REMARKS ON DUALITY, ANALYTIC TORSION AND GAUSSIAN INTEGRATION IN ANTISYMMETRIC FIELD THEORIES

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From a path integral point of view (e.g. [Q98]) physicists have shown how *duality* in antisymmetric quantum field theories on a closed space-time manifold M relies in a fundamental way on Fourier Transformations of formal infinite-dimensional volume measures. We first review these facts from a measure theoretical point of view, stressing the importance of the Hodge decomposition theorem in the underlying geometric picture, ignoring the local symmetry which lead to degeneracies of the action. To handle these degeneracies we then apply Schwarz's Ansatz showing how duality leads to a factorization of the analytic torsion of M in terms of the partition functions associated to degenerate "dual" actions, which in the even dimensional case corresponds to the identification of these partition functions.

Introduction

Antisymmetric field theories are generalizations of electromagnetic theory where the potential 1-form is replaced by a k-form. Some remarkable facts arising in electromagnetism are also observed in general antisymmetric theories, notably T-*duality* on which we will focus here. In electromagnetic theory this type of duality corresponds to the observation that electric and magnetic fields in the theory are interchanged under transformations taking solutions of field equations into solutions of the Bianchi identity, particles into topological defects, weak couplings into strong couplings, etc. (for a review see [O95]).

Consider a theory of antisymmetric tensors on a n-dimensional space-time manifold M equpped with a Riemannian metric. Let $\omega_{i_1 i_2 \ldots i_k}$ be a k-tensor field on M, consider the k-form

$$\omega_k = \omega_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k},$$

for $0 \leq k \leq n$, and the *Euclidean Action* of the theory defined by

$$\mathcal{S}(\omega_k) = \langle d_k \omega_k, d_k \omega_k \rangle, \tag{1}$$

where d_k denotes the exterior derivative on the space Ω^k of k-forms on M and the inner product $(\, ,) : \Omega^k \times \Omega^k \to \mathbb{R}$ is defined by Hodge-star operation on Ω^k, namely

$$(\alpha_k, \beta_k) = \int_M \alpha_k \wedge *\beta_k. \tag{2}$$

This generalizes electromagnetic theory, where the potential is described by a 1-form A, the electromagnetic field by its exterior derivative ($F = dA$), and where by *gauge invariance* of the theory we mean the invariance of F under "gauge" transformations on A of the form

$$A \mapsto A + d\chi, \tag{3}$$

χ being an arbitrary function (0-form) on M.

Following [$Q98$], T-duality in the case of antisymmetric field theories is the statement that two different theories (defined by two different actions $\mathcal{S}$ and $\mathcal{S}^*$) give rise to the same generating function, being therefore (at the quantum level) physically equivalent. As in [$Q98$] [$W99$] and many other references on this topic, in this paper we focus on the identification of partition functions, hoping to complete the discussion on the level of generating functions in some later work.

By partition function we understand the formal object

$$Z(S) = \int \exp\{-\mathrm{k}\, \mathcal{S}(\alpha)\}\ [\mathcal{D}\alpha] \tag{4}$$

where k denotes a (positive) constant (including Planck's and coupling constants), $[\mathcal{D}\alpha]$ denotes a formal measure on the space of all the fields α and $\mathcal{S}(\alpha)$ the classical action of the theory under consideration. Looking for a dual version of a theory means looking for a different action, called *dual* action (on a different set of *dual* fields), giving rise to a *dual* partition function. Starting from a given action (i.e a given theory), a standard procedure to obtain a dual action (i.e. a dual theory) is the so-called *gauging* of the global symmetry of the original theory [$Q98$][$W99$]. This requires introducing new variables into the original action in such a way that integrating them out we can recover the original theory and integrating out the original variables of the action we find the dual one. Unlike in [$Q98$], we only consider local symmetries (namely of the type (3)) of the classical action, having left aside global symmetries because of the acyclicity assumption (see section 1). The very presence of local symmetries leads to degenerate actions. Forgetting about degeneracy of the classical action, as was pointed out in [$Q98$], duality strongly relies on Fourier transformations of measures. We follow this point of view in section 2. On the other hand, if one wants to take into account the presence of local symmetries, a method is required to handle partition functions with degenerate actions. In the context of what is now called Topological Quantum Field Theories, Schwarz proposed an Ansatz to compute such partition functions which we apply in section 3, this leading us to an interpretation of duality of partition functions in terms of a factorization of the analytic torsion of the underlying space-time manifold. In even dimensions this give the expected identification of the partition function of an action with its dual.

Let us describe briefly the contents of this contribution. In section 1 we describe the geometric setting underlying the definition of antisymmetric tensor fields. In section 2 we give a measure theoretical interpretation of the formal path integral manipulations in the case of duality between two antisymmetric field theories defined by non degenerate action functionals and, following Quevedo [$Q98$], we give the heuristic path integral interpretation of duality in terms of Fourier transformation of measures. In section 3 we use the approach proposed by Schwarz [$S79$] to study the partition function of a degenerate functional, and we show how two dual actions yield a factorisation of the analytic torsion on the underlying manifold.
To distinguish between formal (heuristic) equalities from precise mathematical ones we shall use the symbol " = " for the first kind.

