = 0$ and $z \to TR(C(z))$ is holomorphic at zero. Here TR denotes the canonical trace as in section 4. Let us compute its value in zero. We have:

$$\begin{aligned} tr^Q([A,B]) &= \lim_{z\to 0} TR(C(z)) = Res_{z=0}\frac{TR(C(z))}{z} \\ &= Res_{z=0}\frac{TR([Q^{-z},A]B)}{z} \\ &= Res_{z=0}TR([\frac{Q^{-z}-I}{z},A]B) \\ &= \frac{1}{\alpha'(0)}res([\log Q,A]B) \\ &= -\frac{1}{q}res([\log Q,A]B) \quad \text{by} \quad (5.8) \end{aligned}$$

where we have used the fact that $D(z) := [\frac{Q^{-z}-I}{z},A]B$ is also a holomorphic family at zero of order $\alpha(z) = a+b-qz$ and that $\left(\frac{Q^{-z}-I}{z}\right)_{|_{z=0}} = -\log Q$.

The Q-weighted Radul cocycle c_R^Q carries this name because it generalizes the ordinary Radul cocycle [Ra],[CDFW],[Mi] which we recover for $M=S^1$ and $Q=|D|$, D being the Dirac operator on S^1 introduced at the beginning of section 3. Two different weights Q_1,Q_2 lead to two different weighted Radul cocycles $c_R^{Q_i}, i=1,2$ and we have:

$$c_R^{Q_1}(A,B) - c_R^{Q_2}(A,B) = res\left(\left(\frac{\log Q_1}{ordQ_1} - \frac{\log Q_2}{ordQ_2}\right)[A,B]\right).$$

We recall from the end of section 3 that the difference $P := \frac{\log Q_2}{ordQ_2} - \frac{\log Q_1}{ordQ_1}$ is a classical P.D.O. so that we can write $c_R^{Q_1} - c_R^{Q_2} = \delta res(P\,\cdot)$. Thus two weighted Radul cocycles are equivalent since their difference is a coboundary.

•The Q-dependence of tr^Q:

Proposition [CDMP]:*Let $\{Q_t, t\in I\subset I\!R\}$ be a smooth one parameter family of elements of $Ell^{*adm}_{ord>0}(M,E)$ with constant order q. Let $\{A_t \in CL(M,E), t\in I\subset I\!R\}$ be a smooth one parameter family of elements of $CL(M,E)$. Then for any $t_0\in I$,*

$$\frac{d}{dt}_{|_{t=t_0}} tr^{Q_t}(A_t) = tr^{Q_{t_0}}(\frac{d}{dt}_{|_{t=t_0}}A_t) - \frac{1}{q}res(A_{t_0}\frac{d}{dt}_{|_{t=t_0}}\log Q_t). \qquad (5.10)$$

Proof. We shall only prove

$$\frac{d}{dt}_{|t=t_0} tr^{Q_t}(A_{t_0}) = \lim_{z\to 0} \frac{d}{dt}_{|t=t_0} tr(A_{t_0} Q_t^{-z}) = -\frac{1}{q} res(A_{t_0} \frac{d}{dt}_{|t=t_0} \log Q_t)$$

since (5.10) then easily follows.

Let us first notice that although $\log Q_t$ does not in general belong to $CL(M,E)$, the operators $\frac{d}{dt}_{|t=t_0} \log Q_t = \frac{d}{dt}_{|t=t_0} (\log Q_t - \log Q_{t_0})$ does since it is expressed in terms of a difference of logarithms of same order.

The family $C(z) := A_{t_0} \frac{d}{dt}_{|t=t_0} Q_t^{-z} = A \frac{d}{dt}_{|t=t_0} (Q_t^{-z} - Q_{t_0}^{-z})$ is a holomorphic family in $CL(M,E)$ and hence by the fundamental property (5.8), the map $z \to TR(C(z)) = TR(A_{t_0} \frac{d}{dt}_{|t=t_0} Q_t^{-z}) = \frac{d}{dt}_{|t=t_0} TR(A_{t_0} Q_t^{-z})$ is meromorphic with simple poles. In fact the map $z \to TR(C(z))$ is holomorphic at $z=0$ since using again the fundamental property we can write (here $\alpha(z) = a - zq$):

$$Res_{|z=0} tr(C(z)) = \frac{1}{q} res(C(0)) = 0.$$

Since $C(0) = 0$, the family $z \to z^{-1}C(z)$ is holomorphic at zero and we have:

$$\begin{aligned}
\frac{d}{dt}_{|t=t_0} tr^{Q_t}(A_{t_0}) &= \left[\frac{d}{dt}_{|t=t_0} TR(A_{t_0} Q_t^{-z})\right]_{|z=0} \\
&= Res_{z=0} TR\left(\frac{C(z)}{z}\right) \\
&\quad \text{because} \quad z \to TR(C(z)) \quad \text{has no pole at} \quad z=0 \\
&= \frac{1}{q} res\left[\left(A_{t_0} \frac{d}{dt}_{|t=t_0} \left(\frac{Q_t^{-z} - Q_{t_0}^{-z}}{z}\right)_{z=0}\right)\right] \\
&= -\frac{1}{q} res\left[\left(A_{t_0} \frac{d}{dt}_{|t=t_0} \log Q_t\right)\right].
\end{aligned}$$

Notice that unless Q_t or $\frac{d}{dt}_{|t=t_0} Q_t$ commutes with A_{t_0}, in general, one has

$$res(A_{t_0} \frac{d}{dt}_{|t=t_0} \log Q_t) \neq res(A_{t_0} Q_{t_0}^{-1} \frac{d}{dt}_{|t=t_0} Q_t).$$

This generalizes to (smooth) families of operators $\{A_b \in CL(M,E), b \in B\}$ and $\{Q_b \in Ell^{+*}(M,E), b \in B\}$ parametrized by a manifold B [CDMP], [PR]:

$$dtr^Q(A) = tr^Q(dA) - \frac{1}{ordQ} res(A d\log Q). \tag{5.11}$$

5.3 Further obstructions of algebraic type

Let us briefly mention other obstructions of algebraic type that can occur.

