MULTIPLICATIVE ANOMALY FOR THE ζ-REGULARIZED DETERMINANT

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We study new regularized determinants on elliptic pseudo-differential operators. They are of the form "$expTr^Q$Log", where Tr^Q is a weighted trace. We express the obstruction preventing the ζ-determinant from being of this type, this leading us to a general formula for the multiplicative anomaly of the ζ-determinant.

1 Introduction

In this paper we study determinants of elliptic classical pseudo-differential operators, acting on sections of some finite rank vector bundle E based on a closed manifold M. Two types of determinant are involved: determinants associated to weighted traces and the well known ζ-regularized determinant.

Using an auxilary operator Q, one can define a pseudo trace Tr^Q which we call a weighted trace. This weighted trace is a linear form on the algebra of classical pseudo-differential operators. It coincides with the ordinary trace on trace class operators but, in general, it is not tracial i.e. $Tr^Q(AB) \neq Tr^Q(BA)$.

By determinant associated to a weighted trace Tr^Q, we mean a functional det^Q which is defined by a formula of the type: "$det^Q = expTr^Q$Log". We call this determinant a weighted determinant. In general, weighted determinants do not satisfy the multiplicative property i.e. $det^Q(AB) \neq det^Q(A)det^Q(B)$. However for operators of the type $I+K$, with K of trace class, weighted determinants coincide with the Fredholm determinant and the multiplicative property holds.

By restricting to operators for which one can use the same determination of logarithm, we show that the weighted determinant det^Q satisfies the multiplicative property as long as these operators are taken in a subalgebra on which the weighted trace Tr^Q is tracial.

On the other hand, the ζ-regularized determinant det_ζ, introduced by Ray and Singer [RS] in 1971 in a geometric context, also gives rise to a non trivial multiplicative anomaly:

$$F_\zeta(A,B) := det_\zeta(AB)/det_\zeta(A)det_\zeta(B),$$

previously studied by M.Kontsevich and S.Vishik, and independently by K.Okikiolu (see [KV1,2] and [O1,2]). We relate the weighted determinants to the ζ-regularized determinant, this leading us to a general formula for the multiplicative anomaly of the ζ-regularized determinant, which extends formulae established by M.Wodzicki ([W], [K]) for commuting operators, and by K.Okikiolu ([O2]) for operators with scalar leading symbols.

From now on, we consider a C^∞ closed riemaniann manifold M of finite dimension d and a hermitian vector bundle $E \xrightarrow{\pi} M$ of finite rank n based on M. We shall denote by $\Gamma(M,E)$ the set of C^∞ sections of E. The space $\Gamma(M,E)$ is endowed with the hermitian product: $< s,\rho >= \int_M < s(x),\rho(x) >_{E_x} dvol(x)$, where vol is the riemaniann volume measure on M and $<,>_{E_x}$ is the hermitian structure on the fiber E_x of E over $x\in M$.

2 Preliminaries

In this section, we introduce pseudo-differential operators (PDOs) on $\Gamma(M,E)$, we define complex powers and logarithms, and we describe a topology on classical PDOs. We refer the reader to the appendix for notations and basic facts concerning PDOs on an open subset of $\mathbb{R}^d$.

• *Pseudo-differential operators acting on sections of a vector bundle*

Let (U,ϕ,u) be a local chart of E where (U,ϕ) is a local chart of M and u a local trivialization giving rise to a diffeomorphism

$$\begin{aligned} \pi^{-1}(U) &\to \phi(U)\times \mathbb{C}^n \\ z &\to \Big(\phi(\pi(z)), u(z)\Big). \end{aligned}$$