1 The Geometric Setting

Consider a closed (i.e. compact and without boundary) n-dimensional Riemannian manifold M, and let ρ be a representation of the fundamental group of M on an inner product vector space V. Let $E(\rho)$ be the vector bundle over M defined by ρ, and consider the space of k-forms on M with values in $E(\rho)$, for $0 \leq k \leq n$, i.e. C^∞-sections of the vector bundle $\Lambda^k T^*M \otimes E(\rho)$. This space of sections, that we will denote by Ω^k, will be the space of k-antisymmetric tensor fields. The bundle $E(\rho)$ comes with a flat connection that couples with exterior differentiation on k-forms to define an exterior differential (also denoted d_k), on $E(\rho)$-valued k-forms, such that $d_k^2 = 0$. The inner product (2) on $\Lambda^k T^*M$, defined by the Riemannian metric on M, thogether with the inner product on $E(\rho)$, provides Ω^k with an inner product that we will denote by $\langle \, , \rangle$. With respect to that inner product, and by Hodge $*$-duality map, $d_k^* = (-1)^{nk+n+1} * d_{n-k-1} *$ defines the formal adjoint of d_k.

Finally, we assume that the complex

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \cdots \Omega^{k-1} \xrightarrow{d_{k-1}} \Omega^k \xrightarrow{d_k} \Omega^{k+1} \xrightarrow{d_{k+1}} \cdots \Omega^n \xrightarrow{d_n} 0, \tag{5}$$

is acyclic, i.e. all the de Rham cohomology groups of the complex are trivial ($H^k(M,\rho) = \{0\}$, $0 \leq k \leq n$). This representation of $\pi_1(M)$ will be fixed through all the paper and no specific reference to it will be given (in the notation) in the sequel.

Let us focus on the space of k-forms

$$\Omega^{k-1} \xrightarrow{d_{k-1}} \Omega^k \xleftarrow{d_k^*} \Omega^{k+1},$$

where d_k^* denotes the formal adjoint to d_k, and on the Hodge decomposition

$$\Omega^k = \Omega'_k \oplus \Omega''_k \tag{6}$$

where $\Omega'_k = \text{Im}\ d_{k-1} = \text{Ker}\ d_k$ and $\Omega''_k = \text{Im}\ d_k^* = \text{Ker}\ d_{k-1}^*$, as follows from our asumption of acyclicity. Accordingly, $\omega_k \in \Omega^k$ splits into $\omega_k = \omega'_k \oplus \omega''_k$ where $\omega'_k = d_{k-1}\omega_{k-1} \in \Omega'_k$ and $\omega''_k = d_k^*\omega_{k+1} \in \Omega''_k$, for some $\omega_{k-1} \in \Omega^{k-1}$, $\omega_{k+1} \in \Omega^{k+1}$.

Consider the functional

$$\begin{aligned} \mathcal{S}_o : \Omega^k &\to \ \mathbb{R} \\ \omega_k &\mapsto \mathcal{S}_o(\omega_k) = \langle \omega_k, \omega_k \rangle, \end{aligned} \tag{7}$$

on k-antisymmetric tensor fields. Then, from the decomposition (6) and $d_k d_{k-1} = d_{k-1}^* d_k^* = 0$, it follows that

$$\begin{aligned} \mathcal{S}_o(\omega_k) &= \langle d_{k-1}\omega_{k-1} \oplus d_k^*\omega_{k+1},\ d_{k-1}\omega_{k-1} \oplus d_k^*\omega_{k+1} \rangle \\ &= \langle d_{k-1}\omega_{k-1},\ d_{k-1}\omega_{k-1} \rangle \oplus \langle d_k^*\omega_{k+1},\ d_k^*\omega_{k+1} \rangle. \end{aligned}$$

Thus, we find a canonical decomposition of $\mathcal{S}$ in terms of two degenerate action functionals, namely

$$\mathcal{S}_o(\omega_k) = \mathcal{S}(\omega_{k-1}) \oplus \mathcal{S}^*(\omega_{k+1}), \tag{8}$$

where

$$\mathcal{S}(\omega_{k-1}) = \langle d_{k-1}\omega_{k-1},\ d_{k-1}\omega_{k-1} \rangle \tag{9}$$

and

$$\mathcal{S}^*(\omega_{k+1}) = \langle d_k^*\omega_{k+1},\ d_k^*\omega_{k+1} \rangle, \tag{10}$$

which are degenerate on Ω^{k-1} and Ω^{k+1}, respectively. The functionals $\mathcal{S}(\omega_{k-1})$ and $\mathcal{S}^*(\omega_{k+1})$ are degenerate but, by restriction on the respective domains, the maps

$$d_k : \Omega''_k \to \Omega'_{k+1} \tag{11}$$

and

$$d^*_k : \Omega'_{k+1} \to \Omega''_k, \tag{12}$$

are isomorphisms, giving rise to the bijective maps

$$d^*_{k-1} d_{k-1} : \Omega''_{k-1} \;\to\; \Omega''_{k-1},$$

$$d_k d^*_k : \Omega'_{k+1} \to \Omega'_{k+1}.$$

Thus, the functionals

$$\widehat{\mathcal{S}}(\omega''_{k-1}) = \langle d_{k-1}\omega''_{k-1},\, d_{k-1}\omega''_{k-1}\rangle \tag{13}$$

and

$$\widehat{\mathcal{S}}^*(\omega'_{k+1}) = \langle d^*_k\omega'_{k+1},\, d^*_k\omega'_{k+1}\rangle, \tag{14}$$

are non-degenerate on Ω''_{k-1} and Ω'_{k+1}, respectively. These spaces are the ones we shall be working with in section 2 in order to have partition functions of non-degenerate actions.