• **Multiplicative anomalies:** For $A \in Ell^{*+}_{ord>0}(M,E)$ we can define the ζ determinant $det_\zeta(A) := exp(-\zeta'_A(0))$ where $\zeta_A(z) := M(t \to tr(e^{-tA}))(z)$ (see (2.9)). In general it does not have the multiplicative property i.e. given another $B \in Ell^{*+}_{ord>0}(M,E)$ we have $det_\zeta(AB) \neq det_\zeta(A)det_\zeta(B)$. This phenomenum is referred to as the *multiplicative anomaly* [KV1,2], [O] which is also investigated in [D] from a different point of view. The anomaly is an algebraic obstruction which can be expressed in terms of a residue of the type $res(P(\frac{\log Q_1}{ordQ_1} - \frac{\log Q_2}{ordQ_2}))$ where $P \in CL(M,E)$ is of order zero (cf. [KV1,2]).

• **Schwinger functional:** Let $D \in Ell^*_{ord>0}(M,E)$ be a self-adjoint operator. We assume it is invertible for simplicity but this construction can be extended to the non-invertible case. Adopting notations which are frequently used in the context of second quantization, let $\varepsilon(D) := D|D|^{-1}$ (or $\varepsilon(D) := (D+P_D)(|D|+P_D)^{-1}$ if D is not invertible where P_D denotes the orthogonal projection onto its kernel) denote the sign of D which defines a classical P.D.O. of order 0. The operator $Q := |D|$ is a positive self-adjoint operator in $Ell^*_{ord>0}(M,E)$ with strictly positive order. We define the *Schwinger functional*:

$$\begin{aligned} &CL(M,E) \to \mathbb{C} \\ &(A,B) \quad \to c_S^D(A,B) := \frac{1}{2} tr^{|D|}(\varepsilon(D)[\varepsilon(D),A][\varepsilon(D),B]). \end{aligned} \tag{5.12}$$

The terminology "Schwinger functional" is motivated by the fact (as we shall see shortly) that it coincides with the usual Schwinger cocycle on some subalgebra of $CL(M,E)$.

A straight forward computation yields:

$$c_S^D(A,B) = tr^{|D|}([A,\varepsilon(D)]B) = tr^{|D|}(A[\varepsilon(D),B]).$$

Proposition [CDMP]: *Given a subalgebra $\mathcal{A}$ of $CL(M,E)$ the following conditions are equivalent (all the cocycles are Lie algebra cocycles):*

c_S^D *is a 2-cocycle on* $\mathcal{A}$

$\Leftrightarrow c_S^D$ *is an antisymmetric bilinear form on* $\mathcal{A}$

$\Leftrightarrow (A,B) \to c_{TR}^D(A,B) := tr^{|D|}([\varepsilon(D)A, B])$ *defines a 2-cocycle on* $\mathcal{A}$

$\Leftrightarrow c_{TR}^D$ *is an antisymmetric bilinear form on* $\mathcal{A}$

$\Leftrightarrow res\,(\varepsilon(D)[A,[\log|D|,B]]) = 0 \quad \forall A,B \in \mathcal{A}.$

Provided one of these conditions is fufilled, the following relation holds:

$$c_{TR}^D - c_S^D = \delta tr_\varepsilon^D$$

where $tr_\varepsilon^D := tr^{|D|}(\varepsilon(D)\cdot)$ *so that the cocycles* c_{TR}^D *and* c_S^D *are cohomologous.*

The fourth equivalence shows that the obstruction to the cocycle property of the various functionals involved arises as a Wodzicki residue $res(\varepsilon(D)[A,[\log|D|,B]])$. Here again the obstruction expressed in terms of a residue is purely infinite dimensional. If $res(\varepsilon(D)\cdot)$ is tracial on $\mathcal{A}$ i.e if $res(\varepsilon(D)[A,B]) = 0 \quad \forall A, B \in \mathcal{A}$, the obstruction vanishes.

The algebra

$$\begin{aligned} CL_{res}^D(M,E) &:= \left\{ A \in CL(M,E), ordA \leq 0, ord[\varepsilon(D),A] < -\frac{dimM}{2} \right\} \\ &= g_{res}^D(L^2(M,E)) \cap CL(M,E), \end{aligned} \tag{5.13}$$

where

$$g_{res}^D(L^2(M,E)) := \left\{ A \in Hom(L^2(M,E)), [\varepsilon(D),A] \quad \text{is Hilbert-Schmidt} \right\}, \tag{5.14}$$

satisfies the equivalent assumptions listed above and the restriction of the Schwinger functional c_S^D to this subalgebra is therefore a cocycle. There it coincides with the usual Schwinger cocycle [CFNW], [Mi] and is cohomologous to the twisted Radul cocycle c_{TR}^D [CDMP].

6 Geometric obstructions

The purpose of this section is to give some geometric applications of the apparatus of weighted traces developped in the previous sections. The notion of trace underlies many constructions in finite dimensions, some of which were briefly described in the first section. Many of them extend to the infinite dimensional context naively replacing the ordinary trace by a weighted trace. However the algebraic obstructions combined with some geometric ones we are about to describe, impel a naive generalization of some of their properties; typically additional Wodzicki residue terms arise.

This prevents us for example from extending the Chern-Weil construction to build caracteristic classes; only in very special cases (we shall briefly comment on the loop group case) do we obtain closed forms naively replacing the trace in the usual invariant ploynomials by weighted traces.

In the "half weighted" graded case (see [PR]), the "half-weight" being a family of Dirac type operators (the square of which yields a weight), one sees from previous works on families of Dirac operators ([BF], [DF], see also [BGV]) that a better approach to invariant polynomials on such infinite rank complex vector bundles might arise from modifying the initial connection using the "half-weight" into a family of Bismut connections [B]. In finite dimensions the choice of the connection is irelevent at the level of characteristic classes, here it is the very choice of the connection which makes all the difference!

Although because of lack of space, we shall not be able to report on these possible extensions, we hope to give the reader an idea of how weighted traces could be used in the infinite dimensional geometric context and thereby also the limits of their possible use. As a result we allow ourselves a "looser" tone in this section than that used in the previous ones and will only brush over notions without giving technical constructions. We refer the reader to [Pa] for further developments and speculations along these lines.

