AN INTRODUCTION TO SEIBERG-WITTEN THEORY

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We give an elementary introduction to Seiberg-Witten theories in N=2 supersymmetry. Starting with a description of electromagnetic duality and the different forms of monopole, we continue with an introduction to supersymmetry to arrive at the non-perturbative solution of the Moduli space.

Contents

1 Introduction

Recently more importance has been given to finding theories which allow non-perturbative solutions. A relevant development in this direction was initiated by the work of Seiberg and Witten [$SW1, SW2$] on N=2 supersymmetric Yang-Mills theory. There are many developments and review articles in this field. The aim of this course is to offer an introduction to the material handled in these numerous works. Duality plays a very important role since Seiberg-Witten theory showed that it can be interpreted as a quantum symmetry.

In the first part, we present the electromagnetic duality including the introduction of different forms of monopole and its consequences in the different quantization conditions. We follow review articles by Alvarez-Gaume and Hassan [AH] and Figueroa-O'farrill [FIG] as well as classical books like that by Cheng and Li [CHL]. We will not discuss the problem of singularities which are presented in the lectures by Tsou Sheung Tsun [$TSOU$] in this volume. We then introduce supersymmetry following books like [BW], [Fr] and [DF], from which we shall borrow the notation commonly used in supersymmetry. Finally we present the theory of Seiberg and Witten and the description of the moduli space for the $SU(2)$ symmetry as an elliptic curve. In this last part we follow the original articles, the reviews mentioned above and that written by Lerche [L].

2 Electro-Magnetic Duality

In this section we show the consequences of the symmetrization of the Maxwell equation in different formulations. Let us begin with the Maxwell equations

$$\begin{array}{ll} \vec{\nabla}\cdot\vec{\mathbf{E}} = 4\pi\rho \ , & \vec{\nabla}\times\vec{\mathbf{B}} - \partial\vec{\mathbf{E}}/\partial t = \vec{\mathbf{j}} \\ \vec{\nabla}\cdot\vec{\mathbf{B}} = 0 \ , & \vec{\nabla}\times\vec{\mathbf{E}} + \partial\vec{\mathbf{E}}/\partial t = 0 \ . \end{array} \tag{1}$$

This set of equation in absence of charges $\rho = 0$,$\vec{\mathbf{j}} = 0$ is invariant under a transformation

$$\begin{array}{l} \vec{\mathbf{E}} \to \ \vec{\mathbf{B}} \\ \vec{\mathbf{B}} \to -\vec{\mathbf{E}} \ , \end{array}$$

which is a discrete symmetry, called duality symmetry. It is clear that the symmetry is broken in presence of charges. To restore the symmetry Dirac proposed the introduction of a monopole density σ and a monopole current $\vec{\mathbf{k}}$ in the following form

$$\begin{array}{ll} \vec{\nabla}\cdot\vec{\mathbf{E}} = 4\pi\rho \ , & \vec{\nabla}\times\vec{\mathbf{B}} - \partial\vec{\mathbf{E}}/\partial t = \vec{\mathbf{j}} \\ \vec{\nabla}\cdot\vec{\mathbf{B}} = \sigma \ , & \vec{\nabla}\times\vec{\mathbf{E}} + \partial\vec{\mathbf{E}}/\partial t = -\vec{\mathbf{k}} \ , \end{array} \tag{2}$$

where the transformation set is enlarged with

$$\begin{array}{ll} \rho \to \sigma & \vec{\mathbf{j}} \to \vec{\mathbf{k}} \\ \sigma \to -\rho & \vec{\mathbf{k}} \to -\vec{\mathbf{j}} \ . \end{array}$$

The Maxwell Eq. (1) can be also written in terms of the potentials ϕ and $\vec{\mathbf{A}}$. The absence of monopole density allows to write $\vec{\mathbf{B}} = \vec{\nabla}\times\vec{\mathbf{A}}$ and the absence of a monopole current $\vec{\mathbf{E}} = -\vec{\nabla}\phi - \partial\vec{\mathbf{A}}/\partial t$. In presence of a monopole $\vec{\mathbf{A}}$ can not be defined globally, as we will discuss later. The electromagnetic equation can be written in a covariant form when we put the potentials in a 4-vector $A^\mu = \{\phi, \vec{\mathbf{A}}\}$ and define the field strength tensor $F_{\mu\nu}$ as

$$\begin{array}{l} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ F_{0i} = \partial_0 A_i - \partial_i A_0 = -\partial_0 A^i - \partial_i A^0 = E^i \ , \\ F_{ij} = \partial_i A_j - \partial_j A_i = -(\partial_i A^j - \partial^j A_i) = \epsilon_{ijk} B^k \ . \end{array}$$

Note that $B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk} = -\frac{1}{2}\epsilon^{0i\mu\nu}F_{\mu\nu}$ and $E^i = F_{oi}$, (where $\epsilon^{ijk} = 1$ if $i = 1,\ j = 2,\ k = 3$ and all even permutation of them, $= -1$ for any odd permutation and $= 0$ in other case). The choice of the potential A^μ has the gauge freedom ($A^\mu \to A'^\mu = A^\mu + \partial^\mu\chi$). We also need a dual of the strength field tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\delta}F_{\rho\delta} \ ,$$

where $\epsilon^{0123} = 1$. Now the covariant expression of the Maxwell equations is:

$$\partial_\nu F^{\mu\nu} = -j^\mu \ , \quad \partial_\nu \tilde{F}^{\mu\nu} = 0 \ , \tag{3}$$

where $j^\mu = \{\rho, \vec{\mathbf{j}}\}$. The new expression for the duality symmetry of the Maxwell equations without charges is $F \to \tilde{F}$, and $\tilde{F} \to -F$. Introducing the monopole the equation becomes

$$\partial_\nu F^{\mu\nu} = -j^\mu \ , \quad \partial_\nu \tilde{F}^{\mu\nu} = k^\mu \ ,$$

and the symmetry is enlarged with the transformations

$$j^\mu \to k^\mu \ , \ \ and \ \ k^\mu \to -j^\mu \ .$$

2.1 The Dirac Monopole

The inclusion of monopole at first sight destroys the possibility of a gauge theory, but Dirac proposed a very ingenious way to allow for a potential $\vec{\mathbf{A}}$ in presence of monopole. By Stoke's theorem, it cannot be well-defined in a neighborhood around the charge an hence in any sphere around the charge, $\vec{\mathbf{A}}$ must have a singularity. By continuity, the singularities are on a line starting in the charge and ending in a monopole of opposite charge or at infinity. This construction is called the Dirac string. But if we consider the monopole as a point their magnetic field should be radial of the form $\vec{\mathbf{B}}_{mon} = g\vec{\mathbf{r}}/4\pi r^3$, where g is magnetic charge of the monopole. In this formulation Dirac introduced a solenoid along the string such that

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_{mon} + \vec{\mathbf{B}}_{sol} \ , \ \ and \ \ \vec{\nabla} \cdot \vec{\mathbf{B}} = 0 \ .$$

For example, if the solenoid is chosen along the z-axes $B_{sol} = g\theta(-z)\delta(x)\delta(y)\hat{z}$, (where $\theta(x) = 1$ for $x > 0$ and $= 0$ for $x < 0$), it is possible to find a well-defined vector potential $\vec{\mathbf{A}}$, with which one can formulate electromagnetism in a covariant way.

By quantizing this electromagnetic theory with Dirac monopole one finds a condition for the charges. The Schrödinger equation for a particle interacting with an electromagnetic field is:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \vec{\mathbf{D}}^2 \psi \ ,$$

where $\vec{\mathbf{D}} = \vec{\nabla} + ie\vec{\mathbf{A}}$ with $e = q/\hbar$. This equation is gauge invariant under the transformations

$$\vec{\mathbf{A}} \to \vec{\mathbf{A}} + \vec{\nabla}\chi \ \ and \ \ \psi \to exp(-ie\chi)\psi \ .$$

But the wave function in this last expression is not single-valued. In the Aharonov-Bohm experiment the presence of the string will affect the phase factor. If one considers the variation of the phase factor along a closed curve Γ, encircling the string, which is the boundary of a surface S, we have

$$e \oint_\Gamma \vec{\mathbf{A}}_{sol} \cdot d\vec{\mathbf{l}} = e \int_S \vec{\mathbf{B}}_{sol} \cdot d\vec{\mathbf{s}} = eg \ .$$