Then, ϕ and u induce two maps:
$\phi_u^* : C^\infty\Big(\phi(U),\mathbb{C}^n\Big) \to \Gamma(U,E)$, $\phi_{u*} : \Gamma(U,E) \to C^\infty\Big(\phi(U),\mathbb{C}^n\Big)$ defined by: $(\phi_u^* s)(x) = u^{-1}(s(\phi(x)))$, $(\phi_{u*}s)(x) = u(s(\phi^{-1}(x)))$.
Let A be a linear map on $\Gamma(M,E)$ and let A_U be the induced map from the set of C^∞ sections of E with compact support in U $\Gamma_c(U,E)$, to $\Gamma(U,E)$: $A_U := r_U A i_U$, where i_U is the natural embedding $\Gamma_c(U,E)\to\Gamma(M,E)$ and r_U is the natural restriction $\Gamma(M,E)\to\Gamma(U,E)$. Then A is called a PDO, resp.*classical* PDO,resp. *odd-class classical* PDO if the operator $A_U^\sharp := \phi_{u*}A_U\phi_u^*$ is a matricial PDO on $\phi(U)$, resp. classical, resp. odd-class classical. We shall

say that $A^\sharp_U$ is A read in the local chart (U,ϕ,u). Invariance under a change of coordinates (see Appendix) shows that these definitions are independent of the choice of the local chart.

Since the manifold M is compact, by using a partition of unity one can compose PDOs. We shall denote by $PDO(M,E)$, resp. $CL(M,E)$, resp. $CL_{(-1)}(M,E)$ the algebra of PDOs acting on $\Gamma(M,E)$, resp. classical, resp. odd-class classical.

- *Topology on CL(M,E)*

Let (U_i,ϕ_i,u_i) be a finite trivializing covering of M for E and let (V_i) an open covering of M such that $\overline{V_i}\subset U_i$. Let (g_i) be a partition of the unity subordinated to (V_i) and let Mg_i be the multiplication by g_i. Let $A\in CL(M,E)$ of order m. Then we have $A=\sum AMg_i$. Let $A^\sharp_i$ be AMg_i read in the local chart (U_i,ϕ_i,u_i) and let $a^{(i)}$ be the symbol of $A^\sharp_i$ with formal expansion $a^{(i)}\sim\sum_{j\geq 0}a^{(i)}_{m-j}$. Let N be an integer, $N\geq 1$, and let $B_i^{(N)}:=\sum_{j=0}^{j=N-1}Op(a^{(i)}_{m-j})$, $A^\sharp_i:=B_i^{(N)}+R_i^{(N)}$. For $N\geq d+m$, $R_i^{(N)}$ has a kernel $K_i^{(N)}(x,y)$ which is $C^{N-(d+m)}$. We endow the vector space $CL_{ord\leq m}(M,E)$ of classical PDOs of order less than m, with the Fréchet topology given by the following semi norms labelled by multiindices δ,γ and integers $j\geq 0, N\geq d+m$:

$$\|A\|_{\delta,\gamma,j,N}:=max_i\Big(sup_{x\in\overline{V_i},|\xi|=1}|\xi|^{j-m}\|D_x^\delta D_\xi^\gamma a^{(i)}_{m-j}(x,\xi)\|+$$

$$sup_{x,y\in\overline{V_i},|\delta'|+|\gamma'|\leq N-(d+m)}\|D_x^{\delta'}D_y^{\gamma'}K_i^{(N)}(x,y)\|\Big).$$

One can check that this topology does not depend on the choices we made.

- *Leading symbols and elliptic operators*

Let $A\in CL(M,E)$. The first homogeneous component of its symbol (which is only locally defined) not identically equal to zero can be globally defined on T^*M and is called the *leading symbol* $\sigma_L(A)$ of A. For $x\in M$, $\xi\in T^*_xM$, $\sigma_L(A)(x,\xi)$ is an endomorphism of the fiber E_x of E. If $\sigma_L(A)(x,\xi)$ is invertible for $\xi\neq 0$, one says that A is *elliptic.* We endow the set of leading symbols with a norm:

$$\|\sigma_L(A)\|:=sup_{x\in M}sup_{|\xi|=1}\|\sigma_L(A)(x,\xi)\|_{End(E_x)}.$$

We denote by $Ell(M,E)$ the set of elliptic operators in $CL(M,E)$.