6.1 Weighted vector spaces

We shall call a *weighted space* a couple $(\Gamma(M,E),Q)$ where $\Gamma(M,E)$ denotes a Fréchet or resp. a separable Hilbert space of smooth or resp. Sobolev type sections of some finite rank vector bundle E based on a closed Riemannian manifold M and where $Q \in Ell^{*adm}_{ord>0}(M,E)$ is called a *weight*.

Generalized Laplacians provide weighted structures on spaces of type $\Gamma(M,E)$. By generalized Laplacian acting on sections of E, we understand a second order differential operator with leading symbol $\sigma_L(x,\xi) := \|\xi\|^2$. Given a connection ∇^E on E, the Levi-Civita connection on M induces a connection

$\nabla^{T^*M\otimes E}$ on $T^*M \otimes E$ and the operator given by $\sigma \to -tr(\nabla^{T^*M\otimes E}\nabla^E\sigma)$ is a generalized Laplacian.

Notice that when M is reduced to a point, then the vector bundle E reduces to a vector space V and $(\Gamma(M,E),Q)$ coincides with (V,Q) where now V is a vector space of dimension the rank of the vector bundle and Q a matrix with spectral cut. Hence Q can be viewed as a generalized metric, a metric being given by a positive definite matrix. In fact in our infinite dimensional setting, it is useful to think of Q as a kind of "quantized metric": a positive definite matrix has been replaced by an admissible elliptic operator.

From the data $(\Gamma(M,E),Q)$, Q of order $q>0$, in addition to the L^2-hermitian product $\langle\cdot,\cdot\rangle$ mentioned in (4.0), one can build for any $s\in \mathbb{R}^+$ an H^s hermitian product:

$$\langle\sigma,\rho\rangle_s^Q := \langle Q^{\frac{s}{q}}\sigma, Q^{\frac{s}{q}}\rho\rangle \tag{6.1}$$

and each Sobolev space $H^s(M,E)$ can be seen as the closure of $C^\infty(M,E)$ for this scalar product.

Here and in what follows, if Q is not invertible, which is in fact often the case, we can replace Q in the formulae by $Q+P_Q$ where P_Q is the orthogonal projection onto the (finite dimensional) kernel $KerQ$.

To a weighted space $(C^\infty(M,E),Q)$, we can associate a weighted trace tr^Q on the algebra $CL(M,E)$ of classical pseudo-differential operators acting on smooth sections of E as defined in the previous section.

Exercise: Show that whether one takes this weighted trace using the H^s-scalar product defined in (6.1) or the L^2-scalar product does not make any difference. In the following we shall use the L^2-scalar product.

Remark: When $E = M\times V$ is a trivial vector bundle, V being a finite dimensional vector space with scalar product $\langle\cdot,\cdot\rangle_V$, we can view $CL(M,E)$ as the tensor product $CL(M,\mathbb{C})\otimes Hom(V)$. When the weight Q is *scalar*, meaning that $Q := Q_0\otimes I_V$ for some $Q_0\in Ell^{*adm}_{ord>0}(M,\mathbb{C})$ then for any $A\in CL(M,E)$ we have $tr^Q(A) = tr^{Q_0}(tr_V(A))$. If $tr_V(A)\in CL(M,\mathbb{C})$ is trace class then $tr^Q(A) = tr(tr_V(A))$ does not depend on Q any more.

•**Weighted Laplacians:** Let $(C^\infty(M,E),Q)$ be a weighted space and let $f\in C^2(H^s(M,E))$ for some $s>0$. Then for any $x\in H^s(M,E)$ the Hessian $Hessf(x) := D^2f(x)$ defines a bounded bilinear form on $H^s(M,E)$ which by the Riesz theorem can be interpreted as a bounded operator on $H^s(M,E)$ also denoted by $Hessf(x)$. If $Hessf(x)$ differs form a trace class operator on $H^s(M,E)$ by an element of $CL(M,E)$ we can define the Q-*weighted Laplacian*

$$\Delta^{Q,\mu}f(x) := -tr^{Q,\mu}(Hessf(x)) \quad \forall\mu\in\mathbb{R},\forall x\in H^s(M,E) \tag{6.2}$$

Let us point out at this stage that there are many other ways of defining Laplacians in infinite dimensions. Here we have picked out the finite part of the asymptotic expansion $tr(AQ^{-z})$ when $z \to 0$. But one could have instead picked up the residue term using the Wodzicki residue to build $\Delta^{res} f = -res(Hessf)$. Or we could have taken $-lim_{t\to 0}\frac{tr(Hessf(x)e^{-tQ})}{tr(e^{-tQ})}$ which leads to a *Lévy-type* Laplacian [ABP], similar to Lévy Laplacians arising in white noise analysis.

The latter Laplacians are of purely infinite dimensional nature because they vanish on the trace-class part of $Hessf(x)$. The weighted Laplacian however coincides with the usual one whenever $Hessf(x)$ is tracial, which is the reason why we shall keep to it. In particular, if M reduces to a point, only the weighted Laplacian is non trivial.

Example: Let $(C^\infty(M,E),Q)$ be a weighted Hilbert space. Let for $0 < t < s$, $f : H^s(M,E) \to \mathbb{C}$ be the C^2 function defined by $f(x) = \left(\|x\|_l^Q\right)^2$ for $x \in H^s(M,E)$ where $\|\cdot\|_t^Q$ is the norm associated to $\langle\cdot,\cdot\rangle_t^Q$ (see (6.1)). Then $D^2 f(x)(u,v) = \langle Q^{\frac{t-s}{q}}u, Q^{\frac{t-s}{q}}v\rangle_s^Q$. Hence $Hessf(x) = Q^{\frac{2(t-s)}{q}} \in CL(M,E)$ and we can define $\Delta^{Q,\mu} f = -tr^{Q,\mu}(Q^{\frac{2(t-s)}{q}})$ which coincides with $-tr(Q^{t-s})$ if $t - s < -\frac{q dim M}{2}$.