The conditions for a single-valued wavefunction is known as the Dirac charge quantization condition

$$eg = 2\pi n \ .$$

There is another semi classical way to derive a charge quantization condition. Let us consider a particle with electric charge q in presence of the magnetic field $\vec{\mathbf{B}} = g\vec{\mathbf{r}}/4\pi r^3$ of a monopole. The particle experiences a force $m\ddot{\vec{\mathbf{r}}} = q\dot{\vec{\mathbf{r}}} \times \vec{\mathbf{B}}$. Therefore the change of the angular momentum of the electric charge is given by

$$\frac{d}{dt}(m\vec{\mathbf{r}} \times \dot{\vec{\mathbf{r}}}) = m\vec{\mathbf{r}} \times \ddot{\vec{\mathbf{r}}} = \frac{qg}{4\pi r^3}\vec{\mathbf{r}} \times (\dot{\vec{\mathbf{r}}} \times \vec{\mathbf{r}}) = \frac{d}{dt}(\frac{qg}{4\pi}\frac{\vec{\mathbf{r}}}{r}) \ .$$

Hence, the total conserved angular momentum of the system is

$$\vec{\mathbf{J}} = \vec{\mathbf{r}} \times m\dot{\vec{\mathbf{r}}} - \frac{qg}{4\pi}\frac{\vec{\mathbf{r}}}{r} \ .$$

The second term is the angular momentum of the electromagnetic field, calculated from

$$\vec{\mathbf{J}}_{em} = \int d^3x \vec{\mathbf{r}} \times (\vec{\mathbf{E}} \times \vec{\mathbf{B}}) \ .$$

Assuming that the total angular momentum and the particle angular momentum are quantized, we find again the Dirac quantization condition

$$\frac{qg}{4\pi} = \frac{1}{2}n\hbar \ .$$

Now, let us consider particles carrying electric and magnetic charges called *dyons*. The calculation of the interaction between two dyons with charges (q_1, g_1) and (q_2, g_2) is carried out in a similar way. One splits the electric and magnetic field as $\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2$ and $\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2$ and finds the Dirac-Schwinger-Zwanziger quantization condition

$$\frac{q_1 g_2 - q_2 g_1}{4\pi\hbar} = \frac{1}{2}n_{12} \ .$$

Note that this condition is invariant under an $SO(2)$ transformation written as $(q + ig) \to e^{i\phi}(q + ig)$.

In 1974, 't Hooft $['tH]$ and Polyakov $[Pol]$ independently discovered that the bosonic part of Georgi-Gashow model admits finite energy solutions that look like Dirac monopole from afar. In contrast with the Dirac monopole, these solutions are everywhere regular and do not requires the introduction of a magnetic charge source.

2.2 The Bosonic part of the Georgi-Glashow model

The Georgi-Glashow model describes the electromagnetic interaction. Here we will consider only the bosonic part of this model with a rotation ($SO(3)$) symmetry. The Lagrange density of the model is given by

$$L = -\frac{1}{4}\vec{\mathbf{G}}^{\mu\nu} \cdot \vec{\mathbf{G}}_{\mu\nu} + \frac{1}{2}D^{\mu}\vec{\phi} \cdot D_{\mu}\vec{\phi} - V(\phi) \ , \tag{4}$$

where the gauge strength field $\vec{\mathbf{G}}_{\mu\nu} = \partial_{\mu}\vec{\mathbf{W}}_{\nu} - \partial_{\nu}\vec{\mathbf{G}}_{\mu} - e\vec{\mathbf{W}}_{\mu} \times \vec{\mathbf{W}}_{\nu}$. Here the vector notation means the component in the Lie algebra $\vec{\phi} = \sum_a \phi^a T^a$, with the Lie algebra generators T^a and the vector product have the same structure as the commutator. The Higgs field $\vec{\phi}$ is a vector in the adjoint representation which has a minimal coupling with the gauge field via the gauge covariant derivative

$$D_{\mu}\vec{\phi} = \partial\vec{\phi} - ie\vec{\mathbf{W}}_{\mu} \times \vec{\phi} \ . \tag{5}$$

The Higgs potential $V(\phi)$ is given by

$$V(\phi) = \frac{1}{4}\lambda(\phi^2 - a^2)^2 \ , \tag{6}$$

where $\phi^2 = \vec{\phi} \cdot \vec{\phi}$ and λ is assumed non negative.

The Lagrangian density L is invariant under the $SO(3)$ gauge transformations

$$\begin{aligned} \vec{\phi} &\to \vec{\phi}' = g(x)\vec{\phi} \\ \vec{\mathbf{W}}_{\mu} &\to \vec{\mathbf{W}}'_{\mu} = g(x)\vec{\mathbf{W}}_{mu}g(x)^{-1} + \tfrac{1}{e}\partial_{\mu}g(x)\ g(x)^{-1} \ , \end{aligned} \tag{7}$$

where $g(x)$ is a x-dependet rotation, ie. a possibly x-dependet 3×3 orthogonal matrix in $SO(3)$ with unit determinant.

The classical dynamics of the fields is determined from the Lagrangian

$$D_{\mu}\vec{\mathbf{G}}^{\mu\nu} = -e\vec{\phi} \times D^{\mu}\vec{\phi} \ , \quad D^{\mu}D_{\mu}\vec{\phi} = -\lambda(\phi^2 - a^2)\vec{\phi} \, ,$$

and from the Bianchi identity

$$D_{\mu}\vec{\tilde{\mathbf{G}}}^{\mu\nu} = 0 \ ,$$

where $\vec{\tilde{\mathbf{G}}}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\delta}\vec{\mathbf{G}}_{\rho\delta}$. The canonical conjugate momentum to the gauge field $\vec{\mathbf{W}}_\mu$ and the Higgs $\vec{\phi}$ are given by

$$\vec{\mathbf{E}}^i = -\vec{\mathbf{G}}^{0i} \ , \quad \vec{\mathbf{\Pi}} = D_0\vec{\phi} \ .$$

Defining $\vec{\mathbf{B}}^i$ by

$$\vec{\mathbf{G}}_{ij} = -\epsilon_{ijk}\vec{\mathbf{B}}^k = +\epsilon_{ijk}\vec{\mathbf{B}}_k \ ,$$

we can write the energy density as

$$H = \frac{1}{2}\vec{\mathbf{E}}_i \cdot \vec{\mathbf{E}}_i + \frac{1}{2}\vec{\mathbf{\Pi}} \cdot \vec{\mathbf{\Pi}} + \frac{1}{2}\vec{\mathbf{B}}_i \cdot \vec{\mathbf{B}}_i + \frac{1}{2}D_i\vec{\phi} \cdot D_i\vec{\phi} + V(\phi) \ , \tag{8}$$

which is manifestly positive-semidefinite and also gauge invariant.

We call *vacuum* the configuration for which the energy density vanishes. That means

$$\vec{\mathbf{G}}_{\mu\nu} = 0 \ , \ \ D_\mu\vec{\phi} = 0 \ , \quad V(\phi) = 0 \ . \tag{9}$$

The second two equations define the Higgs vacua. The last one is satisfied if $\phi^2 = a^2$ which defines a sphere in the Higgs space. The choice of one vacuum can be $\vec{\phi} = a\vec{\mathbf{e}}_3$. By the choice of the vacuum we break the $SO(3)$ symmetry to a subsymmetry, which are the rotation $SO(2) = U(1)$ around the $\vec{\phi}$ axis. This procedure is called the *spontaneous symmetry breaking.*

After the choice of the vacuum we can rewrite the Lagrange function to study the oscillation around this vacuum, redefining the Higgs field by $\vec{\phi} = \vec{\mathbf{a}} + \vec{\varphi}$, where $\vec{\mathbf{a}}$ is a constant vector obeying $\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = a^2$. Expanding the Lagrangian density in terms of $\vec{\varphi}$ and writing the scalar field as $\varphi = \frac{1}{a}\vec{\mathbf{a}} \cdot \vec{\varphi}$ and the vector boson as $A_\mu = \frac{1}{a}\vec{\mathbf{a}} \cdot \vec{\mathbf{W}}_\mu$ one can read off the mass of the particles from the expression

$$L = \cdots + \frac{1}{2}\left(\frac{M_H}{\hbar}\right)^2 \varphi^2 + \frac{1}{2}\left(\frac{M_W}{\hbar}\right)^2 W_\mu^+ W^{\mu -} + \cdots \ .$$

This leads to the following mass spectrum

	Mass	*Charge*
A_μ	0	0
φ	$M_H = a\sqrt{2\lambda}\hbar$	0
$W_\mu^\pm$	$M_W = ae\hbar$	$\pm e\hbar$

2.3 't Hooft- Polyakov Monopole

The energy of a given field configuration is the spatial integral of the Hamiltonian density $E = \int d^3x\ H$. In this model the Hamiltonian density is given by (8). Finite energy means that the integral exists. Hence the field must approach a vacuum configuration asymptotically when $r \to \infty$. This means the Higgs field approaches the Higgs vacuum $\mathcal{M}_0$ which is topologically a sphere

$$\phi_\infty(\hat{r}) \equiv \lim_{r\to\infty} \vec{\phi}(\vec{\mathbf{r}}) \in \mathcal{M}_0 \ .$$