• *Admissible elliptic operators*

Let $A \in Ell^*_{ord\geq 0}(M,E)$. For $\theta \in \mathbb{R}$, we denote by L_θ the half line $\{z \in \mathbb{C} : Arg z = \theta\}$ and we call L_θ a *spectral cut* of A if $L_\theta \cap Sp(A) = \emptyset$. If $A \in Ell^*_{ord>0}(M,E)$, then A admits a spectral cut when A satisfies the Agmon-Nirenberg condition: there exists a closed angle Λ in the complex plane with vertex 0 such that Λ contains no eigenvalues of the leading symbol of A, $\sigma_L(A)(x,\xi)$ for any $(x,\xi) \in S^*M$. In that case, $\Lambda \cap Sp(A)$ is a finite set (see [Sh] p79). We shall call such an angle an *Agmon-Nirenberg angle.* Furthermore, we shall call A an *admissible* operator if A is of zero order and admits a spectral cut, or if A is of positive order and satisfies the Agmon-Nirenberg condition. We denote by $Ell^{*,adm}_{ord\geq 0}(M,E)$ the set of admissible operators in $Ell^*_{ord\geq 0}(M,E)$.

• *Complex powers, logarithms of admissible elliptic operators* ([Se], [KV1], [O1])

Let $A \in Ell^{*,adm}_{ord>0}(M,E)$ with a spectral cut L_θ and a positive order m. Let Γ_θ be the contour in $\mathbb{C}$ defined by $\Gamma^1_\theta \cup \Gamma^2_\rho \cup \Gamma^3_\theta$ where
$\Gamma^1_\theta := \{re^{i\theta} : +\infty > r \geq \rho\}$
$\Gamma^2_\rho := \{\rho e^{i\phi} : \theta \geq \phi \geq \theta - 2\pi\}$
$\Gamma^3_\theta := \{re^{i\theta} : \rho \leq r \leq +\infty\}$
and where ρ is a small positive real such that $\Gamma_\theta \cap Sp(A) = \emptyset$.

Following R.Seeley ([Se]), for $Re z < 0$ we set:

$$A^z_\theta := \frac{i}{2\pi}\int_{\Gamma_\theta} \lambda^z (A - \lambda I)^{-1} d\lambda$$

where λ^z is defined by determinations of complex logarithms given by the contour Γ_θ, and where $(A-\lambda I)^{-1}$ is the resolvent of A . The integral makes sense since $\|(A-\lambda I)^{-1}\| = O(1/|\lambda|)$ when $|\lambda| \to +\infty$ (see [Sh] p 77), and since $\int_\rho^{+\infty} \frac{\lambda^z}{|\lambda|} d\lambda$ is finite for $Re z < 0$.

Using the semi group property and the fact that $A^{-k}_\theta = (A^{-1})^k$ for any positive integer k, the definition of complex powers can be extended to the whole complex plane. For any $z \in \mathbb{C}$, $A^z_\theta \in Ell_{ord=mz}(M,E)$ and for any $k \in \mathbb{Z}$ and $s \in \mathbb{R}$, the map:

$$\begin{aligned} \{Re z < k\} &\to L(H^s(M,E), H^{s-mk}(M,E)) \\ z &\to A^z_\theta \end{aligned}$$

is a holomorphic function, where $L(H^s(M,E), H^{s-mk}(M,E))$ denotes the space of bounded linear operators from the Hilbert space $H^s(M,E)$ of sections of E of Sobolev class H^s to the space $H^{s-mk}(M,E)$ of sections of E of Sobolev class H^{s-mk} . Following [KV1,2] and [O1], we now define logarithms.

Definition 1 For $A \in Ell^{*,adm}_{ord>0}(M,E)$ with spectral cut L_θ we set

$$Log_\theta A := D_z A^z_\theta|_{z=0}.$$

The operator $\mathrm{Log}_\theta A$ is a PDO but it is not a classical one as the next proposition shows:

Proposition 1 [KV1], [O1]: *$Log_\theta A$ is a PDO of order ε for any $\varepsilon > 0$. In some local chart, the symbol of $Log_\theta A$ is of the form:*
$\sigma(Log_\theta A)(x,\xi) = m log|\xi| I + \sigma^A_0(x,\xi)$, where σ^A_0 is a classical symbol of zero order.