With the same notations as above, let $f, g \in C^2(H^s(M,E),\mathbb{C})$ such that for any $x \in M$, the operators $Hessf(x)$ and $Hessg(x)$ both differ from a trace-class operator by an element of $CL(M,E)$. Then $tr(Df(x)\otimes Dg(x)) := \sum D_{e_n} f(x) \otimes D_{e_n} g(x)$ is well defined for any C.O.N.S. $(e_n)_{n\in\mathbb{N}}$ of $H^s(M,E)$ and

$$\begin{aligned}\Delta^{Q,\mu}(fg)(x) &= \Delta^{Q,\mu}(f)(x)g(x) \\ &\quad -2tr(Df(x)\otimes Dg(x)) + f(x)\Delta^{Q,\mu}g(x) \quad \forall \mu \in \mathbb{R}. \qquad (6.3)\end{aligned}$$

This easily follows from the property of the Hessian $Hess(fg) = (Hessf)g + 2Df \otimes Dg + f(Hessg)$. It shows that Δ^Q behaves like a second order differential operator.

6.2 Families of weighted vector spaces

Instead of a fixed weighted space we now consider families of weighted spaces which leads us to the notion of weighted vector bundle.

• **A class of vector bundles:** We first need to restrict ourselves to a class of vector bundles, namely vector bundles $\pi : \mathcal{E} \to X$ based on a manifold X with fibres modelled on a Fréchet, resp. Hilbert space $\Gamma(M,E)$ of smooth, resp. Sobolev type sections of a fixed finite rank vector bundle $\pi : E \to M$ based on a closed Riemannian manifold M and such that there is a local trivialization of the bundle with *transition maps given by elliptic classical pseudo-differential operators.* (We will also allow P.D.O.s which are locally given by multiplication operators by a Sobolev type function).

For a vector bundle $\mathcal{E}$ in that class with fibres modelled on some $\Gamma(M, E)$, it makes sense to call a linear operator A acting (possibly only densely) fibrewise on $\mathcal{E}$ a (classical) P.D.O. whenever, given a local trivialization (U, ϕ, Φ) where (U, ϕ) is a local chart on the base space, its induced local representation $\Phi^\sharp A := \Phi^{-1} A \Phi$ lies in $CL(M, E)$. This way we can define the bundle $CL(\mathcal{E})$ of operators acting densely fibrewise which are locally given by an operator in $CL(M, E)$. This definition is independent of the choice of local trivialization, since two local representations of A in different trivializations (U, ϕ, Φ) and (V, ψ, Ψ) with $U \cap V \neq \phi$ are related by

$$\Psi^\sharp A = \Phi \circ \Psi^{-1} (\Phi^\sharp A) \Phi^{-1} \circ \Psi. \tag{6.4}$$

The transition maps $\Phi \circ \Psi^{-1}$ being classical P.D.O.s by assumption, whenever $\Psi^\sharp A$ is a classical P.D.O. so is $\Phi^\sharp A$.

Notice that the order of the operator is also unchanged under a change of local trivialization so that it makes sense to talk about the order of an element of $CL(\mathcal{E})$.

• **Weighted vector bundles and weighted manifolds:** Recall that a Riemannian structure on a finite rank vector bundle $\pi : E \to X$ corresponds to a reduction of the frame bundle $GL(E)$ to an orthonormal frame bundle $O(E)$. A metric g on a finite rank vector bundle E can be seen as a section of $g_x, x \in X$ of $Hom(E, E^*)$ which is locally a positive definite matrix.

Similarly we can equip a bundle $\mathcal{E}$ modelled on some $\Gamma(M, E)$ where E is a Riemannian finite rank bundle, with some data Q given by a section of $CL(\mathcal{E})$ which is locally an invertible elliptic operator of positive order and with positive leading symbol; i.e, there is a local trivialization (U, ϕ, Φ) of $\mathcal{E}$ where (U, ϕ) is a local chart on the base manifold, such that $\Phi^\sharp Q \in Ell^{*+}_{ord>0}(M, E)$. Provided $\mathcal{E}$ is Riemannian, we can reduce the structure group $Ell^*(M, E)$ of the bundle $\mathcal{E}$ (notice that it coincides with $GL(V)$ when $E = \{*\} \times V$) to the subgroup $Oll(M, E) := \{A \in Ell^*(M, E), A^* A = A A^* = Id\}$ (notice that it coincides with $O(V)$ when $E = \{*\} \times V$). In that case, the condition $\Phi^\sharp Q \in Ell^{*+}_{ord>0}(M, E)$ does not depend on the choice of the local trivialization (U, ϕ, Φ) since, given another trivialization (V, ψ, Ψ) with $U \cap V \neq \phi$, we have $\Phi^\sharp Q = C \Phi^\sharp Q C^{-1}$ for some $C \in Oll(M, E)$ so that

$\sigma_L(\Psi^\sharp Q) = \sigma_L(C)\sigma_L(\Phi^\sharp Q)\sigma_L(C)^{-1}$ is also positive and hence $\Psi^\sharp Q$ also lies in $Ell^{*+}_{ord>0}(M,E)$. (Notice that the order is in fact independent of the choice of local trivialization.) This confirms the fact that Q can be seen as a generalization of a metric.

However, if we do not want to restrict ourselves to Riemannian bundles, we need to weaken the requirements on the data Q; *a weight* Q on a vector bundle $\mathcal{E}$ modelled on some $\Gamma(M,E)$ is a section of $CL(\mathcal{E})$ which is locally an admissible (see section 3) invertible elliptic operator; i.e, there is a local trivialization (U,ϕ,Φ) of $\mathcal{E}$ where (U,ϕ) is a local chart on the base manifold, such that $\Phi^\sharp Q \in Ell^{*adm}_{ord>0}(M,E)$. Admissibility being a condition on the spectrum, it is invariant under a transformation of type $A \to CAC^{-1}$ induced by a change of trivialization, where $C \in Ell^*(M,E)$.

A *weighted vector bundle* is a couple $(\mathcal{E},Q)$ where $\mathcal{E}$ is a Fréchet or a Hilbert vector bundle modelled on some Fréchet or Hilbert space $\Gamma(M,E)$ of sections of some finite rank vector bundle E based on a closed Riemannian manifold M and where Q is a weight on $\mathcal{E}$. A *weighted manifold* is a couple (X,Q) where X is a Fréchet or Hilbert manifold modelled on some Fréchet or Hilbert space of sections $\Gamma(M,E)$ and where Q a weight on its tangent bundle TX in such a way that (TX,Q) is a weighted bundle.