The identification between two dual antisymmetric field theories involves identifying formal integrals, which we will interpret as gaussian integrals since they are defined using quadratic actions. In section 2 we study the "equivalence" between two such partition functions from a measure theoretical point of view in the case in which the action functionals involved are not degenerate and, following Quevedo [Q98], we give the heuristic path integral interpretation of duality. The case of degenerate action functionals will be studied in section 3.

2 Duality and Gaussian Measures on Antisymmetric Tensor Fields

2.1 Some Facts about Gaussian Measures

A *characteristic function* on a topological vector space E is a continuous (on every finite dimensional subspace of E) function χ satisfying

$$\sum_{j,k=1}^{N} \alpha_j \bar{\alpha_k}\, \chi(\xi_j - \xi_k) \geq 0$$

for $\alpha_k \in \mathbb{C}$, $\xi_j \in E \quad (j,k = 1,...,N)$. In a finite dimensional vector space E, with inner product $\langle\, , \rangle$, *Bochner's theorem* assures a one-to-one correspondence between characteristic functions and measures [$Y85$]. In particular, to the function

$$\chi(\xi) = \exp\left\{-\frac{1}{2}\langle \xi, \xi\rangle\right\}$$

there corresponds a unique Borel measure on E, called *Gaussian Measure* and denoted by μ, such that

$$\chi(\xi) = \int_E \exp\{i\langle \xi, \phi\rangle\}\, d\mu(\phi)$$

and $\mu(E) = 1$. In infinite dimensions, starting from a characteristic function χ on a topological vector space E, one typically ends up with a measure with support in a larger space. Even in the case of a Hilbert space $\mathcal{H}$, the corresponding measure to a characteristic function lies in some Hilbert-Schmidt extension of $\mathcal{H}$. Bochner's theorem holds in the case of continuous characteristic functions on a nuclear Hilbert space (a topological vector space whose topology is defined by a family $\{||\cdot||_\alpha\}$ of Hilbertian semi-norms such that $\forall \alpha\ \exists \alpha' : ||\cdot||_\alpha$ is Hilbert-Schmidt with respect to $||\cdot||_{\alpha'}$) [$GV64$].

The case we are dealing with is that of a Hilbert Space $\mathcal{H}$ (with inner product $\langle\, , \rangle_{\mathcal{H}}$) and, for $a > 0$, where we consider the characteristic function

$$\chi_{a,G}(\xi) = \exp\left\{-\frac{1}{2a}\langle G\xi, \xi\rangle_{\mathcal{H}}\right\}, \tag{15}$$

where G is a positive bounded operator on $\mathcal{H}$, corresponding to the infinite dimensional gaussian measure (with covariance G) formally written

$$d\mu_{a,G}(\phi) \text{``}=\text{''} \frac{1}{Z_{a,G}} \exp\left\{-\frac{a}{2}\langle G^{-1}\phi, \phi\rangle_{\mathcal{H}}\right\}[\mathcal{D}\phi], \tag{16}$$

(where $Z_{a,G}$ is a constant such that $\mu_{a,G}(\mathcal{H}) = 1$) the support of which lies in a Hilbert-Schmidt extension of $\mathcal{H}$. All this can be sumarized in the single equation

$$\chi_{a,G}(\xi) = \int_{\mathcal{H}} \exp\{i\langle \xi, \phi\rangle_{\mathcal{H}}\}\, d\mu_{a,G}(\phi), \tag{17}$$

where the left hand side involves G (see (15)) and the right hand side involves G^{-1} (see (16)). This generalizes the very well known relation

$$\exp\left\{-\frac{1}{2}\langle A^{-1}\vec{x}, \vec{x}\rangle\right\} = k(\det A)^{-\frac{1}{2}} \int_{\mathrm{I\!R}^n} \exp\left\{i\langle \vec{x}, \vec{y}\rangle - \frac{1}{2}\langle A\vec{y}, \vec{y}\rangle\right\} d\vec{y}, \tag{18}$$

where $\vec{x}, \vec{y} \in \mathbb{R}^n$, k is a constant, A denotes a positive matrix and $\langle\, , \rangle$ denotes the inner product in this space. Equation (17) defines the function $\chi_{a,G}$ as the *Fourier Transform* of the gaussian measure $\mu_{a,G}$, which we will denote by $\widehat{\mu}_{a,G}$.