Because the term $m\log|\xi|I$ behaves "badly" under change of coordinates, σ^A_0 is not a symbol of a PDO. But for two operators $A, B \in Ell^{*,adm}(M,E)$ with resp. L_{θ_1}, L_{θ_2} as spectral cuts, then $\sigma^A_0 - \sigma^B_0$ is a symbol of a PDO and we have:

$$\frac{\mathrm{Log}_{\theta_1} A}{o(A)} - \frac{\mathrm{Log}_{\theta_2} B}{o(B)} \in CL_{ord=0}(M,E).$$

Let us specify that if the orders of A and B are both even integers and if A and B are both of odd-class, then $\frac{\mathrm{Log}_{\theta_1} A}{o(A)} - \frac{\mathrm{Log}_{\theta_2} B}{o(B)} \in CL_{(-1)}(M,E)$ (see [KV1,2]).

If an operator $A \in Ell^{*,adm}(M,E)$ is of zero order, then the spectrum of A is bounded. In this case, we can define complex powers and logarithms of A directly using a Cauchy integral formula. The logarithms of A belong to $CL_{ord=0}(M,E)$.

In what follows, we denote by Q an operator of $Ell^{*,adm}_{ord>0}(M,E)$ and we denote by q its order. When we take logarithms, we shall drop the mention of the chosen determination.

3 Weighted traces

Concerning weighted traces, we refer to [MN], [CDMP].

In order to study weighted traces and weighted determinants, let us introduce the Wodzicki Residue ([W], [K]) by its local formula, which we take as a definition.

Definition 2: Let $A \in CL(M, E)$. The Wodzicki Residue of A is:

$$res(A) = \frac{1}{(2\pi)^d} \int_M \int_{S^*M} tr\ a_{-d}(x, \xi) d\xi\ dx,$$

where a_{-d} is the homogeneous component of degree $-d$ of a symbol of A.

This definition is independent of the chosen local chart. From this expression of the Wodzicki Residue, it easily follows that the Wodzicki Residue vanishes on the algebra $CL_{(-1)}(M, E)$ whenever M is odd dimensional.

Let us now give a fundamental result on meromorphic extensions of traces for holomorphic family of classical PDOs. For that, let us recall a few facts about traces and trace class operators. A PDO A is of trace class if and only if $o(A) < -d$. In that case, the Schwartz Kernel $A(x, y)$ of A is a continuous section and the trace of A is given by $Tr(A) = \int_M TrA(x, x)dx$.

Theorem 1 [KV1]:*Let f be an holomorphic function on $\mathbb{C}$ such that $f'(s) \neq 0$ for $s \in f^{-1}\{j - d : j \in \mathbb{N}\}$. Let $(B(s))_{s \in \mathbb{C}}$ be a holomorphic family of operators in $CL(M, E)$ with $B(s)$ of order $f(s)$.*

The function $s \to Tr(B(s))$ is holomorphic in the domain $\{s \in \mathbb{C} : Ref(s) < -d\}$ and can be extended to a meromorphic function on the whole complex plane with only simple poles.

The poles may occur at the points $s_j := f^{-1}(j - d)$, $j \in \mathbb{N}$. At $s = s_j$, the residue is:

$$Res_{s=s_j} Tr(B(s)) = -\frac{1}{f'(s_j)} res\Big(\sigma(B(s_j))\Big).$$

Proof. In what follows, we consider a local chart of E, and we identify the domain of this chart with the corresponding open subset of $\mathbb{R}^d$.

For every $s \in \mathbb{C}$, let $\sum_{j \geq 0} b^{(s)}_{f(s)-j}$ be the formal symbol of $B(s)$ and let $B^{(s)}_{f(s)-j}$ be a PDO of formal symbol $b^{(s)}_{f(s)-j}$. These PDOs can be defined with the help of a cut-off function ψ.