• **Current groups:** They are manifolds $C^\infty(M,G)$ of smooth, resp. $H^s(M,G), s > \frac{dim M}{2}$ of Sobolev maps from a closed Riemannian manifold M into aa matrix Lie group G. They have a Fréchet, resp. a Hilbert Lie group structure with Lie algebra given by the space of sections $C^\infty(M, Lie(G))$, resp. $H^s(M; Lie(G))$ where $Lie(G)$ is the Lie algebra of G. Here the model space is therefore of the type $\Gamma(M,E)$ with E the trivial bundle $E := M \times Lie(G)$. The couple $(C^\infty(M,G),Q)$, resp. $(H^s(M,G),Q)$ with $Q_\gamma := L_\gamma \Delta L_\gamma^{-1}$ is a weighted manifold, where L_g denotes left multiplication by $g \in G$, $L_\gamma(x) := L_{\gamma(x)}$ for any $x \in M$ and for any $\gamma : M \to G$ and where Δ is the scalar Laplacian on M.

6.3 Weighted traces on weighted vector bundles

Let $(\mathcal{E},Q)$ be a weighted vector bundle. For a section A of $CL(\mathcal{E})$ and given a local trivialization (U,ϕ,Φ) of the bundle around $b \in B$, in a similar way to the finite dimensional setting (see section 1) and using the notations of (4.12) we set:

$$tr^{Q,\mu}(A)(b) := tr^{\Phi^\sharp(b)Q_b,\mu}(\Phi^\sharp(b)A_b), \quad \mu \in \mathbb{R} \tag{6.6}$$

where $\Phi^\sharp(b)A_b$, resp. $\Phi^\sharp(b)Q_b$ denotes the operator A_b, resp. Q_b in the local trivialization (U,ϕ,Φ). This definition is independent of the choice of local trivialization. Indeed, let C be an invertible elliptic operator and let $Q \in Ell^{*adm}_{ord>0}(M,E)$, $A \in CL(M,E)$. Then provided $Re(z)$ is chosen large enough so that $CAQ^{-\frac{z}{2}}$ and $Q^{-\frac{z}{2}}C^{-1}$ are Hilbert Schmidt and using the fact that $tr(K_1K_2) = tr(K_2K_1)$ for Hilbert Schmidt operators K_1, K_2, with the notations of section 4 we have:

$$\begin{aligned}
tr^{CQC^{-1},\mu}(CAC^{-1}) &= LIM^{\mu}_{z\to 0} tr\left(CAC^{-1}\left(CQC^{-1}\right)^{-z}\right)\\
&= LIM^{\mu}_{z\to 0} tr\left(CAC^{-1}C\,(Q)^{-z}\,C^{-1}\right)\\
&= LIM^{\mu}_{z\to 0} tr\left(CA\,(Q)^{-z}\,C^{-1}\right)\\
&= LIM^{\mu}_{z\to 0} tr\left(CA\,(Q)^{-\frac{z}{2}}\,(Q)^{-\frac{z}{2}}\,C^{-1}\right)\\
&= LIM^{\mu}_{z\to 0} tr\left((Q)^{-\frac{z}{2}}\,C^{-1}CA\,(Q)^{-\frac{z}{2}}\right)\\
&= LIM^{\mu}_{z\to 0} tr\left((Q)^{-\frac{z}{2}}\,A\,(Q)^{-\frac{z}{2}}\right)\\
&= LIM^{\mu}_{z\to 0} tr\left(A\,(Q)^{-\frac{z}{2}}\,(Q)^{-\frac{z}{2}}\right)\\
&= LIM^{\mu}_{z\to 0} tr\left(AQ^{-z}\right)\\
&= tr^{Q,\mu}(A),
\end{aligned}$$

where we have also used the fact that $\left(CQC^{-1}\right)^{-z} = C\,(Q)^{-z}\,C^{-1}$ which follows from the very definition of complex powers (see formula (3.10)). Applying this to $C := \Phi(b)\circ\Psi(b)^{-1}$ where (V, ψ, Ψ) is another local trivialization we have for $b \in U \cap V$:

$$\begin{aligned}
tr^{\Phi^\sharp(b)Q_b,\mu}(\Phi^\sharp(b)A_b) &= tr^{C\left(\Psi^\sharp(b)Q_b\right)C^{-1},\mu}(C\left(\Psi^\sharp(b)A_b\right)C^{-1})\\
&= tr^{\Psi^\sharp(b)Q_b,\mu}(\Psi^\sharp(b)A_b).
\end{aligned}$$

• **Variation of the weighted trace:** We saw in section 5, formula (5.11) that given a smooth family $\{A_b, b \in B\}$ of P.D.O.s and a smooth family $\{Q_b, b \in B\}$ of invertible positive elliptic operators with constant order q, B being a smooth (possibly infinite dimensional) manifold and for any $\mu \in I\!R$ we have: $dtr^{Q,\mu}(A) = tr^{Q,\mu}(dA) - \frac{1}{q}res(\mathrm{Ad}\ \log Q)$. We want to generalize this to sections of $CL(\mathcal{E})$ where $(\mathcal{E}, Q)$ is a weighted vector bundle.

The weighted traces $tr^{Q,\mu}, \mu \in I\!R$ extend to $CL(\mathcal{E})$ valued forms:

$$\begin{aligned}
tr^{Q,\mu} : \Omega(X)\otimes CL(\mathcal{E}) &\to \Omega(X)\\
\alpha\otimes A \qquad &\to tr^{Q,\mu}(A)\alpha.
\end{aligned}$$

In a similar way, we can extend the notion of Wodzicki residue to $CL(\mathcal{E})$-valued forms.