2.2 Gaussian Measures and Duality

Consider the acyclic complex (5) and Hodge decomposition (6) on the space of k-forms. We take $\mathcal{H}_k = L^2(\Omega^k)$, where the closure is taken with respect to the L^2-hermitian product $\langle\, , \rangle$ defined by the Riemannian metric on M and the inner product structure of $E(\rho)$, and we consider the decomposition $\mathcal{H}_k \cong \mathcal{H}'_k \oplus \mathcal{H}''_k$ induced by (6). For $a, b > 0$ consider the gaussian measures μ_a on Ω^k and μ_b on Ω'_k defined by the characteristic functions

$$\widehat{\mu}_a(\alpha_k) = \exp\left\{-\frac{a}{2}\langle \alpha_k, \alpha_k\rangle\right\} \tag{19}$$

and

$$\widehat{\mu}'_b(\eta'_k) = \exp\left\{-\frac{b}{2}\langle \eta'_k, \eta'_k\rangle\right\}. \tag{20}$$

Proposition 1 *Let $a, b > 0$, then*

$$\int_{\Omega'_k} d\mu'_b(\eta'_k)\widehat{\mu}'_a(\eta'_k) = \int_{\Omega^k} d\mu_a(\alpha_k)\widehat{\mu}'_b(\alpha'_k). \tag{21}$$

Proof. By (27), $\widehat{\mu}_a(\xi) = \int \exp\{i\langle \xi, \phi\rangle_{\mathcal{H}}\}\, d\mu_a(\phi)$, so

$$\begin{aligned}
\int_{\Omega'_k} d\mu'_b(\eta'_k)\widehat{\mu}'_a(\eta'_k) &= \int_{\Omega'_k} d\mu'_b(\eta'_k) \int_{\Omega^k} \exp\{i\langle \eta'_k, \alpha_k\rangle\}\, d\mu_a(\alpha_k) \\
&= \int_{\Omega^k} d\mu_a(\alpha_k) \int_{\Omega'_k} \exp\{i\langle \eta'_k, \alpha'_k\rangle\}\, d\mu'_b(\eta'_k) \\
&= \int_{\Omega^k} d\mu_a(\alpha_k)\widehat{\mu}'_b(\alpha'_k).
\end{aligned}$$

□

Let us see that equality between the partition functions corresponding to the action functionals (13) and (14) can be seen as the *heuristic* limit of (21) when b goes to infinity. Let ϵ be a positive real number and take $b = \frac{1}{\epsilon}$. Then,

$$\int_{\Omega'_k} d\mu'_{\frac{1}{\epsilon}}(\eta'_k)\widehat{\mu}'_a(\eta'_k) = \int_{\Omega^k} d\mu_a(\alpha_k)\widehat{\mu}'_{\frac{1}{\epsilon}}(\alpha'_k)$$

and taking the limit $\epsilon \to 0$ (in the sense of distributions) of the gaussian characteristic function $\widehat{\mu}'_b$ we find a Dirac delta function forcing α'_k to vanish. The corresponding limit of the associated gaussian measure on Ω'_k is *heuristically* (proportional to a) Lebesgue measure on that space. Thus, if we write the formal expression (all these calculations are formal, the measures are of course ill-defined "Lebesgue measures" on L^2 spaces of forms)

$$\int_{\Omega'_k} \widehat{\mu}'_a(\eta'_k)\ [\mathcal{D}\eta'_k] \text{ “ = ” } \int_{\Omega^k} d\mu_a(\alpha_k)\ \delta[\alpha'_k = 0], \tag{22}$$

we find, by using of the formal relation (16),

$$\begin{aligned}\int_{\Omega'_k} \exp\left\{-\frac{a}{2}\langle\eta'_k,\eta'_k\rangle\right\}[\mathcal{D}\eta'_k] \text{ “ = ” } &\int_{\Omega^k} \exp\left\{-\frac{1}{2a}\langle\alpha_k,\alpha_k\rangle\right\}\delta[\alpha'_k=0][\mathcal{D}\alpha_k] \\ \text{ “ = ” } &\int_{\Omega''_k} \exp\left\{-\frac{1}{2a}\langle\alpha''_k,\alpha''_k\rangle\right\}[\mathcal{D}\alpha''_k].\end{aligned}$$

Now let us do the change of variables defined by the maps (11) and (12), $\eta'_k = d_{k-1}\omega''_{k-1}$ and $\alpha''_k = d^*_k\omega'_{k+1}$, then we find

$$\begin{aligned}&\mathcal{J}_{k-1}\int_{\Omega''_{k-1}} \exp\left\{-\frac{a}{2}\widehat{\mathcal{S}}(\omega''_{k-1})\right\}[\mathcal{D}\omega''_{k-1}] \\ &\qquad\text{ “ = ” }\quad \mathcal{J}_{k+1}\int_{\Omega'_{k+1}} \exp\left\{-\frac{1}{2a}\widehat{\mathcal{S}}^*(\omega'_{k+1})\right\}[\mathcal{D}\omega'_{k+1}],\end{aligned} \tag{23}$$

where $\mathcal{J}_{k-1}$ and $\mathcal{J}_{k+1}$ denotes the associated jacobian determinants $\mathcal{J}_{k-1} := \sqrt{\det(d^*_{k-1}d_{k-1})}$ and $\mathcal{J}_{k+1} := \sqrt{\det(d_k d^*_k)}$.