Let $N \in \mathbb{N}^*$ and let us set:

$$B^{(s)}_N := \sum_{j=0}^{N-1} B^{(s)}_{f(s)-j},\ R^{(s)}_N := B(s) - B^{(s)}_N.$$

On the set $\{s \in \mathbb{C} : Ref(s) < N - d\}$, the operator $R^{(s)}_N$ is of trace class and the function $s \to Tr(R^{(s)}_N)$ is holomorphic.

Let $0 \leq j \leq N-1$, and let $s \in \mathbb{C}$ such that $Ref(s) < j - d$.

The kernel of $B^{(s)}_{f(s)-j}$ restricted to the diagonal is given by:

$$B^{(s)}_{f(s)-j}(x,x) = \int \psi(|\xi|) b^{(s)}_{f(s)-j}(x,\xi) d\xi.$$

Going to polar coordinates gives:

$B^{(s)}_{f(s)-j}(x,x) = \int_0^\infty \psi(r) r^{f(s)-j+d-1} dr \int_{|\xi|=1} b^{(s)}_{f(s)-j}(x,\xi) \frac{d\xi}{(2\pi)^d}$.

We can split the first integral $\int_0^\infty \Psi(r) r^{f(s)-j+d-1} dr = \int_0^1 \Psi(r) r^{f(s)-j+d-1} dr + \int_1^\infty \Psi(r) r^{f(s)-j+d-1} dr$.

Both complex functions defined by $\int_{|\xi|=1} b^{(s)}_{f(s)-j}(x,\xi) \frac{d\xi}{(2\pi)^d}$ and $\int_0^1 \psi(r) r^{f(s)-j+d-1} dr$ are analytic on $\mathbb{C}$.

The integral $\int_1^\infty \psi(r) r^{f(s)-j+d-1} dr$ is equal to $\int_1^\infty r^{f(s)-j+d-1} dr$ and hence we have:

$$\int_1^\infty \psi(r) r^{f(s)-j+d-1} dr = -\frac{1}{f(s) - (j-d)}.$$

Then $B^{(s)}_{f(s)-j}(x,x)$ can be extended in a meromorphic function with a simple pole at $s_j = f^{-1}(j-d)$ and the residue at this point is given by:

$$Res_{s=s_j} B^{(s)}_{f(s)-j}(x,x) = -\frac{1}{f'(s_j)} \int_{|\xi|=1} b^{(s_j)}_{-d}(x,\xi) \frac{d\xi}{(2\pi)^d}.$$

Taking traces yields the theorem.

Applying Theorem 1 to the holomorphic family A^{-s}, where $A \in Ell^{*,adm}_{ord>0}(M,E)$, leads to the ζ-function of A, $\zeta_A(s) := Tr(A^{-s})$, which is holomorphic at $s=0$.

Applying Theorem 1 to the holomorphic family AQ^{-s}, where $A \in CL(M,E)$, leads to a linear functional which we now define.

Definition 3: Let $A \in CL(M,E)$ of order $a \in \mathbb{R}$. The expression:

$$Tr^Q(A) := Tr(AQ^{-s}) - \frac{1}{qs} res(A)\Big|_{s=0}$$

is called the Q-weighted trace of A.

In a similar way, we define the Q-weighted trace of logarithms.

Proposition 2: *Let* $A \in Ell^{*,adm}_{ord>0}(M,E)$ *of order* a *and let* $\epsilon > 0$. *The function* $s \to Tr((LogA)Q^{-s})$ *is holomorphic in the domain* $\{Re(s) > \frac{d+\epsilon}{q}\}$ *and can be extended to a meromorphic function on the whole complex plane with a simple pole at* $s = 0$. *The residue of which is:*

$$Res_{s=0} Tr((\mathrm{Log}A)Q^{-s}) = \frac{1}{q} res(\mathrm{Log}A - \frac{a}{q}\mathrm{Log}Q).$$