It is well-known that the space of continuous maps from a sphere to a sphere has a infinite number of connected components indexed by an integer, called the degree of the map. Heuristically, the degree is the number of times one sphere wraps around the other. It is a topological number, which is a invariant under continuous deformations and is an element of the homotopy group $\pi_2(S^2)$. Let us relate this with the definitions of a Higgs vacuum ie.

$$\vec{\phi}_{vac} \cdot \vec{\phi}_{vac} = a^2 \ , \tag{10}$$

$$\partial_\mu \vec{\phi}_{vac} - e\vec{\mathbf{W}}_\mu \times \vec{\phi}_{vac} = 0 \ . \tag{11}$$

The most general solution of the second equation is given by

$$\vec{\mathbf{W}}_\mu = \frac{1}{ea^2}\vec{\phi}_{vac} \times \partial_\mu \vec{\phi}_{vac} + \frac{1}{a}\vec{\phi} A_\mu \ . \tag{12}$$

Using this solution, we can calculate the gauge field strength tensor $\vec{\mathbf{G}}_{\mu\nu}$ and with it the unbroken part which corresponds to the electromagnetic field strength tensor $F_{\mu\nu}$

$$\begin{aligned} F_{\mu\nu} &= \tfrac{1}{a}\vec{\phi}_{vac} \cdot \vec{\mathbf{G}}_{\mu\nu} \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + \tfrac{1}{ea^3}\vec{\phi}_{vac} \cdot (\partial_\mu \vec{\phi}_{vac} \times \partial_\nu \vec{\phi}_{vac}) \ . \end{aligned} \tag{13}$$

The equation of motion of the Higgs satisfies:

$$\partial_\nu F^{\mu\nu} = 0 \ , \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 \ .$$

We can now calculate the magnetic charges g_Σ associated to this solution, using $B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk}$, :

$$g_\Sigma = \int_\Sigma B^i ds^i = -\frac{4\pi}{e}\left[\frac{1}{4\pi a^3}\int_\Sigma \frac{1}{2}\epsilon_{ijk}\vec{\phi}_{vac} \cdot (\partial^j \vec{\phi} \times \partial^k \vec{\phi}_{vac}) ds^i\right] \ . \tag{14}$$

Since Σ is also topologically equivalent to S^2, ϕ_{vac} is a map from S^2 to $\mathcal{M}_0$ and the expression in brackets is the degree of this map and is an integer n.

Hence, we have a topological quantization condition

$$g_\Sigma = -\frac{4\pi\, n}{e} \; . \tag{15}$$

The Ansatz proposed by 't Hooft $['tH]$ and Polyakov $[Pol]$ for a construction of a monopole solution in the Georgy-Glashow model is given by

$$\begin{array}{r} \vec{\phi}(\vec{\mathbf{r}}) = \frac{\vec{\mathbf{r}}}{er^2}\; H(aer) \\ W_a^i = -\epsilon_{aij}\frac{r^j}{er^2}(1-K(aer)) \\ W_a^0 = 0 \end{array} \tag{16}$$

for some arbitrary functions H and K.

Plugging the Ansatz into the energy density expression one finds

$$E = \frac{4\pi a}{e}\int_0^\infty d\xi \left(\xi^2\frac{dH}{d\xi} + \frac{1}{2}(\xi\frac{dH}{d\xi} - H)^2 + \frac{1}{2}(K^2-1)^2 + K^2H^2 + \frac{\lambda}{4e^2}(H^2-\xi^2)^2\right) \; , \tag{17}$$

which is finite in the boundary condition if: when $\xi = aer \to \infty$

$$\begin{array}{l} K \to \;\; 0 \\ \frac{H}{\xi} \to 1 \; , \end{array}$$

and when $\xi \to \; 0$

$$\begin{array}{ccc} K-1 \leq O(\xi) \\ H \quad \leq \quad O(\xi) \; , \end{array}$$

sufficiently fast. Notice that, with the above asymptotic conditions

$$\vec{\phi}_\infty(\vec{\mathbf{r}}) \equiv \lim_{r\to\infty}\frac{\vec{\mathbf{r}}}{er^2}H(aer) = a\vec{\mathbf{r}} \; .$$

With this Ansatz it is possible to show that asymptotically $F_{ij} = \frac{1}{a}\vec{\phi}$ $cdotG_{ij}$ is given by

$$F_{ij} = \epsilon_{ijk}\frac{r^k}{er^3} \; ,$$

and then

$$\vec{\mathbf{B}} = -\frac{1}{e}\frac{\vec{\mathbf{r}}}{r^3} \; .$$

2.4 The Bogomol'nyi Bound and the BPS States

In the center of mass frame, all the energy of the monopole is concentrated in its mass. Therefore

$$\begin{aligned} M &= \int_{\mathbb{R}^3}\Big(\tfrac{1}{2}\vec{\mathbf{E}}_i \cdot \vec{\mathbf{E}}_i + \tfrac{1}{2}\vec{\mathbf{B}}_i \cdot \vec{\mathbf{B}}_i + \tfrac{1}{2}\vec{\mathbf{\Pi}}_i \cdot \vec{\mathbf{\Pi}}_i + \tfrac{1}{2}D_i\vec{\phi} \cdot D_i\vec{\phi} + V(\phi)\Big) \\ &\geq \tfrac{1}{2}\int_{\mathbb{R}^3}\Big(\vec{\mathbf{E}}_i \cdot \vec{\mathbf{E}}_i + \vec{\mathbf{B}}_i \cdot \vec{\mathbf{B}}_i + D_i\vec{\phi} \cdot D_i\vec{\phi}\Big)\ . \end{aligned}$$

Introducing an angular parameter θ and adding and subtracting $\vec{\mathbf{E}}_i \cdot D_i\ \vec{\phi}\ sin\theta$ and $\vec{\mathbf{B}}_i \cdot D_i\ \vec{\phi}\ cos\theta$ we find

$$\begin{aligned} M &\geq \tfrac{1}{2}\int_{\mathbb{R}^3}\Big(\|\vec{\mathbf{E}}_i = D_i\vec{\phi} sin\theta\|^2 + \|\vec{\mathbf{B}}_i - D_i\vec{\phi} cos\theta\|^2\Big) \\ &\quad + sin\theta \int_{\mathbb{R}^3} D_i\vec{\phi} \cdot \vec{\mathbf{E}}_i + cos\theta \int_{\mathbb{R}^3} D_i\vec{\phi} \cdot \vec{\mathbf{B}}_i \\ &\geq sin\theta \int_{\mathbb{R}^3} D_i\vec{\phi} \cdot \vec{\mathbf{E}}_i + cos\theta \int_{\mathbb{R}^3} D_i\vec{\phi} \cdot \vec{\mathbf{B}}_i\ . \end{aligned}$$

But now note that

$$\begin{aligned} \int_{\mathbb{R}^3} D_i\vec{\phi} \cdot \vec{\mathbf{B}}_i &= \int_{\mathbb{R}^3} \partial_i(\vec{\phi} \cdot \vec{\mathbf{B}}_i)\ \ by\ \text{Bianchi identity} \\ &= \int_{\Sigma_\infty} \vec{\mathbf{B}}_i ds_i\ \ \text{by Stokes} \\ &= a\int_{\Sigma_\infty} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} \equiv ag\ , \end{aligned}$$

where Σ_∞ is the sphere at spatial infinity and g is the magnetic charge of the solution. Similarly, using the equation of motion instead of Bianchi identity, one find

$$\int_{\mathbb{R}^3} D_i\vec{\phi} \cdot \vec{\mathbf{E}}_i = a\int_{\Sigma_\infty} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} \equiv aq\ ,$$

where q is the electric charge of the solution. Therefore for all angles θ we have the following bound on the mass:

$$M \geq agcos\theta + aqsin\theta\ .$$

The sharpest bound occurs when the right hand side is a maximum, which happens for $q\ cos\theta = g\ sin\theta$ or in terms of the electric and magnetic charges

$$M \geq a\sqrt{q^2 + g^2}\ . \tag{18}$$

This is known as the *Bogomol'nyi bound* for the mass. The saturation for static solution with $\vec{\mathbf{E}}_i = 0$ and $D_0\vec{\phi}$, require that $V(\phi)$ should vanish and in addition the solution must satisfy the Bogomol'nyi equation

$$\vec{\mathbf{B}}_i = \pm D_i\vec{\phi}\ . \tag{19}$$

2.5 The Witten effect

In order to study the different vacua Witten introduced a term in the Lagrangian density which does not affect the equation of motion because it is locally a total derivative and is an integer (called the *instanton number*) times a parameter θ which parametrizes inequivalent vacua. This term is

$$L_\theta = -\frac{e^2\theta}{32\pi^2}\,\vec{\tilde{\mathbf{G}}}^{\mu\nu}\cdot\vec{\mathbf{G}}_{\mu\nu} = \frac{1}{2}\frac{e^2\theta}{32\pi^2}\,\epsilon^{\alpha\beta\mu\nu}\,\vec{\mathbf{G}}_{\alpha\beta}\cdot\vec{\mathbf{G}}_{\mu\nu}\ . \tag{20}$$