Proposition [CDMP], [PR]: *Let $(\mathcal{E}, Q)$ be a weighted vector bundle based on B and let α be a smooth $CL(\mathcal{E})$-valued form on B. Let us assume $\mathcal{E}$ is equipped with a connection $\nabla^{\mathcal{E}}$ such that $[\nabla^{\mathcal{E}}, \alpha]$ is also a $CL(\mathcal{E})$-valued form on B. Then*

$$[\nabla^{\mathcal{E}}, tr^{Q,\mu}](\alpha) := dtr^{Q,\mu}(\alpha) - tr^{Q,\mu}([\nabla^{\mathcal{E}}, \alpha]) = (-1)^{|\alpha|+1}\frac{1}{q} res(\alpha[\nabla^{\mathcal{E}}, \log Q]) \quad \forall \mu \in \mathbb{R} \tag{6.7}$$

where α is the degree of the form α. *Proof:* We only prove the result for a bundle morphism A i.e. a $CL(\mathcal{E})$ valued zero form A on B. The proof easily generalizes to forms of higher degree keeping track of the signs. We write $\nabla^{\mathcal{E}} = d + \theta^{\mathcal{E}}$ in a local trivialization (U, ϕ, Φ) (see notations of section 1). Since $A \to [\nabla^{\mathcal{E}}, A]$ defines a bundle morphism of $CL(\mathcal{E})$, for $A \in \phi(U) \times CL(M, E)$, $[\theta^{\mathcal{E}}, A]$ lies in $\phi(U) \times CL(M, E)$ and it follows from the results of section 3 that $[\theta^{\mathcal{E}}, \log Q]$ lies in $\Omega_1(\phi(U)) \otimes CL(M, E)$. We now combine the fact proved in section 5 formula (5.9) that $tr^{Q,\mu}([\theta^{\mathcal{E}}, A]) = \frac{1}{q} res(A[\theta^{\mathcal{E}}, \log Q])$ with (5.11) to write for any $\mu \in \mathbb{R}$:

$$\begin{aligned}
dtr^{Q,\mu}(A) &= tr^{Q,\mu}(dA) - \frac{1}{q} res(A d \log Q) \\
&= tr^{Q,\mu}([\nabla^{\mathcal{E}}, A]) - tr^{Q,\mu}([\theta^{\mathcal{E}}, A]) - \frac{1}{q} res(A d \log Q) \\
&= tr^{Q,\mu}([\nabla^{\mathcal{E}}, A]) - \frac{1}{q} res(A[\theta^{\mathcal{E}}, \log Q]) - \frac{1}{q} res(A d \log Q) \\
&= tr^{Q,\mu}([\nabla^{\mathcal{E}}, A]) - \frac{1}{q} res(A[\nabla^{\mathcal{E}}, \log Q]).
\end{aligned}$$

We shall refer to the bracket:

$$\begin{aligned}
[\nabla^{\mathcal{E}}, tr^{Q,\mu}] : \Omega(B) \otimes CL(\mathcal{E}) &\to B \times \mathbb{R} \\
\alpha &\to -\frac{1}{q} res(\alpha[\nabla^{\mathcal{E}}, \log Q])
\end{aligned}$$

as a *geometric obstruction* of the weighted trace $tr^{Q,\mu}$. Notice that it is independent of $\mu \in \mathbb{R}$.

6.4 Some geometry in infinite dimensions

Going back to section 1, let us see how properties such as the Bochner-Weitzenböck formula (1.12) or the Chern-Weil construction briefly described in section 1 could be naively extended using weighted traces.

Let (X, Q) be a Hilbert weighted manifold equipped with a Riemannian metric and the corresponding Levi-Civita connection ∇. Then the tangent bundle TX is also equipped with the weight Q.

• **A weighted Ricci curvature:** Let Ω be the curvature of ∇. If the map $(U, V) \to \Omega(U, \cdot)V$ defines a section of $TX \otimes TX \otimes CL(TX)$, we can define for any $\mu \in I\!R$ a *Q-weighted Ricci curvature*

$$Ricc^{Q,\mu}(U, V) := tr^{Q,\mu}(\Omega(U, \cdot), V). \tag{6.8}$$

•**A generalized Bochner Weitzenböck formula [ABP]:** We use the notation of formula (6.2) for the Q-weighted Laplacian.

$$[\Delta^{Q,\mu}, \nabla]f(V) = Ricc^{Q,\mu}(V, \nabla f) + \frac{1}{ordQ} res(Hessf[\nabla, \log Q]) \quad \forall \mu \in I\!R \tag{6.9}$$

for any a vector field V on X, for any C^2 function f such that the two tensors $Hessf$, $(\nabla)^3(\cdot, \cdot, V)$, $\nabla_V Hessf$ on X can be seen (via Riesz theorem) as sections of $CL(TX)$. Notice that if X is finite dimensional, then the Wodzicki residue vanishes and we recover (1.12).

Proof. The proof of (6.9) closely follows that of (1.12) (see [ABP]) up to the fact that exchanging the covariant differentiation with the weighted trace (which goes without saying in the finite dimensional setting) requires more care here; it gives rise to a residue term by formula (6.7). Let $V \in T_x X$ and let $(e_i, i \in I\!N)$ be a local orthonormal frame at point x of eigenvectors of Q_x.

$$\begin{aligned}
([-\Delta^Q, \nabla]f)(V) &= (-\Delta^Q \nabla f)(V) + \nabla_V \Delta^{Q,\mu} f \\
&= Lim^{\mu}_{\varepsilon \to 0} \sum_i \nabla^3 f(e^{-\varepsilon Q} e_i \otimes e^{-\varepsilon Q} e_i \otimes V) - \nabla_V tr^Q(Hessf) \\
&= Lim_{\varepsilon \to 0} \sum_i \nabla^3 f(e^{-\varepsilon Q} e_i \otimes V \otimes e^{-\varepsilon Q} e_i) - tr^Q(\nabla_V \nabla^2 f) + \\
&\qquad + \frac{1}{ordQ} res(Hessf[\nabla, \log Q])
\end{aligned}$$

where we have applied (6.7) to $\alpha = Hessf$

$$= LIM^{\mu}_{\varepsilon \to 0} \sum_i \left(\nabla^3 f(e^{-\varepsilon Q} e_i \otimes V \otimes e^{-\varepsilon Q} e_i) - \nabla^3 f(V \otimes e^{-\varepsilon Q} e_i \otimes e^{-\varepsilon Q} e_i) \right) +$$

$$+ \frac{1}{ordQ} res(Hessf[\nabla, \log Q])$$

$$= LIM^{\mu}_{\varepsilon\to 0}\sum_i \nabla^2\nabla f((e^{-\varepsilon Q}e_i\wedge V)\otimes e^{-\varepsilon Q}e_i)+\frac{1}{ordQ}res(Hessf[\nabla,\log Q])$$