Finally let us write down the formal calculations usually used to arrive to relation (23); they involve Fourier Transforms, usual properties of gaussian integrals and changing the order of integration [$Q98$]:

$$\begin{aligned}&\int_{\Omega'_k} \exp\left\{-\tfrac{a}{2}S_o(\eta'_k)\right\}[\mathcal{D}\eta'_k] \\ &\text{ “ = ” }\quad \int_{\Omega'_k}[\mathcal{D}\eta'_k]\int_{\Omega^k}[\mathcal{D}\alpha_k]\exp\left\{-\frac{1}{2a}S_o(\alpha_k)\right\}\exp\left\{i\langle\eta'_k,\alpha_k\rangle\right\} \\ &\text{ “ = ” }\quad \int_{\Omega^k}[\mathcal{D}\alpha_k]\exp\left\{-\frac{1}{2a}S_o(\alpha_k)\right\}\int_{\Omega'_k}[\mathcal{D}\eta'_k]\exp\left\{i\langle\eta'_k,\alpha_k\rangle\right\} \\ &\text{ “ = ” }\quad \int_{\Omega^k}[\mathcal{D}\alpha_k]\exp\left\{-\frac{1}{2a}S_o(\alpha_k)\right\}\int_{\Omega'_k}[\mathcal{D}\eta'_k]\exp\left\{i\langle\eta'_k,\alpha'_k\rangle\right\}\end{aligned}$$

$$\text{“ = ”} \qquad \int_{\Omega^k} [\mathcal{D}\alpha_k] \exp\left\{-\frac{1}{2a} S_o(\alpha_k)\right\} \delta\left[\alpha'_k = 0\right]$$
$$\text{“ = ”} \qquad \int_{\Omega''_k} \exp\left\{-\frac{1}{2a} S_o(\alpha''_k)\right\} [\mathcal{D}\alpha''_k],$$

which, after the change of variables defined by the maps (11) and (12), is equivalent to (23).

Let us make a few comments on this computation which, although very formal, gives the gist of the dualization procedure.

1. *Hodge decomposition* in the case of an acyclic complex splits the space of k-antisymmetric tensor fields (6) and then, through isomorphisms (11) and (12),
$$\Omega^k \cong \Omega''_{k-1} \oplus \Omega'_{k+1}. \tag{24}$$
The L^2 scalar product on Ω^k then gives rise to two (non degenerate) actions $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}}^*$ ((13) and (14)), on Ω''_{k-1} and Ω'_{k+1} respectively, which are related by a *Fourier transform*. The non-degeneracy in the actions comes from the fact we restrict ourselves to
$$\Omega''_{k-1} \xrightarrow{d_{k-1}} \Omega^k \xleftarrow{d^*_k} \Omega'_{k+1}. \tag{25}$$
Thus, the field $\omega_k \in \Omega_k$ splits into
$$\omega_k = d_{k-1}\omega''_{k-1} \oplus d^*_k \omega'_{k+1}, \tag{26}$$
giving rise to two new “fields” (gauge potentials) $\omega''_{k-1}, \omega'_{k+1}$.

2. In the process of taking the Fourier Transform, the coefficient of the quadratic action is inverted ($a \mapsto a^{-1}$), a fact often observed in duality and typical for Fourier transforms of gaussian functions. A strong coupling can thus be turned into a weak coupling [$D98$].

3. Finally, if we consider *Hodge star duality* on the complex, through the relation
$$d^*_k \omega_{k+1} = (-1)^{nk+n+1} * d_{n-k-1} * \omega_{k+1},$$
we recover the usual “moral” of duality in antisymmetric fields [$Q98$]: a $(k-1)$-rank antisymmetric tensor field (the “gauge potential” ω_{k-1}) is dual to a $(n-k-1)$-rank antisymmetric tensor field (η_{n-k-1} =

$*\omega_{k+1}$) or, in "brane" language, a $(k-2)$(electric)-brane is dual to a $(n-k-2)$(magnetic)-brane. In fact observe that

$$\begin{aligned}\langle d_k^* \omega_{k+1}, d_k^* \omega_{k+1}\rangle &= \langle *d_{n-k-1} * \omega_{k+1}, *d_{n-k-1} * \omega_{k+1}\rangle \\ &= *^2 \langle d_{n-k-1}\eta_{n-k-1}, d_{n-k-1}\eta_{n-k-1}\rangle\end{aligned}$$

where $*^2$ denotes a $\pm$ sign depending on k and the dimension of M.

¿From a physical point of view this formal computation tells us that if we consider an antisymmetric field theory modelling physical fields by k-forms, given by the action (7) then (in this acyclic case) we find two possible "potentials" associated to that field: ω''_{k-1} and ω'_{k+1}, the first one for the exterior differential d_{k-1}, the second one for d_k^* (see (26)). Writing the partition function of the theory with respect to one or the other give us "dual" formulations of the same theory.

3 Duality and the Analytic Torsion of the de Rham Complex

Unlike in the previous section, we now want to take into account local symmetries of the type (3) and "dual ones" obtained replacing d by d^*. Thus we now consider the degenerate actions (9) and (10) computing their corresponding partition functions and we show how from this point of view duality leads to a factorization of the analytic torsion of the space-time manifold.
The analytic torsion of a Riemannian manifold M is a topological invariant defined by some spectral properties of the Laplacian operators acting on spaces of differential forms on M. These properties are a consequence of the one to one correspondence $\Omega''_k \xrightarrow{d_k} \Omega'_{k+1}$ used previously, and its "dual" $\Omega'_{k+1} \xrightarrow{d_k^*} \Omega''_k$ (both of them defined in the acyclic case), together with the Hodge star duality map. In this section we will study the relation between two dual antisymmetric tensor field actions, their partition functions and the analytic torsion of the space-time manifold on which such fields are defined. We will use zeta-regularization techniques [G95] and an Ansatz introduced by Schwarz to define the partition function associated to a degenerate action functional [S79].