Lagrange density reads

$$L + L_\theta = -\frac{1}{4e^2}\vec{\mathbf{G}}_{\mu\nu}\cdot\vec{\mathbf{G}}^{\mu\nu} + \frac{\theta}{32\pi^2}\vec{\mathbf{G}}_{\mu\nu}\cdot\vec{\tilde{\mathbf{G}}}^{\mu\nu} + \frac{1}{2}D^\mu\vec{\phi}\cdot D_\mu\vec{\phi} - V(\phi)\ . \tag{21}$$

The first two terms can be rewritten in a suggestive form defining $\vec{\mathcal{G}}_{\mu\nu} \equiv \vec{\mathbf{G}}_{\mu\nu} + i\vec{\tilde{\mathbf{G}}}_{\mu\nu}$ and $\tau \equiv \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$

$$-\frac{1}{32\pi}Im\,\left(\tau\,\vec{\mathcal{G}}_{\mu\nu}\cdot\vec{\mathcal{G}}^{\mu\nu}\right)\ . \tag{22}$$

Let us see one of the consequences of the presence of this term in the mass bound. For this we consider an operator N which implements a gauge rotation around the $\vec{\phi}$ direction. Under this transformation a vector $\vec{\mathbf{v}}$ and the gauge field transform as follows:

$$\begin{aligned} \delta\vec{\mathbf{v}} &= \tfrac{1}{a}\vec{\phi}\times\vec{\mathbf{v}} \\ \delta\vec{\mathbf{W}}_\mu &= -\tfrac{1}{ea}D_\mu\vec{\phi}\ . \end{aligned} \tag{23}$$

At large distance where $|\phi| = a$, the operator $e^{2\pi iN}$ is a 2π-rotation around $\vec{\phi}$ an therefore exp $(2\pi iN) = 1$. Elsewhere, the rotation angle is $2\pi|\phi|/a$. However, by Gauss' law, if the gauge transformation is 1 at ∞, it leaves the physical states invariant. Thus it is only the large distance behavior of the transformation, which matters and the eigenvalues of N are quantized in integer units. Now we use Noether's formula to compute N:

$$N = \int d^3x\left(\frac{\delta L}{\delta\partial_0\vec{\mathbf{W}}_\mu}\cdot\delta\vec{\mathbf{W}}_\mu + \frac{\delta L}{\delta\partial_0\vec{\phi}}\cdot\delta\vec{\phi}\right)\ . \tag{24}$$

Using the expression above as well as $\delta\vec{\phi} = 0.$ we find

$$N = -\frac{1}{ae}\int_{\mathbb{R}^3}\frac{\delta L}{\delta\partial_0\vec{\mathbf{W}}_i}\cdot D_i\vec{\phi}\ . \tag{25}$$

Since $\vec{\mathbf{E}}_i = \vec{\mathbf{G}}^{0i}$ and $1/2\epsilon^{0i\alpha\beta}\vec{\mathbf{G}}_{\alpha\beta} = B_i$ then

$$N = \frac{1}{ae}\int_{\mathbb{R}^3} E_i \cdot D_i\vec{\phi} + \frac{e\theta}{8\pi^2 a}\int_{\mathbb{R}^3} B_i \cdot D_I\vec{\phi} = \frac{q}{e} + \frac{e\theta}{8\pi^2} g \ . \qquad (26)$$

For the 't Hooft-Polyakov monopole, eg=-4π, hence the integrality of N means that

$$q = ne + \frac{e\theta}{2\pi} \ . \qquad (27)$$

The bound mass of a dyon with charges (q,g) given in (18), can be now written using the quantization condition for monopole charge $g = n_m 4\pi/e$ and for the electric charge $q = n_e e + n_m e\theta/2\pi$

$$M \geq 4\pi a^2 \vec{\mathbf{n}}^t \cdot A(\tau) \cdot \vec{\mathbf{n}} \ , \qquad (28)$$

where $\vec{\mathbf{n}} = (n_e, n_m)^t$ and

$$a(\tau) = \frac{1}{Im\ \tau}\begin{pmatrix} 1 & Re\ \tau \\ Re\ \tau & |\tau|^2 \end{pmatrix} \ . \qquad (29)$$

From here one can read the $SL(2,Z)$-invariance when A transforms as follows:

$$A(\tau) \ \rightarrow \ A(M \cdot \tau) = (M^{-1})^t \cdot A(\tau) \cdot M^{-1} \ , \qquad (30)$$

and

$$\vec{\mathbf{n}} \rightarrow \ M \cdot \vec{\mathbf{n}} \ , \qquad (31)$$

which is related with the Montonen-Olive duality conjecture.

3 Supersymmetry

In nature particles are classified according to their statistics into Fermions and Bosons. Consequently it is natural to divide the Hilbert space into two subspaces $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Supersymmetry is the symmetry between Bosons and Fermions, in this case the algebra must include commutators and anticommutators, building a graduated Lie algebra $[BW]$. The extension of the Lie algebra to a graded algebra allowed Haag, Lopuzanski and Sophnius to avoid the Coleman Mandula no-go theorem and to combine space-time operators and internal symmetry operators in the same algebra.

The algebra of Supersymmetry includes hermitian operators Q_i, $i = 1, \ldots, N$, which map $\mathcal{H}^+$ into $\mathcal{H}^-$ and vice-versa. These operators, like any symmetry operators, must commute with the Hamiltonian operator

$$Q_i H - H Q_i = 0\ .$$

Since the Hamilton Operator is the time translation generator, in order to build a relativistic quantum field theory Q must also commute with the momentum operator P. In four dimensions the sypersymmetry algebra is $[BW, DF]$

$$\begin{aligned} \{Q_a, Q_{\dot{b}}\} &= 2\Gamma^{\mu}_{a\dot{b}} P_\mu\ , \\ \{Q_a, Q_b\} &= \{Q_{\dot{a}}, Q_{\dot{b}}\} = 0\ , \\ [P_\mu, Q_a] &= [P_\nu, Q_{\dot{b}}] = 0\ , \\ [P_\mu, P_\nu] &= 0\ , \end{aligned} \tag{32}$$

where the Latin indices $(a, b, \ldots, \dot{a}, \dot{b}, \ldots)$ run from one to two and denote the spinors indices. The Greek indices $(\mu, \nu, \ldots)$ run from one to four and denote the space-time indices. Γ is a map from the time-like cone V to its spinor space S (spin V) and its dual S^* given by the Pauli Matrices $\Gamma^i, i = 1, 2, 3$ and the Identity. This is the usual vector representation in the spinor algebra.

One way to formulate a Supesymmetry theory is to look for a realization of this algebra in local coordinates and their vector fields, i. e. a local parameterization of the tangent bundle, this method is called superspace mechanism.

In Supersymmetry the usual space must be extended to a superspace which in local coordinate is parameterized by the usual coordinates x^i with $i = 1, \ldots, n$ and a set of Grassmann coordinates θ^i $((\theta^i)^2 = 0)$. When the number of θ coordinates and spacetime coordinates are the same we call it N=1 superspace. In other cases, it is called extended superspace. When the number of θ coordinates are twofold it is called N=2 superspace, fourfold N=4 superspace and so on. A function on superspace is usual written as a polynomial

$$F(x,\theta) = f^0(x) + f^1(x)\cdot\theta + f^2(x)(\theta)^2 + \ldots + f^n(x)(\theta)^n \ , \tag{33}$$

where the $f^i(x)$ are functions under the usual spacetime coordinates, and are called the components of the superfunction $F(x,\theta)$.

In four dimensions, we can array the spacetime coordinates, as a vector in the spinor representation i.e. in a complex 2 x 2 matrix $y^{a\dot{b}} = \Gamma^{a\dot{b}}_{\mu} x^{\mu}$ as follows:

$$\begin{aligned} y^{1\dot{1}} &= \tfrac{x^0+x^1}{2} \ , \\ y^{2\dot{2}} &= \tfrac{x^0-x^1}{2} \ , \\ y^{1\dot{2}} &= \tfrac{x^2+ix^3}{2} \ , \\ y^{2\dot{1}} &= \tfrac{x^2-ix^3}{2} \ . \end{aligned} \tag{34}$$

the Grassmann coordinate as a complex set $\theta^a, \bar{\theta}^{\dot{b}}$, where $(\theta^a)^* = \bar{\theta}^{\dot{a}}$ and the vector fields denoted by $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\partial_a = \frac{\partial}{\partial \theta^a}$ and $\partial_{\dot{b}} = \frac{\partial}{\partial \bar{\theta}^{\dot{b}}}$. This set of elements obey the commutation and Leibniz rules condensed in the following differential algebra calculus

$$\begin{array}{rl} [x^\mu, x^\nu] = 0 & [\partial_\mu, x^\nu] = \delta^\nu_\mu \ , \\ \{\theta^a, \theta^b\} = 0 & \{\partial_a, \theta^b\} = \delta^b_a \ , \\ [\theta^a, x^\mu] = 0 & [\partial_a, \partial_\mu] = 0 \ , \\ [\partial_a, x^\mu] = 0 & [\partial_\mu, \theta^a] = 0 \ , \\ \{\theta^a, \bar{\theta}^{\dot{b}}\} = 0 & \{\partial_a, \bar{\theta}^{\dot{b}}\} = 0 \ . \end{array} \tag{35}$$

All other relations then follow from complex conjugation.