Proof. For $Re(s) > \frac{d+\epsilon}{q}$, the operator $(\mathrm{Log}A)Q^{-s}$ is of trace class and we have: $Tr((\mathrm{Log}A)Q^{-s}) = Tr((\mathrm{Log}A - \frac{a}{q}\mathrm{Log}Q)Q^{-s}) + \frac{a}{q}Tr((\mathrm{Log}Q)Q^{-s})$. From the definition of Log, it follows that $\frac{d}{ds}Q^{-s} = (\mathrm{Log}Q)Q^{-s}$ and since Tr is continuous on trace class operators, we have $Tr((\mathrm{Log}Q)Q^{-s}) = \zeta'_Q(s)$. Hence $Tr((\mathrm{Log}Q)Q^{-s})$ admits a meromorphic extension to $\mathbb{C}$, holomorphic at $s = 0$. The proposition now follows from Theorem 1.

Remark: Let us point out that, with our notations, we have $\zeta'_A(0) = -Tr^A(\mathrm{Log}A)$, for any $A \in Ell^{*,adm}_{ord>0}(M,E)$.

Definition 4: Let $A \in Ell^{*,adm}_{ord>0}(M,E)$ of order a. The Q-weighted trace of $\mathrm{Log}A$ is:

$$Tr^Q(\mathrm{Log}A) := Tr((\mathrm{Log}A)Q^{-s}) - \frac{1}{qs} res(\mathrm{Log}A - \frac{a}{q}\mathrm{Log}Q)\Big|_{s=0}$$

The obstruction preventing Tr^Q from being tracial and the dependence of Tr^Q in relation to Q can be expressed in terms of a Wodzicki Residue.

Proposition 3 :

1. Let $A, B \in CL(M,E)$, *then* $[logQ, A]B \in CL(M,E)$ *and:*

$$Tr^Q[A,B] = -\frac{1}{q} res([logQ, A]B).$$

2. Let $A \in CL(M,E)$ *and let* $Q_1, Q_2 \in Ell^{*,adm}_{ord>0}(M,E)$, *then*

$$Tr^{Q_1}(A) - Tr^{Q_2}(A) = res\Big((logQ_2/o(Q_2) - logQ_1/o(Q_1))A\Big).$$

The assertion 1. was first proved independently in [CDMP] and in [MN]. The second one can easily be deduced from Theorem 1 and was proved in [KV1] and [O2].

This proposition provides us with two examples of subalgebras of $CL(M,E)$ on which Tr^Q is tracial:

- the subalgebra $CL^Q(M,E)$ of operators which commute with Q modulo a regularizing operator (it is easy to check that it is an algebra since the product of a PDO by a regularizing operator is a regularizing operator).

- the subalgebra $CL_{(-1)}(M,E)$ of odd-class operators when M is odd dimensional, Q is of odd-class and of even integer order. In that case, Tr^Q is independent of Q and is denoted by $Tr^{(-1)}$ as in [KV1,2].

The first assertion follows from the fact that the Wodzicki Residue vanishes on regularizing operator, as its local expression shows. The second one relies upon the fact that the two residues in Proposition 3 are then taken on odd-class operators and hence vanishes in odd dimensions.

The following result will be used later.

Proposition 4: *Let $(A_t)_{t\in\mathbb{R}}$ be a family in CL(M,E) of constant order. If (A_t) is continuous w.r. to t then $lim_{t\to t_o} res(A_t) = res(A_{t_o})$ and $lim_{t\to t_o} Tr^Q(A_t) = Tr^Q(A_{t_o})$, where $t_o \in \mathbb{R}$.*

Proof. Let m be the constant order of the family $(A_t)_{t\in\mathbb{R}}$. W.r.t. the topology on $CL_{ord\leq m}(M,E)$ described above, the first result is obvious. The second one relies upon the expression of the meromorphic extension of $Tr(AQ^{-s})$ given in the proof of Theorem 1.

4 Determinants associated to weighted traces

Definition 5: Let $A \in Ell^{*,adm}_{ord\geq 0}(M,E)$. The determinant associated to the Q-weighted trace Tr^Q, also called the Q-weighted determinant, is defined by:

$$det^Q(A) := exp Tr^Q(\mathrm{Log} A).$$

It coincides with the usual determinant when M is reduced to a point and with the Fredholm determinant for suitable operators as the next proposition shows.