3.1 Zeta-Regularized Determinants and Analytic Torsion on Riemannian Manifols

Let us take again a closed n-dimensional Riemannian manifold M and the acyclic de Rham complex (5) on it, with the Hodge decomposition (6) of the space of ($E(\rho)$-valued) k-forms on M. The Laplacian operator on k-forms,

$\Delta_k = d_{k-1}d_{k-1}^* + d_k^* d_k$, is a positive selfadjoint elliptic operator, and its determinant can be defined by the zeta-function regularization method as,

$$\det\nolimits_\zeta \Delta_k = \exp\left\{-\zeta'_{\Delta_k}(0)\right\}, \tag{27}$$

where the zeta-function is given by

$$\zeta_{\Delta_k}(s) = \sum_\lambda \frac{1}{\lambda^s}, \tag{28}$$

and the sum is over all the eigenvalues λ of Δ_k. Indeed, it can be shown using properties of elliptic operators on closed manifolds that this function is analytic for $s \in \mathbb{C}$ with $Re(s) >> 0$, and extends by analytical continuation to a meromorphic function on $\mathbb{C}$, regular at $s = 0$ (see e.g. [$G95$]).

The *analytic torsion* of the Riemannian manifold M [$RS71$] (see also [$R97$] for a review) is defined in terms of the (regularized) determinant of the Laplacian operators on Ω^k, $0 \leq k \leq n$, as

$$T(M) := \exp\left(\frac{1}{2}\sum_{k=0}^{n}(-1)^k k \log(\det\nolimits_\zeta \Delta_k)\right). \tag{29}$$

Since Hodge star duality $*\Delta_k = \Delta_{n-k}*$ implies that Δ_k and Δ_{n-k} are isospectral, so that $\zeta_{\Delta_k}(s) = \zeta_{\Delta_{n-k}}(s)$ and hence $\sum_k(-1)^k k\zeta_{\Delta_k}(s) = \frac{n}{2}\sum_k(-1)^k\zeta_{\Delta_k}(s) = 0$, $T(M)$ is equal to 1 if n is even. In the odd case it can be shown to be independent of the metric.

Let us recall some simple observations on the decomposition of the eigenspace of Δ_k for a given eigenvalue λ. Let us set $\mathcal{E}_k(\lambda) := \mathrm{Ker}\ (\Delta_k - \lambda)$, then Hodge decomposition (6) induces a decomposition of such eigenspaces $\mathcal{E}_k(\lambda) = \mathcal{E}'_k(\lambda) \oplus \mathcal{E}''_k(\lambda)$, where

$$\mathcal{E}'_k(\lambda) := \{\omega_k \in \mathcal{E}_k(\lambda), d_k\omega_k = 0\} = \mathcal{E}_k(\lambda) \cap \Omega'_k$$

and

$$\mathcal{E}''_k(\lambda) := \left\{\omega_k \in \mathcal{E}_k(\lambda), d_{k-1}^*\omega_k = 0\right\} = \mathcal{E}_k(\lambda) \cap \Omega''_k.$$

Let $\omega_k \in \mathcal{E}_k(\lambda)$, and suppose $\omega_k \in \Omega''_k$. Then $d_k\omega_k \in \Omega'_{k+1}$ and

$$\Delta_{k+1}d_k\omega_k = d_k d_k^* d_k\omega_k = d_k\Delta_k\omega_k = \lambda d_k\omega_k,$$

so d_k maps $\mathcal{E}''_k(\lambda)$ bijectively into $\mathcal{E}'_{k+1}(\lambda)$, giving us a bijective correspondence between (non-zero) eigenvalues (and their corresponding eigenvectors) of the operators $d_k^* d_k$ and $d_{k+1}d_{k+1}^*$.

The zeta-regularization techniques used to define the determinant of the Laplacian operators can also be used to define the "regularized determinants" of the maps $d_{k-1}d^*_{k-1}$ and $d^*_k d_k$. From the decomposition (6) of each space Ω^k, and the nilpotency of the d^*_k and d_k operators, it follows that the set of eigenvalues of the Laplacian operator Δ_k is the union of the eigenvalues of $d_{k-1}d^*_{k-1}$ and $d^*_k d_k$. Since the eigenvalues of $d_{k-1}d^*_{k-1}$ and $d^*_{k-1}d_{k-1}$ are the same, the set of eigenvalues of Δ_k is the union of the $d^*_{k-1}d_{k-1}$ and $d^*_k d_k$ eigenvalues. So, if we define the zeta-function associated to the operators $d_{k-1}d^*_{k-1}$ and $d^*_k d_k$ by

$$\zeta_{d_{k-1}d^*_{k-1}}(s) := \sum_{\lambda''} \frac{1}{\lambda''^s} \tag{30}$$

and

$$\zeta_{d^*_k d_k}(s) := \sum_{\lambda'} \frac{1}{\lambda'^s}, \tag{31}$$

where λ'' and λ' denote (non zero) eigenvalues of $d_{k-1}d^*_{k-1}$ and $d^*_k d_k$, respectively, it follows that, for $0 \leq k \leq n$,

$$\zeta_{\Delta_k}(s) = \zeta_{d_{k-1}d^*_{k-1}}(s) + \zeta_{d^*_k d_k}(s). \tag{32}$$