Usual integration can be also extended by a linear and translation invariant integral in the θ coordinates:

$$\int \theta^a d\theta^b = \delta^{ab} \ , \quad \int d\theta = 0 \ . \tag{36}$$

In this algebra we can find a representation of the supersymmetry algebra in the form

$$\begin{aligned} Q_a &= \partial_a + \Gamma^\mu_{a\dot{b}} \bar{\theta}^{\dot{b}} \partial_\mu \\ Q_{\dot{b}} &= \partial_{\dot{b}} + \theta^a \Gamma^\mu_{a\dot{b}} \partial_\mu \\ P_\mu &= \partial_\mu \ . \end{aligned} \tag{37}$$

There is a second set of operators

$$D_a = \frac{\partial}{\partial\theta^a} - \Gamma^{\mu}_{a\dot{b}}\bar{\theta}^{\dot{b}}\partial_\mu \qquad (38)$$
$$\bar{D}_{\dot{a}} = \frac{\partial}{\partial\bar{\theta}^{\dot{a}}} - \theta^b\Gamma^{\nu}_{a\dot{b}}\partial_\nu \ ,$$

satisfying the following relations

$$\{D_a, \bar{D}_{\dot{b}}\} = -2\Gamma^{\mu}_{a\dot{b}}\partial_\mu \qquad (39)$$
$$\{D_a, D_b\} = 0 = \{\bar{D}_{\dot{a}}, \bar{D}_{\dot{b}}\} \ .$$

The operators D and Q anticommute

$$\{D_a, Q_b\} = \{D_a, \bar{Q}_{\dot{b}}\} = \{\bar{D}_{\dot{a}}, Q_b\} = \{\bar{D}_{\dot{a}}, \bar{Q}_{\dot{b}}\} = 0 \ .$$

Now let us develop some simple supersymmtric models.

3.1 Supersymmetric Quantum Mechanics

A simple example of a supersymmetric model can be written for one particle dimension. To built it let us begin with the description of one dimensional model in Classical Mechanics. In this case the superspace coordinates are (t, θ), and the superpoint particle is described by the superfield

$$\Phi(t, \theta) = \phi(t) + \theta\psi(t) \ . \qquad (40)$$

The supersymmetry operators are of the form

$$Q = \partial_\theta + \theta\partial_t \quad \text{and} \quad D = \partial_\theta - \theta\partial_t$$

with the relations

$$\{Q, Q\} = 2q^2 = 2\partial_t \quad \text{and} \quad \{D, D\} = 2D^2 = -2\partial_t \ .$$

The supersymmetric transformation can be written by introducing a Grassmannian parameter η, in order to have a scalar variation $\delta_\eta = -\eta Q$. Under this transformation the superfield transforms as

$$\delta_\eta\Phi = (-\eta Q)\Phi = -\eta\psi(t) + \theta\eta\partial_t\phi(t) \ , \qquad (41)$$

hence, we can write the transformation of the components field as

$$\delta_\eta\phi = -\eta\psi(t) \quad \text{and} \quad \delta_\eta\psi = \eta\partial_t\phi = \eta\dot{\phi} \ . \qquad (42)$$

A supersymmetry invariant Lagrange function can be found from the superspace action

$$S = \int dt d\theta \hat{L} \ ,$$

with $\hat{L} = \frac{1}{2} D\Phi D(D\Phi)$. After the θ integration we find

$$S = \int dt (\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\psi\dot{\psi}) = \int dt L \ .$$

The invariance of this Lagrangian under a supersymmetry transformation can be checked from (42):

$$\delta_\eta L = \partial_t(-\eta\psi\dot{\phi}) \ . \tag{43}$$

In the Lagrange function only the kinetic energies appear. To introduce a potential we could try adding a function of the superfield in the superlagrangian

$$V(\Phi(t,\theta)) = V(\phi(t) + \theta\psi(t)) = V(\phi(t)) + \frac{\partial V}{\partial \Phi}|_{\theta=0}\theta\psi(t) \tag{44}$$

but the term $V(\phi(t))$, whose supersymmetrization we seek, disappears upon θ-integration. This difficulty can be corrected by going to the N=2 supersymmetry. We extend the superspace coordinates to (t, θ^1, θ^2). It can be considered as a reduction of the four dimensional superspace, when $x^1 = x^2 = x^3 = 0$ and the real line in Grassmann coordinates $\theta^a = \bar{\theta}^{\dot{a}}$. In this case we have two generators

$$\begin{aligned} Q_a &= \partial_a + \theta^b \Gamma^0_{ab}\partial_t \\ D_a &= \partial_a - \theta^b \Gamma^0_{ab}\partial_t \end{aligned} \tag{45}$$

with the algebra

$$\{Q_a, Q_b\} = \delta_{ab} 2\partial_t \quad \text{and} \quad \{D_a, D_b\} = -\delta_{ab} 2\partial_t \ , \tag{46}$$

and the superfields are

$$\Phi(t, \theta^1, \theta^2) = \phi(t) + \theta^1\psi_1(t) + \theta^2\psi_2(t) + \theta^1\theta^2 F \ . \tag{47}$$

The action for N=2 now becomes

$$S = \int dt d\theta^1 d\theta^2 \ \frac{1}{2}\epsilon^{ab} D_a\Phi D_b\Phi \ , \tag{48}$$

which after carrying out the θ-integrations, yields

$$S = \int dt \ \frac{1}{2}(\dot{\phi}^2 + \dot{\psi}_1\psi_1 + \dot{\psi}_2\psi_2 + F^2) \ . \tag{49}$$

Here appears a new boson variable F without a kinetic term called *auxiliary field* in physics. The Euler-Lagrange equations yields $F = 0$.

We can now add an interaction term

$$S_{int} = \int dtd\theta^1\theta^2 \; W(\Phi(t,\theta^1,\theta^2)) \;, \tag{50}$$

with an arbitrary function of the superfield Φ, which will maintain the supersymmetric invariance. Expanding and carrying out the θ-integration we find

$$S_{int} = \int dt \; \left\{ \frac{\partial W(\phi(t))}{\partial \phi(t)} F + \frac{\partial^2 W(\Phi)}{\partial \Phi^2}|_{\Phi=\phi(t)} \; \psi_1\psi_2 \right\} \;. \tag{51}$$

The equation of motion for the auxiliary field now gives

$$F = -\frac{\partial W(\phi)}{\partial \phi} \equiv V(\phi) \;. \tag{52}$$

Inserting the interaction in the action we have:

$$S = \int dt \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\psi}_a\psi_a - \frac{1}{2}V(\phi)^2 + \frac{\partial V(\Phi)}{\partial \Phi}|_{\Phi=\phi(t)} \; \psi_1\psi_2 \right] \;. \tag{53}$$

More generally, for a superparticle in a n-dimensional manifold M, with metric $g_{\alpha\beta}$ the Lagrange function reads:

$$L = \frac{1}{2}\dot{\phi}^\alpha\dot{\phi}_\alpha + \frac{1}{2}\psi^\alpha\dot{\psi}_\alpha \;,$$

where the ϕ^α and ψ^α are the bosons and fermion point particle coordinates at t.

A canonical quantization of this model $[DIJ]$ is based on $[\phi_\mu, P_{\phi_\beta}] = i\delta_{\mu\beta}$ and $\{\psi_\alpha, P_{\psi_\beta}\} = i\delta_{\alpha\beta}$ and the superstates are functions in the Hilbert space given by the expansion

$$\Psi(\phi,\psi) = \sum_{k=o}^{n} \Psi_{\alpha_1} \ldots \alpha_k(\phi)\psi^{\alpha_1} \ldots \psi^{\alpha_k} \;, \tag{54}$$

the supersymmetry transformation are given by

$$\delta_\eta\phi^\alpha = -\eta\psi^\alpha, \quad \delta_\eta\psi_\alpha = \eta P_\alpha \;. \tag{55}$$

So that $Q = -i\psi^\alpha P_\alpha = -\psi^\alpha \frac{\partial}{\partial\phi^\alpha}$ acting in the state functions. We find that $Q^2 = 0$ and its cohomology gives the de Rahm cohomology of M.