Proposition 5: *Let $K \in CL(M,E)$ of order strictly less than $-d$. Then the Q-weighted determinant of $I+K$ is independent of Q and is equal to the Fredholm determinant of $I+K$.*

Proof. Since K is of trace class, $\mathrm{Log}(I+K)$ is also of trace class and via Hadamard's Theorem, we have $det_{Fred}(I+K) = exp Tr \mathrm{Log}(I+K)$.

• *Multiplicative anomaly*

Proposition 6:*Let $\mathcal{A}$ be a subalgebra of $CL(M,E)$ on which Tr^Q is tracial and such that $\mathcal{A}\cap Ell^*(M,E)$ is a group.*
Let (A_t) be a differentiable one-parameter family of operators in $\mathcal{A}\cap Ell^{*,adm}_{ord\geq 0}(M,E)$, of constant order, and such that there exists a same spectral cut contained in a common Agmon-Nirenberg angle for all A_t. Then:

$$\frac{d}{dt}Tr^Q(\mathrm{Log}A_t) = Tr^Q(\dot{A}_t A_t^{-1}).$$

Proof. Let us denote by Λ the Agmon-Nirenberg angle for all A_t and let be $t\in\mathbb{R}$. We need to show that for any $\lambda\notin Sp(A_t)$, $\lambda\in\Lambda$,

$$Tr^Q\left[(A_t-\lambda I)^{-1}\dot{A}_t(A_t-\lambda I)^{-1}\right] = Tr^Q\left[\dot{A}_t(A_t-\lambda I)^{-2}\right].$$

Then we conclude by integrating by parts. By Proposition 4, we have $\lim_{h\to 0}Tr^Q\left[(A_t-\lambda I)^{-1}\left(\frac{A_{t+h}-A_t}{h}\right)(A_t-\lambda I)^{-1}\right] = Tr^Q\left[(A_t-\lambda I)^{-1}\dot{A}_t(A_t-\lambda I)^{-1}\right]$ and an analogous expression for the r.h.s.. These traces being limits of traces of operators in $\mathcal{A}$, the proposition follows from the fact that Tr^Q is tracial on $\mathcal{A}$, switching the order of the terms in the l.h.s. using the tracial property.

Theorem 2 [D]: *Let $\mathcal{A}$ be a subalgebra of $CL(M,E)$ satisfying the assumptions of Proposition 6 and such that if $A\in\mathcal{A}\cap Ell^{*,adm}_{ord=0}(M,E)$, then $A^t\in\mathcal{A}$ for any $t\in\mathbb{R}$.*
Let $A,B\in\mathcal{A}\cap Ell^*_{ord\geq 0}(M,E)$ of integer orders and such that their leading symbols are sufficiently close to positive definite self-adjoint ones. Then A, B, and AB admit a same spectral cut and:

$$det^Q(AB) = det^Q(A)det^Q(B).$$

Proof. Let F^Q be the multiplicative anomaly of det^Q defined as F_ζ. Let $A,B\in\mathcal{A}\cap Ell^*_{ord\geq 0}(M,E)$ of integer orders, B of order 1. We consider the family $A_t := \left(AB^{-o(A)}\right)^t B^{o(A)}$, $0\leq t\leq 1$. If A and B have their leading symbols sufficiently close to positive definite self-adjoint ones, then A, B, AB et A_t for all t are admissible and admit a same spectral cut contained in a same Agmon-Nirenberg angle. The assumptions on $\mathcal{A}$ imply that $A_t\in\mathcal{A}\cap Ell^*_{ord\geq 0}(M,E)$. We have $\frac{d}{dt}\log\, F^Q(A_t,B) =$