$$= LIM^{\mu}_{\varepsilon\to 0}\sum_i (\nabla^2_{e^{-\varepsilon Q}e_i\wedge V}\nabla f)e^{-\varepsilon Q}e_i$$

$$= LIM^{\mu}_{\varepsilon\to 0}\sum_i \langle\Omega(e^{-\varepsilon Q}e_i,V)\nabla f, e^{-\varepsilon Q}e_i\rangle+\frac{1}{ordQ}res(Hessf[\nabla,\log Q])$$

$$= -Ricc^{Q,\mu}(V,\nabla f)+\frac{1}{ordQ}res(Hessf[\nabla,\log Q]).$$

• Obstructions to a (weighted) Chern-Weil theory

Let us go back to a general weighted vector bundle $(\mathcal{E},Q)$ based on X and equipped with a connection $\nabla^{\mathcal{E}}$. When α in formula (6.7) is the curvature tensor $\Omega^{\mathcal{E}}$ of $\nabla^{\mathcal{E}}$ seen as a $CL(\mathcal{E})$ valued two form, then by the Bianchi identity we have $[\nabla^{\mathcal{E}},\Omega^{\mathcal{E}}]=0$ but this does not imply anymore that $dtr^Q(\Omega^{\mathcal{E}})=0$ (as it did in the finite dimensional case, see (1.15)) since it follows from (6.7) that $dtr^Q(\Omega^{\mathcal{E}})=-\frac{1}{q}res(\Omega^{\mathcal{E}}[\nabla^{\mathcal{E}},\log Q])$. Hence in general, $tr^Q(\Omega^{\mathcal{E}})$ does not define a characteristic class because of the above obstructions unless Q and $\nabla^{\mathcal{E}}$ are chosen in such a way that $res(\Omega^{\mathcal{E}}[\nabla^{\mathcal{E}},\log Q])=0$.

One can generalize this observation to traces of higher order polynomials in the curvature tensor. Let us see how the Chern-Weil construction we briefly recalled in section 1 could be extended to our infinite dimensional framework with the help of weighted traces. Since we have replaced the algebra of matrices by the algebra $CL(M,E)$ and the trace on the algebra of matrices by weighted traces tr^Q where $Q\in Ell^{*adm}_{ord>0}(M,E)$, it is natural to introduce the following definitions. Given a weighted vector bundle $(\mathcal{E},Q)$ based on some manifold X where the model space of $\mathcal{E}$ is a space of sections $\Gamma(M,E)$, M being the base manifold of E, and given a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, to a polynomial P on $CL(M,E)$ we can associate a form $tr^{Q,\mu}(P(\Omega^{\mathcal{E}}))$ on X where $\mu\in\mathbb{R}$. It is a well defined expression since we can write in a similar way to the finite dimensional case (see section 1) and with similar notations:

$$tr^{\Psi^\sharp Q,\mu}(P(\Psi^\sharp\Omega^{\mathcal{E}}))=tr^{C\Phi^\sharp QC^{-1},\mu}(CP(\Phi^\sharp\Omega^{\mathcal{E}})C^{-1})=tr^{\Phi^\sharp Q,\mu}(P(\Phi^\sharp\Omega^{\mathcal{E}}))$$

using the covariance property $tr^{CQC^{-1}}(CAC^{-1})=tr^Q(A)$. However obstructions come into the way at various stages, and to start with when trying to generalize (1.16). As in section 1 we take $P(X)=X^k$ and consider its canonical form $\tilde{P}(X_1,\cdots,X_k)=X_1\cdots X_k$. Differentiating w.r.to t the covariance property just mentioned applied to $C := e^{tB}$ and using (5.9) we find:

$$\sum_{j=1}^{k} tr^{Q,\mu}(\tilde{P}_k(A_1,\cdots,A_{i-1},[B,A_j],A_{i+1},\cdots A_k)) =$$

$$= \frac{1}{ordQ} res(\tilde{P}_k(A_1,\cdots,A_k)[A,\log Q]) \quad (6.10)$$

for any $B \in CL(M,E), A_j \in CL(M,E), j = 1,\cdots,k$. Furthermore we come across geometric obstructions of the type (6.7) and we have (compare with (1.17)):

$$dtr^{Q,\mu}(P(\Omega^{\mathcal{E}})) = -\frac{1}{ordQ} res(P(\Omega^{\mathcal{E}})[\nabla^{\mathcal{E}},\log Q]). \quad (6.11)$$

Proof. Let us try to mimick the proof of the corresponding finite dimensional result as shown in section 1. As we did there, we first restrict ourselves to a monomial $P(A) := A^k$ and compute $dtr(P(\Omega^{\mathcal{E}}))$. Let $\nabla^{\mathcal{E}} = d + \theta^{\mathcal{E}}$ be the local representation of the connection in some given local trivialization (U,Φ,ϕ). Applying (6.10) to $B := \theta^{\mathcal{E}}(\phi_* V)$, $A_i = \Phi^\sharp\Omega^{\mathcal{E}}(\phi_* V_1, \phi_* V_2)$ for any $V, V_1, V_2 \in TX$, the tangent space to the base manifold of $\mathcal{E}$, we can write:

$$dtr^{Q,\mu}(P(\Omega^{\mathcal{E}})) = tr^{Q,\mu}(d\tilde{P}(\Phi^\sharp\Omega^{\mathcal{E}},\cdots,\Phi^\sharp\Omega^{\mathcal{E}})) - \frac{1}{ordQ} res(P(\Omega^{\mathcal{E}}) d\log Q)$$

$$= \sum_{i=1}^{k} tr^{Q,\mu}(\tilde{P}(\Phi^\sharp\Omega^{\mathcal{E}},\cdots,d(\Phi^\sharp\Omega^{\mathcal{E}}),\cdots,\Phi^\sharp\Omega^{\mathcal{E}})) - \frac{1}{ordQ} res(P(\Omega^{\mathcal{E}}) d\log Q)$$

where $\quad d(\Phi^\sharp\Omega^{\mathcal{E}}) \quad$ is at place i

$$= \sum_{i=1}^{k} tr(\tilde{P}(\Phi^\sharp\Omega^{\mathcal{E}},\cdots,d(\Phi^\sharp\Omega^{\mathcal{E}}),\cdots,\Phi^\sharp\Omega^{\mathcal{E}}))$$