In the case N=2 we have two supercharges Q_1 and Q_2. Complexifying them $\tilde{Q} = (Q_1 + iQ_2)/2$ shows that $\tilde{Q}$ and $\bar{\tilde{Q}}$ can be interpreted as d and d^* and the ground states

$$|\tilde{Q}||\Psi >= \bar{\tilde{Q}}|\Psi >= 0 \ ,$$

hence are harmonic differential forms.

3.2 Supersymmetric Field Theory in 1+1 Dimension

Let us construct an N=1 supersymmetric model with the superspace mechanism. We begin by identifying the two-dimensional superspace a reduction of the four dimensional one by setting $x^2 = x^3 = 0$ and real fermion coordinates $\theta^a = \bar{\theta}^{\dot{a}}$. We shall work in the light cone coordinates described by $x^+ = x^1 + x^2$ and $x^- = x^1 - x^2$. In this model a superfield can be described by

$$\Phi(\mathbf{x}, \theta^1, \theta^2) = \phi(\mathbf{x}) + \theta^1\psi_1 + \theta^2\psi_2 + \theta^1\theta^2 F \tag{56}$$

and the supersymmetry generators are

$$\begin{array}{ll} Q_1 = \partial_1 + \theta^1\partial_+ & Q_2 = \partial_2 + \theta^2\partial_- \\ D_1 = \partial_1 - \theta^1\partial_+ & D_2 = \partial_2 - \theta^2\partial_- \ . \end{array} \tag{57}$$

The kinetic part of the Lagrange function is of the form

$$S = \int dx d\theta^1 d\theta^2 \frac{1}{4}\epsilon^{ab} D_a\Phi D_b\Phi \tag{58}$$

and the interaction can be introduced, like in the one dimensional case, by the superpotential $W(\Phi)$. After the integration of the fermionic coordinates and solving the auxiliary field with the Euler-Lagrange equations ($F = -W(\phi) = V(\phi)$) we find

$$S = \int d\mathbf{x}[\frac{1}{2}\partial_+\phi\ \partial_-\phi + \frac{1}{2}\partial_-\psi_1\ \psi_1 + \frac{1}{2}\partial_+\psi_2\ \psi_2 - \frac{1}{2}V(\phi)^2 - \frac{\partial V}{\partial\Phi}(\phi)\ \psi_1\ \psi_2] \ . \tag{59}$$

TThe conserved supersymmetric current can be calculated and its charges are

$$\begin{array}{l} Q_1 = \int dx \partial_-\phi\psi_1 - V(\phi)\psi_2 \\ Q_2 = \int dx \partial_+\phi\psi_2 + V(\phi)\psi_1 \ . \end{array} \tag{60}$$

By a change of coordinates of the ψ-space $\psi_- = -\psi_1, \ \ \psi_+ = \psi_2$ we find the usual chiral charges $[OW]$

$$Q_- = \int dx \mid \partial_- \phi \psi_- + V(\phi)\psi_+ \mid \qquad (61)$$
$$Q_+ = \int dx \mid \partial_+ \phi \psi_+ - V(\phi)\psi_- \mid \ .$$

Witten and Olive showed a Bogomol'nyi bound for this model that we will reproduce in this lectures. The supersymmetric algebra relations for these charges have an extension:

$$Q_+ \, Q_- + Q_- \, Q_+ = \int dx \; 2V(\phi)\partial\phi/\partial x \ , \qquad (62)$$

which can also be written

$$Q_+ \, Q_- + Q_- \, Q_+ = \int dx \; \partial/\partial x \; (2H(\phi)) \equiv Z \ , \qquad (63)$$

where $H(\phi)$ is a function such that $H'(\phi) = V(\phi)$. For a soliton state this integral is not necessarily zero.

In fact, it is the operator Z on the r.h.s. which is used to extend the supersymmetry algebra. A matrix element Z, is the difference between the expectation value of $2H(\phi)$ at $x = +\infty$ and $x = -\infty$.

For a typical example, we may let $V = -\lambda(\phi^2 - a^2)$, so the potential energy is $\lambda^2(\phi^2 - a^2)^2$. The theory has two ground states, $\phi = \pm a$. $H = a^2\lambda\phi - \frac{1}{3}\lambda\phi^3$, and this model has

$$Z = \int_{-\infty}^{+\infty} dx \frac{\partial}{\partial x}(2a^2\lambda \, \phi - \frac{2}{3}\lambda\phi^3) \ . \qquad (64)$$

Z vanishes in a topologically trivial state, has a positive value in a kink state, a negative value in an antikink state.

Our modified algebra is now

$$Q_+^2 = P_+, \quad Q_-^2 = P_-, \quad Q_+ \, Q_- + Q_- \, Q_+ = Z \ ,$$

and has an interesting consequence. From the algebra we find

$$P_+ + P_- = Z + (Q_+ - Q_-)^2 \ , \qquad (65)$$
$$P_+ + P_- = -Z + (Q_+ + Q_-)^2 \ .$$

But $(Q_+ + Q_-)^2 \ \geq \ 0$, so $P_+ + P_- \ \geq \ |Z|$. For a single particle the mass M at rest, $P_+ = P_- = M$, implies

$$M \geq \ \frac{1}{2} \, |Z| \qquad (66)$$

called the Bogomol'nyi bound or BPS bound. This equation is saturated by the states $|\alpha >$ such that $(Q_+ - Q_-)|\alpha >= 0$ or $(Q_+ + Q_-)|\alpha >= 0$

which is satisfied at least classically by the soliton and antisoliton ϕ ($\partial\phi/\partial t = 0$, $and\ \partial\phi/\partial x = \pm V(\phi)$) states.

3.3 In 4 dimensions

In four dimension the procedure is more complicated but in principle it follows the same mechanism. It is known to work better in a subspace of the superfields [BW] called scalar or chiral field ($\bar{D}\Phi = 0$) or vector field ($V^+ = V$). It is also possible to write a scalar field from a vector field ($W_\alpha = \frac{1}{4}\bar{D}\bar{D}D_\alpha V$) and with them in a special gauge choice (Wess-Zumino gauge) we have in a component the Yang-Mill field strength part arise as a component.

The class of renormalized Lagrangians may be restricted by a global invariance called R-invariance. R acts on chiral multiplets as follows:

$$R\ \Phi(x,\theta) = e^{2in\alpha}\Phi(x, e^{-i\alpha}\theta)$$

$$R\ \Phi^+(x,\bar{\theta}) = e^{-2in\alpha}\Phi^+(x, e^{i\alpha}\bar{\theta})\ ,$$

where $n \in N$ and $\alpha \in R$. In N=1 Supersymmetry the full Lagrangian can be written as:

$$\begin{aligned} L = & \tfrac{1}{8\pi} Im\left(Tr\ \tau \int d^2\theta W^\alpha W_\alpha\right) + \int d^2\theta d^2\bar{\theta}\ Tr\ \Phi^\dagger e^{-2V}\Phi \\ & + \int d^2\theta \mathcal{W} + \int d^2\bar{\theta}\bar{\mathcal{W}}\ . \end{aligned} \tag{67}$$

In components the expansion after θ-integration looks like

$$\begin{aligned} L = & -\tfrac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \tfrac{\theta}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} - \tfrac{i}{g^2}\lambda^a\sigma^\mu D_\mu\bar{\lambda}^a + \tfrac{1}{2g^2}D^aD^a \\ & + (D_\mu\phi)^\dagger D^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu D_\mu - D^a\phi^\dagger T^a\phi - i\sqrt{2}\phi^\dagger T^a\lambda\psi \\ & + i\sqrt{2}\bar{\psi}T^a\lambda\phi\bar{\lambda}^a + F_i^\dagger F_i + \tfrac{\partial\mathcal{W}}{\partial\phi_i}F_i + \tfrac{\partial\bar{\mathcal{W}}}{\partial\phi_i^\dagger}F_i^\dagger \\ & - \tfrac{1}{2}\tfrac{\partial^2\mathcal{W}}{\partial\phi_i\partial\phi_j}\psi_i\psi_j - \tfrac{1}{2}\tfrac{\partial^2\mathcal{W}}{\partial\phi_i^\dagger\partial\phi_j^\dagger}\bar{\psi}_i\bar{\psi}_j\ , \end{aligned} \tag{68}$$

where $\mathcal{W}$ denote the scalar component of the potential and the T^a are the generators in the group. The auxiliary fields F and D^a can be eliminated by equations of motion to get the scalar potential

$$V = \sum_i \|\frac{\partial\mathcal{W}}{\partial_i}\|^2 - \frac{1}{2}g^2(\phi^\dagger T^a\phi)^2\ . \tag{69}$$