Hence

$$\zeta_{d^*_k d_k}(s) = \zeta_{\Delta_k}(s) - \zeta_{\Delta_{k-1}}(s) + \zeta_{\Delta_{k-2}}(s) - \cdots + (-1)^k \zeta_{\Delta_0}(s), \tag{33}$$

and from the properties of the zeta-function of the Laplacian, it follows that $\zeta_{d_{k-1}d^*_{k-1}}$ and $\zeta_{d^*_k d_k}$ are well defined and analytic for $s \in \mathbb{C}$ with $Re(s) >> 0$, and extend by analytic continuation to meromorphic functions on $\mathbb{C}$, regular at the origin. Moreover, using the fact that $\mathcal{E}''_k(\lambda) \cong \mathcal{E}'_{k+1}(\lambda)$, we find

$$\zeta_{d^*_{k-1}d_{k-1}}(s) = \zeta_{d_k d^*_k}(s). \tag{34}$$

It is clear now that we can write

$$\begin{aligned} \det{}_\zeta \Delta_k &= \exp\left\{-\zeta'_{\Delta_k}(0)\right\} \\ &= \exp\left\{-\zeta'_{d_{k-1}d^*_{k-1}}(0) - \zeta'_{d^*_k d_k}(0)\right\} \\ &= \det{}_\zeta(d_{k-1}d^*_{k-1}) \ \det{}_\zeta(d^*_k d_k), \end{aligned}$$

where $\det_\zeta(d_{k-1}d^*_{k-1})$ and $\det_\zeta(d^*_k d_k)$ are defined in a similar way as the zeta determinant of Laplacian operators (equation (27)), using (30) and (31), respectively.

3.2 *The Partition Function of a Degenerate Action Functional (Following Schwarz)*

Let us come back to the action $\mathcal{S}_o(\omega_k) = \langle \omega_k, \omega_k \rangle$ on Ω^k and its decomposition (8), $S_o(\omega_k) = \mathcal{S}(\omega_{k-1}) \oplus \mathcal{S}^*(\omega_{k+1})$, induced by Hodge decomposition of Ω^k, into the degenerate action functionals ((9) and (10)),

$$\mathcal{S}(\omega_{k-1}) = \langle d_{k-1}^* d_{k-1} \omega_{k-1}, \, \omega_{k-1} \rangle$$

and

$$\mathcal{S}^*(\omega_{k+1}) = \langle d_k d_k^* \omega_{k+1}, \, \omega_{k+1} \rangle,$$

on $\Omega^{k-1} = \Omega'_{k-1} \oplus \Omega''_{k-1}$ (= Ker $(d_{k-1}^* d_{k-1}) \oplus$ Ker $(d_{k-1}^* d_{k-1})^\perp$) and $\Omega^{k+1} = \Omega'_{k+1} \oplus \Omega''_{k+1}$ (= Ker $(d_k d_k^*) \oplus$ Ker $(d_k d_k^*)^\perp$), respectively. In section 2 we were dealing with non degenerate actions $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}}^*$, since we had restricted ourselves to Ω''_k and Ω'_{k+1}. As we pointed out there, the degeneracy leads to some formal volume of an infinite dimensional space, and the non degeneracy condition restrict us to look at only a part of the complex (5), namely (25). Schwarz suggested an Ansatz, inspired from the well known Faddeev-Popov procedure, to "compute" this volume and give a meaning to the partition function of a degenerate action functional [S79]. Following Schwarz's method, provided we can associate a chain of vector spaces and maps, called *resolvent*, to the degenerate action, then the partition function associated to that action can be defined in terms of (regularized) determinants of the maps appearing in the resolvent. In our particular case, as we will see, this means to consider the whole complex (5) and not only a part of it, as we done in the non degenerate case.

In the case of $\mathcal{S}(\omega_{k-1})$ the resolvent is the given by (for details about the definition of the resolvent associated to a degenerate functional see e.g. [S79] or [BT91])

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \cdots \xrightarrow{d_{k-3}} \Omega^{k-2} \xrightarrow{d_{k-2}} \Omega'_{k-1} \xrightarrow{d_{k-1}^* d_{k-1}} 0, \tag{35}$$

and we define the partition function associated to that action (and resolvent) as

$$Z(\mathcal{S}) = \det{}_\zeta (d_{k-1}^* d_{k-1})^{\frac{-1}{2}} \prod_{j=1}^{k-1} |\det{}_\zeta (d_{k-j-1})|^{(-1)^{j+1}}. \tag{36}$$

Notice that when $k=1$, $Z(S)$ gives back the usual Ansatz to compute the partition function (compares with (4) and (18))

$$Z(S) \text{``}=\text{''} \int_{\Omega^0} \exp\{-\frac{1}{2}\langle d_0 f, d_0 f\rangle\}[\mathcal{D}f] \text{``}=\text{''} (\det \Delta_0)^{-\frac{1}{2}}.$$