$$+ \sum_{i=1}^{k} tr(\tilde{P}(\Phi^\sharp\Omega^{\mathcal{E}},\cdots,[\theta^{\mathcal{E}},\Phi^\sharp\Omega^{\mathcal{E}}],\cdots,\Phi^\sharp\Omega^{\mathcal{E}}))$$

where $\quad d(\Phi^\sharp\Omega^{\mathcal{E}}) \quad$ and $\quad [\theta^{\mathcal{E}},\Phi^\sharp\Omega^{\mathcal{E}}] \quad$ are at place i

$$- \frac{1}{ordQ} res(P(\Omega^{\mathcal{E}})[\theta^{\mathcal{E}},\log Q]) - \frac{1}{ordQ} res(P(\Omega^{\mathcal{E}}) dlogQ)$$

$$= \sum_{i=1}^{k} tr^{Q,\mu}(\tilde{P}(\Omega^{\mathcal{E}},\cdots,[\nabla^{\mathcal{E}},\Omega^{\mathcal{E}}],\cdots,\Omega^{\mathcal{E}})) - \frac{1}{ordQ} res(P(\Omega^{\mathcal{E}})[\nabla^{\mathcal{E}},\log Q])$$

where $\quad [\nabla^{\mathcal{E}},\Omega^{\mathcal{E}}] \quad$ is at place i

$$= -\frac{1}{ordQ} res(P(\Omega^{\mathcal{E}})[\nabla^{\mathcal{E}},\log Q])$$

where we have used Bianchi identity $[\nabla^{\mathcal{E}}, \Omega^{\mathcal{E}}] = 0$ in the last line.

Given another connection $\tilde{\nabla}^{\mathcal{E}}$ on $\mathcal{E}$, one can check (we leave this as an **exercise** for the reader, see [Pa]) that $tr^{Q,\mu}(P(\Omega^{\mathcal{E}})) - tr^{Q,\mu}(P(\tilde{\Omega}^{\mathcal{E}}))$ is an exact form whenever the residue $res(P(\Omega^{\mathcal{E}})[\nabla^{\mathcal{E}}, \log Q])$ vanishes for any connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$.

Thus, whenever $res(P(\Omega^{\mathcal{E}})[\nabla^{\mathcal{E}}, \log Q])$ vanishes for any connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ (this is the case in finite dimensions of course), then $tr^{Q,\mu}(P(\Omega^{\mathcal{E}}))$ defines a characteristic class given by its cohomology class in the de Rham cohomology which is independent of the choice of connection. Under such an assumption, these *weighted Chern-Weil invariants* indeed define topological invariants as they do in the finite dimensional case.

- **Cases for which the obstructions vanish:**

  Finite rank bundles. As expected, when $\mathcal{E}$ reduces to a finite rank vector bundle, tr^Q coincides with the ordinary trace and this residue which is a purely infinite dimensional obstruction vanishes. Hence we recover the finite dimensional Chern-Weil invariants.

  Weighted loop groups. Let G be a Lie group, M a compact manifold and let us take $B := C_0^\infty(S^1, G)$ and $\mathcal{E} := TB$ where the index 0 means the loops are based at a fixed point. As before we choose a weight given by a left invariant family of Laplacians $Q_\gamma := L_\gamma \Delta L_\gamma^{-1}$ where Δ is the scalar Laplacian on $C^\infty(S^1, Lie(G))$ and L_γ left multiplication. Following [F] let us consider a left invariant $H^{\frac{1}{2}}$ metric built up from the Killing form on the Lie algebra on $C^\infty(S^1, G)$ and the corresponding Levi-Civita connection $\nabla^{\frac{1}{2}}$. There is a natural almost complex structure J on $H^{\frac{1}{2}}(S^1, G)$ which commutes with the connection and $H^{\frac{1}{2}}(S^1, G)$ is a Hilbert Kähler manifold (see [F], [Pr]). The curvature applied to smooth vector fields is a classical P.D.O. of order -1 and hence has divergent trace. However, its trace along the Lie algebra is a P.D.O. of order -2 which is therefore trace-class. For smooth vector fields U, V, W, one can show that $res\left(\Omega^{\frac{1}{2}}(U, V)[\nabla_W^{\frac{1}{2}}, \log Q]\right) = 0$ [Ma] so that $tr^Q(\Omega^{\frac{1}{2}})$ defines a closed form and gives rise to the first Chern class of the weighted complex bundle $(\mathcal{E}, Q)$. Since the curvature is a P.D.O. of order -1, its k-th power is of order $-k$ and therefore of trace-class in dimension 1 for $k \geq 2$. Higher Chern classes are thus well-defined as ordinary traces, or equivalently the weighted traces $tr^Q\left(\left(\Omega^{\mathcal{E}}\right)^k\right)$ coincide with ordinary traces for $k \geq 2$ and hence give rise to higher Chern classes.

6.5 Conclusion

It is rather frustrating to come across so many obstacles arising as Wodzicki residues preventing a straight forward generalization to the infinite dimensional context of some basic geometric concepts and their properties involving traces. These obstructions are due to the additional data given by the weight, so one might ask whether it is possible to avoid using this extra data. My guess is that it is not since the weight, even if not always explicit in the various approaches (at least the ones that I know of) to the geometry of infinite dimensional (continuous, non discrete) manifolds or vector bundles, it always seems to be hidden in some form or other. Just to quote two striking examples, when using Brownian motion to tackle path spaces, or "conditioned traces" as in [DL] and [F] to handle the geometry of loop groups, there is a weight hidden behind the construction in both cases, the Laplacian in the first case as a basic ingredient to construct Brownian motion, and the "conditioning" of the traces in the second case which can be interpreted as a choice of a weighting procedure for the otherwise ill-defined traces. So there seems little chance of getting rid of these weights.

Then a natural question arises, namely whether we can compensate the Wodzicki residues due to these weights in the traces. In certain cases, noncommutative geometry provides a way to counterbalance these obstructions, at least when weights arise as squares of some family of Dirac type operator as in the case of infinite rank bundles built up from families of Dirac operators as in the context of the family index theorem. It would be out of the scope of this article to describe such mechanisms here and we refer the reader to [PR], [Pa] for further developments along these lines.

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