In N=2 SUSY we have to introduce a second copy of the field and the two fields can be organized in a multiplet under N=1. The most general Lagrangian can be constructed using the chiral superfield Ψ

$$L = \frac{1}{4\pi} Im\; Tr \int d^2\theta d^2\tilde{\theta} \mathcal{F}(\Psi) \;\text{ or}$$

$$L = \frac{1}{8\pi} Im \left(\int d^2\theta \mathcal{F}_{rs} W^{r\alpha} W^s_\alpha + 2\int d^2\theta d^2\bar{\theta} (\Phi^\dagger e^{2gV})^r \mathcal{F}_s(\Phi) \right) \;, \tag{70}$$

where $\mathcal{F}$ is a holomorphic function, $\mathcal{F}_s(\Phi) = \partial\mathcal{F}/\partial\Phi^s$, $\mathcal{F}_{rs}(\Phi) = \partial^2\mathcal{F}/\partial\Phi^r\partial\Phi^s$, Ψ is a chiral field in N=1 and $\mathcal{F}$ is referred to as the N=2 prepotential. From here we can read a Kähler structure with a Kähler prepotential $\mathrm{Im}\Phi^\dagger\mathcal{F}_s(\Phi)$. Writing in components the electromagnetic and scalar part with a potential $\mathcal{F} = \tau\phi^2$ yields

$$L = \frac{1}{g^2} Tr(-\frac{1}{4}F_{\mu\nu}F + g^2\frac{\theta}{32\pi^2}F_{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi - \frac{1}{2}[\phi^\dagger,\phi]^2 + FERMIONS) \;. \tag{71}$$

In the case of a SU(2) gauge group the potential is

$$V(\phi) = \frac{1}{2}[\phi^\dagger,\phi]^2 = \frac{1}{2}(f^{abc}\phi^a\phi^b)^2 \;. \tag{72}$$

4 The Seiberg-Witten Analysis

In this section we show the analysis of Seiberg-Witten using duality and the BPS bound of the mass (M$\geq \sqrt{2}\|a(n_e + i/\alpha\ n_m)\|$) to describe the Moduli space for the SU(2) symmetry as an elliptic curve.

4.1 Moduli space

The scalar potential in N=2 has a Higgs vacuum in $[\phi, \phi^\dagger] = 0$, which implies that ϕ takes values in the Cartan subalgebra of the gauge group $\phi = \phi_i H^i$. Thus, the gauge group G is broken to the subgroup H which is generated by elements from the Cartan subalgebra. Elements in G/H do not leave the Higgs vacuum invariant, but being gauge transformations they relate the inequivalent vacua. However, there is still an invariance of the potential under Weyl reflections. Therefore, the correct parameterization of the different vacua is not given by $\phi = \frac{1}{2}\ a\ \sigma_3$, but by Weyl invariant functions. These are obtained by the coefficients of the characteristic equation

$$det(\lambda - \phi) = 0\ .$$

In the case of SU(2) the range is one. The Weyl invariant parametrization of the vacua is given by $u = (Tr\phi^2) = a^2$.

4.2 Duality

To find the duality transformation we consider the gauge terms in the bosonic part of the action.These terms can be written:

$$\frac{1}{32\pi} Im \int \tau(a)(F + i\tilde{F})^2 = \frac{1}{16\pi} Im \int \tau(a)(F^2 + i\tilde{F}F)\ .$$

Now, as in the above first part, we regard F as an independent field and implement the Bianchi Identity dF=0 by introducing a Lagrange multiplier vector field V_D which is coupled with a monopole:

$$\frac{1}{8\pi}\int V_{D\mu}\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = \frac{1}{8\pi}\int \tilde{F_D}F = \frac{1}{16\pi} Re \int (\tilde{F}_D - iF_D)(F + i\tilde{F})\ ,$$

where $F_{D\mu\nu} = \partial_\mu V_{D\nu} - \partial_\nu V_{D\mu}$. Adding this term to the gauge field action and integrating over F, gives the dual theory:

$$\frac{1}{32\pi} Im \int \left(-\frac{1}{\tau}\right)(F_D + i\tilde{F}_D)^2 = \frac{1}{16\pi} Im \int \left(-\frac{1}{\tau}\right)(F_D^2 + i\tilde{F}_D F_D)\ .$$

Thus the effect of the duality transformation is to replace a gauge field which couples to electric charge by a dual gauge field which couples to magnetic charges, and at the same time, transforms the gauge coupling:

$$\tau \to \tau_D = -\frac{1}{\tau} \ .$$

The action is also invariant under the transformation $\tau \to \tau + 1$. Both transformations generate the $SL(2,Z)$ group, which is identified to the full duality group. This group acts on τ by:

$$\tau \to \frac{a\tau + b}{c\tau + d} \ ,$$

where, ab-cd=1 and a, b, c, d $\in$ Z.

In the above $N = 2$ supersymmetric formulation let $\tau(\phi) = \partial h(\phi)/\partial\phi$ where $h(\phi) = \partial\mathcal{F}/\partial\phi$. In this description we identify $A_D = h = \partial\mathcal{F}/\partial\phi$ and $h_D = -\phi$. The kinetic term which appears as second term in (70)

$$Im \int d^4\theta h(\phi)\phi^\dagger = Im \int d^4\theta h_D(\phi_D)\phi_D^\dagger \ ,$$

is also invariant.

The action of this group on ϕ and on ϕ_D can be rewritten as follows

$$\begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \to \begin{pmatrix} a \ b \\ c \ d \end{pmatrix} \begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \ .$$

The transformations $\tau \to -1/\tau$ and $\tau \to \tau + 1$ are implemented by the matrices

$$S = \begin{pmatrix} 0 \ 1 \\ 1 \ 0 \end{pmatrix} \ , \qquad T = \begin{pmatrix} 1 \ 1 \\ 0 \ 1 \end{pmatrix} \ ,$$

which are the generators of SL(2,Z).

Let us see how this invariance can be interpreted in the Moduli space. From the scalar part of the Lagrange action we interpret the vacuum space as a Kähler space where the metric is

$$ds^2 = g_{i\bar{j}} \ da^i \ d\bar{a}^j = Im \ \frac{\partial^2\mathcal{F}}{\partial a_i \partial a_j} \ da^i \ da^j \ . \tag{73}$$

Now we can use the abpve identification to write

$$ds^2 = Im \ da_D \ d\bar{a} = -\frac{i}{2} \left(da_D \ d\bar{a} - da \ d\bar{a}_D \right) \ , \tag{74}$$

which is invariant under SL(2,Z). From here we can identify a and a_D as coordinates on a space $X \cong \mathbb{C}$ on which we can choose a symplectic form

$w = Im\ da_D \wedge d\bar{a}$. The functions $(a_D(u), a(u))$ gives a map from $\mathcal{M}$ to X. In other words, they determine a section of X regarded as an $SL(2, Z)$ bundle over $\mathcal{M}$. In terms of the coordinate u the metric on the moduli space takes the form

$$ds^2 = Im\ \frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}}\,dud\bar{u} = -\frac{i}{2}\left(\frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}} - \frac{d\bar{a}_D}{d\bar{u}}\frac{da}{du}\right)dud\bar{u}\ . \tag{75}$$

4.3 Seiberg-Witten analysis

In the N=2 Supersymmetry when we choose a vacuum $\phi = \frac{1}{2}\ a\sigma_3$ with a$\neq 0$ the SU(2) is broken into a U(1) symmetry and by the Higgs mechanism $W^{\pm}$ acquires a mass. It is clear that this theory only has a meaning outside a neighborhood of $u = 0$, since at this point the $W^{\pm}$ become massless. This tells us that $u = 0$ is a singular point in the classical moduli space $\mathcal{M}_{\mathcal{C}}$. The other singularity is at $u = \infty$ where the coupling is weak because of asymptotic freedom.

It is known that a N=2 supersymmetric Yang-Mills theory is not exact but the quantum corrections are under complete control$[SEI]$. It is governed by the holomorphic prepotential $\mathcal{F}$ and it receives all corrections at the one loop level. As one could expect from a mathematical point of view, the non triviality of the metric arises from a (logarithmic) singularity of F:

$$\mathcal{F}(\phi) = \frac{1}{2}\tau_c\phi^2 + \frac{i}{\pi}ln[\frac{\phi^2}{\Lambda^2}]\ , \tag{76}$$

where Λ is a scale factor. From this we can compute in the region of weak coupling, near to $u = \infty$

$$\tau(u) = \tau_c + \frac{2i}{\pi}ln[\frac{u}{\Lambda^2}]. \tag{77}$$

This implies that if we loop around $u = \infty$ in moduli space, the logarithm will produce an extra shift of $2\pi i$ and thus

$$\tau \rightarrow \tau - 4\ .$$

From this it is clear that τ and $\mathcal{F}$ are multivalued function, moreover $Im\ \tau(u)$ is a metric must be positive definitive but it is a harmonic function therefore if it is globally defined it must be constant. The question is how many and what kind of singularities the exact quantum moduli space should have, and what the physical significance of these singularities might be. Seiberg and Witten $[SW1]$ proposed two singularities at $u = \pm\Lambda^2$and the classical singularity at the origin disappears. This proposition is very difficult to prove rigorously and it was postulated on the grounds of an analysis of the strong behavior guided by the 't Hooft ideas about confinement. Like in classical u=0 singularity, the strong coupling singularities in the quantum moduli space were attributed to certain excitations with the gauge bosons becoming massless as well.