In the same way, taking the resolvent associated to $\mathcal{S}^*(\omega_{k+1})$,

$$0 \longrightarrow \Omega^n \overset{d_{n-1}^*}{\longrightarrow} \cdots \overset{d_{k+2}^*}{\longrightarrow} \Omega^{k+2} \overset{d_{k+1}^*}{\longrightarrow} \Omega''_{k+1} \overset{d_k d_k^*}{\longrightarrow} 0, \tag{37}$$

we define the associated "dual" partition function by

$$Z(\mathcal{S}^*) = \det{}_\zeta(d_k d_k^*)^{\frac{-1}{2}} \prod_{j=1}^{n-k-1} |\det{}_\zeta(d_{k+j}^*)|^{(-1)^{j+1}}. \tag{38}$$

Here, $\det_\zeta(d_k)$ and $\det_\zeta(d_k^*)$ are defined by

$$|\det{}_\zeta(d_k)| := \sqrt{\det{}_\zeta(d_k^* d_k)},$$

$$|\det{}_\zeta(d_k^*)| := \sqrt{\det{}_\zeta(d_k d_k^*)}$$

and, as we remarked in the previous section,

$$\det{}_\zeta(d_k^* d_k) = \det{}_\zeta(d_k d_k^*).$$

Therefore, the two "dual" partition functions are given by

$$Z(\mathcal{S}) = \prod_{j=1}^{k} [\det{}_\zeta(d_{k-j}^* d_{k-j})]^{\frac{(-1)^j}{2}} \tag{39}$$

and

$$Z(\mathcal{S}^*) = \prod_{j=0}^{n-k-1} [\det{}_\zeta(d_{k+j} d_{k+j}^*)]^{\frac{(-1)^{j+1}}{2}}. \tag{40}$$

3.3 Analytic Torsion on Riemannian Manifolds and Duality

The relation between the analytic torsion of the manifold M and the partition function of an antisymmetric field theory defined on it, by the method discussed above, was pointed out by Schwarz ([S79], see also [ST84]), studying quantization of antisymmetric tensor field theories defined by the degenerate action (1) on Ω^k. It has been also used in the context of *Topological Quantum Field Theories* [W89][BT91][AS95]. Schwarz shows that the Hodge star duality map and (34) imply a factorization of the analytic torsion $T(M)$ in terms of the two partition functions, corresponding to the actions $\mathcal{S}(\omega_{k-1})$ and $\mathcal{S}(\omega_{n-k-1})$.

In this section we want to relate the partition functions $Z(\mathcal{S})$ and $Z(\mathcal{S}^*)$ ((36) and (38)), corresponding to the two "dual actions" $\langle d_{k-1}\omega_{k-1}, d_{k-1}\omega_{k-1}\rangle$ and $\langle d_k^*\omega_{k+1}, d_k^*\omega_{k+1}\rangle$, with the analytic torsion of the manifold M on which these antisymmetric field theories are formulated. Such a relation is clear if we look at the splitting in the de Rham complex (5) induced by the two resolvents, (35) and (37), associated with the partition functions of the dual theories defined by the given actions, namely

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \cdots \xrightarrow{d_{k-2}} \Omega^{k-1} \xrightarrow{d_{k-1}} \Omega^k \xleftarrow{d_k^*} \Omega^{k+1} \xleftarrow{d_{k+1}^*} \cdots \xleftarrow{d_{n-1}^*} \Omega^n \longleftarrow 0. \quad (41)$$

Observe that, from (39) and (40),

$$\begin{aligned}\log \frac{Z(\mathcal{S})}{Z(\mathcal{S}^*)} = \quad & \log\left[\det_\zeta(d_0^* d_0)\right]^{\frac{(-1)^k}{2}} + \log\left[\det_\zeta(d_1^* d_1)\right]^{\frac{(-1)^{k-1}}{2}} + \cdots \\ & + \log\left[\det_\zeta(d_{k-1}^* d_{k-1})\right]^{\frac{-1}{2}} - \log\left[\det_\zeta(d_k d_k^*)\right]^{\frac{-1}{2}} \\ & - \log\left[\det_\zeta(d_{k+1} d_{k+1}^*)\right]^{\frac{1}{2}} - \cdots - \log[\det_\zeta(d_{n-1} d_{n-1}^*)]^{\frac{(-1)^{n-k}}{2}},\end{aligned}$$

and from (27) and definition (29)

$$T(M) = \exp\left(\frac{1}{2}\sum_{k=0}^{n-1}(-1)^k \zeta'_{\Delta_k}(0)\right)$$

then,

$$\begin{aligned}\log T(M) &= \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k \zeta'_{d_k^* d_k}(0) \\ &= \sum_{k=0}^{n-1} \log(\det_\zeta d_k^* d_k)^{\frac{(-1)^k}{2}}\end{aligned}$$

so,

$$Z(\mathcal{S})Z(\mathcal{S}^*)^{-1} = T(M)^{(-1)^{n-k-1}}. \quad (42)$$

Thus, we can say that two dual actions yield a factorization of the analytic torsion of the space-time manifold (coming from the splitting (41)) in terms of their corresponding partition functions. Hence in even dimensions we get the expected identification of the partition function with its dual. Note that the analytic torsion is a topological invariant of M, but there is no reason for $Z(\mathcal{S})$ and $Z(\mathcal{S}^*)$ to have this property.

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