The BPS bound is a powerful tool to handle soliton masses. Like in the 1+1 dimensional supersymmetric model above, the bound is given by the central charge of the algebra in question:

$$m^2 \geq |Z^2| \tag{78}$$

For N=2 supersymmetric Yang-Mills theories, it take the form

$$Z = qa + ga_D \tag{79}$$

where q, resp. g are the magnetic, resp. electric quantum numbers of the BPS states under consideration. From the expression for the quantum $\mathcal{F}$ at $u = \Lambda^2$, $a \neq 0$ but $a_D = 0$ so that a monopole hypermultiplet with charges (q,g)=$(\pm 1, 0)$ would be massless. On the other hand at $u = -\Lambda^2$ obtain from $u \to -u$ and is similar to that obtain at $u = \Lambda^2$ replacing a_D in $\mathcal{F}_D(a_D)$ by $a_D - 2a$.

The key idea is to patch together the local information in a consistent global way to fix the theory. The local term determines the local monodromy M around a given singularity, and this acts on the section as

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \to M \begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} .$$

In particular, the asymptotic behavior at large $\|a\|$ is given by

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \cong \begin{pmatrix} \frac{i}{\pi}\sqrt{\frac{u}{2}}\, ln(u/\Lambda^2) \\ \sqrt{u/2} \end{pmatrix} . \tag{80}$$

The monodromy at ∞ is calculated by a loop on the u-plane around u=0. We get $ln\ u + 2\pi i$, hence

$$\begin{aligned} a_D &\to -a_d + 2a \ , \\ a &\to \quad -a \ . \end{aligned} \tag{81}$$

The monodromy matrix implements this

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} . \tag{82}$$

The magnetic and the electric quantum numbers of BPS states undergo a change $(q, g) \to (-q - 2g, -g)$ so that the mass formula is unchanged. For the other monodromies at M_{Λ^2} and $M_{-\Lambda^2}$ we use other strategies. Under the transformation the mass bound must be invariant therefore the charges (q,g) transform under the inverse matrix i.e if

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} \alpha a_D + \beta a \\ \gamma a_D + \delta a \end{pmatrix} \quad \text{then} \quad \begin{pmatrix} g \\ q \end{pmatrix} \to \begin{pmatrix} \delta g - \gamma a \\ -\beta g + \alpha q \end{pmatrix} , \tag{83}$$

where $\alpha\delta - \beta\gamma = 1$. Now we choose the parameters α, β, γ, δ in order to have g=0 and q=1 after the transformation. So we can identify the matrix in the form

$$M_\infty = \begin{pmatrix} 1 + 2gq & 2q^2 \\ -2g^2 & 1 - 2qg \end{pmatrix} . \tag{84}$$

Since we assume that the singularity are only at $u = \infty$ and $u = \pm\Lambda^2$,

$$M_{\Lambda^2}(g,q) M_{-\Lambda^2}(g',q') = M_\infty \tag{85}$$

A simple solution to this equation is given by

$$M_{\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} , \quad M_{-\Lambda^2} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} , \tag{86}$$

where the monopole $(q, g) = (0, \pm 1)$ and the dyons $(q, g) = (-2, 1)$ becomes massless

4.4 Solving the Monodromy Problem

In Mathematics the problem of finding multivalued functions a(u), $a_D(u)$ with this monodromy at the singularities has a unique solution and it is called the Riemann-Hilbert problem. We can consider $a(u)$, $a_D(u)$ as certain periodic integrals related to some surface X, which can be seen as a spectral surface.

The matrices $M_{\pm\Lambda^2,\infty}$ generate a subgroup $\Gamma_0(4)$ of $SL(2, Z)$ built from matrices of the form

$$\Gamma_0(4) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z), b = 0 \, mod \, 4 \} .$$

The quantum moduli space can thus be viewed as the upper half-plane modulo the monodromy group:

$$\mathcal{M} \cong H^+/\Gamma_0(4) .$$

The appearance of a subgroup of the modular group suggests that one can formulate the manifold as a toroidal Riemann surfaces of an elliptic curve. This is algebraically characterized by

$$y^2(x,u) = \prod_{i=1}^{4}(x - e_i(u,\Lambda)) = (x^2 - u)^2 - \Lambda^4 \ . \tag{87}$$

The period matrix of the torus is interpreted as the gauge coupling. From the theory of Riemann surface, the periodic matrix is defined by

$$\tau(u) = \frac{\bar{w}_D(u)}{\bar{w}(u)} \ , \tag{88}$$

where

$$\bar{w}_D(u) = \oint_\beta w, \quad \bar{w}(u) = \oint_\alpha w \ .$$

Here $w = \frac{1}{\sqrt{2}\pi}\frac{dx}{y(x,u)}$ and α, β are the canonical basis homological cycles of the torus.

From the relation $\tau = \partial a_D/\partial a$ we find

$$\bar{w}_D(u) = \frac{\partial a_D(u)}{\partial u} \ , \quad \bar{w}(u) = \frac{\partial a(u)}{\partial u} \ ,$$

which implies

$$a_D(u) = \oint_\beta \lambda_{SW}, \quad a(u) = \oint_\alpha \lambda_{SW} \ ,$$

where

$$\lambda_{SW} = \frac{1}{\sqrt{2}\pi} x^2 \frac{dx}{y(x.u)} \ . \tag{89}$$

The four zeroes of $y^2(x,u) = 0$ are

$$\begin{array}{ll} e_1 = -\sqrt{u+\Lambda^2} \ , & e_2 = -\sqrt{u-\Lambda^2} \\ e_3 = -\sqrt{u-\Lambda^2} \ , & e_4 = \ \ \sqrt{u+\Lambda^2} \end{array} \ .$$

The singularities in the quantum moduli space arise when the torus degenerates, and this happens when any two of the zeros e_i coincide.

The calculation for other groups can be read in $[KLT, L]$.

References

[AH] Alvarez-Gaume, L. and Hassan, S.F., *Introduction to S-Duality in N=2 Supersymmetric Gauge Theories*, Fortsch.Phys 45 (1997) 159-236, hep-th 9701069

[BW] Bagger, J. and Wess, J., *Supersymmetry Supergravity*, Princeton Univresity Press 1998

[CHL] Cheng, T., Li, L., *Gauge Theory of Elementary Particle Physics*, Oxford University Press, 1989

[DF] Deligne, D., Freed, S., *Supersolutions*, Quantum Fields and Strings: A Course for Mathematicians, Am. Math Society and Institute for Advanced Study 1999

[DIJ] Dijkgraaf, R., *Fields, String and Duality*, Les Houches 1995

[FIG] Figueroa-O'Farrill, J.M. ,*Electromagnetic Duality for Children*, Draft, 1997.

[Fr] Freund, P., *Supersymmetry*, Cambridge Monographs on Mathematics Physics 1988

[KLT] Klemm, A., Lerche, and Theisen, S., *Nonperturbative Effective Actions of N=2 Supersymmetric Gauge Theories*, Int.J.Mod. A11 (1996)1929-1974, hep-th 9505150

[L] Lerche, W., *Introduction to Seiberg-Witten Theory and its String Origin*, Nucl.PhysProc.Suppl 55B (1979) 83-117; Fortsch. Phys. 45 (1997) 293-340, hp-th 9611190

[OW] Olive, D. and Witten, E., *Supersymmetry Algebras That Include Topological Charges*, Phy. Lett. B78 (1978) 97-101.

[Pol] Polyakov A. M., *Particles spectrum in the quantum field theory*, JETP Lett. 20 (1974) 194-195

['tH] 't Hooft G., *Magnetic monopoles in unified gauge theories*, Nucl. Phys B79 (1974) 276-284.

[TSOU] Tsou Sheung Tsun, *Concepts in Gauge Theory Leading to Electromagnetic Duality*, In this volume

[SEI] Seiberg, N., *Supersymmetry and Non-Perturbative Beta Functions*, Phy. Lett B206 (1988) 75-80.

[SW1] Seiberg, N. and Witten, E., *Electromagnetic Duality, Monopole Condensation, and Confinement In N=2 Supersymmetric Yang-Mills Theory*, Nucl. Phys, B426 (1994) 19, hep-th 9407087.

[SW2] Seiberg, N. and Witten, E., *Monopole, Duality and Chiral Symmetry Breaking in N=2 Supersymmetric QCD* , Nucl. Phys. B431 (1994) 484, hep-th 9408099.