$\frac{d}{dt}Tr^Q(\mathrm{Log}(A_tB) - \frac{d}{dt}Tr^Q(\mathrm{Log}A_t)$. Since $\dot{\widetilde{A_tB}}(A_tB)^{-1} = \dot{A}_tA_t^{-1}$, by Proposition 6, we have $\frac{d}{dt}\log\ F^Q(A_t, B) = 0$ for all $0 \leq t \leq 1$. One can check that $F^Q\Big(B^{o(A)}, B\Big) = 1$, hence $F^Q(A, B) = 1$. If $o(B) > 1$, the multiplicative property still holds. Since $det^Q(AB^{1/o(B)}) = det^Q(A)det^Q(B^{1/o(B)})$, we have

$$det^Q(AB^{2/o(B)}) = det^Q(AB^{1/o(B)}B^{1/o(B)}) = det^Q(A)det^Q(B^{2/o(B)})$$

and by iterating this argument we obtain $det^Q(AB) = det^Q(A)det!!^!!Q!!(B)$.

Examples: The two subalgebras $CL^Q(M, E)$ and $CL_{(-1)}(M, E)$ described in section 3 satisfy the assumptions of Theorem 2.

5 ζ-regularized Determinant

In this section we essentially relate the weighted determinants to the ζ-regularized determinant, and thus we deduce a general formula for the multiplicative anomaly of the ζ-regularized determinant. We also explain how weighted determinants extend M.Kontsevich and S.Vishik's ζ-determinant for zero order odd-class operators when the manifold is odd dimensional.

Definition 6 : For any operator $A \in Ell^{*,adm}_{ord>0}(M, E)$, the ζ-regularized determinant of A is :

$$det_\zeta(A) := exp\Big(-\zeta'_A(0)\Big).$$

The next result is fundamental for the sequel.

Proposition 7 : *For any operator $A \in Ell^{*,adm}_{ord>0}(M, E)$ of order a,*

$$Tr^A(\mathrm{Log}A) - Tr^Q(\mathrm{Log}A) = -\frac{1}{2a}res\Big((\mathrm{Log}A - \frac{a}{q}\mathrm{Log}Q)^2\Big).$$

A proof can be carried out by deforming A, differentiating the two sides of the equality and using Proposition 3. This is a particular case of a more general result given in [O2].

Using Proposition 7 one can check that in the odd case, i.e. odd-class operators and odd dimensional manifold, the trace $Tr^{(-1)}$ (defined at the end of §3) extends to logarithms of even integer order.

Let be $A \in Ell^{*,adm}_{ord>0}(M, E)$. We recall that $det_\zeta(A) = expTr^A(\mathrm{Log}A)$ (see the last remark in §3). Let us assume that A is an odd-class operator with even integer order and that M is an odd dimensional manifold. Then we have $det_\zeta(A) = expTr^{(-1)}(\mathrm{Log}A)$. In [KV1,2], M.Kontsevich and S.Vishik have noted that for any operator $A \in Ell^*_{ord=0}(M,E)$ with leading symbol sufficiently close to a positive definite self-adjoint one, the ratio $det_\zeta(AC)/det_\zeta(C)$ does not depend on the odd-class positive self-adjoint operator C of even integer order chosen in $Ell^*_{ord>0}(M,E)$. This comes from the fact that since $Tr^{(-1)}$ is tracial, we have $Tr^{(-1)}(\mathrm{Log}(AC)) = Tr^{(-1)}(\mathrm{Log}A) + Tr^{(-1)}(\mathrm{Log}C)$. For such an operator A, they define : $det(A) := det_\zeta(AC)/det_\zeta(C)$. In fact, $det(A) = expTr^{(-1)}(\mathrm{Log}A)$. It follows that any weighted determinant det^Q on zero order operators, with Q of odd-class and even integer order, extends the determinant of M.Kontsevich and S.Vishik defined in the odd case.

Now since again $det_\zeta(A) = expTr^A(\mathrm{Log}A)$, we obtain:

Corollary 1 [D]: *For any operator $A \in Ell^{*,adm}_{ord>0}(M,E)$ of order a,*

$$\frac{det_\zeta(A)}{det^Q(A)} = exp\left\{-\frac{1}{2a}\left[res\Big((\mathrm{Log}A - \frac{a}{q}\mathrm{Log}Q)^2\Big)\right]\right\}.$$

Corollary 1 can be understood as the obstruction preventing det_ζ from being of the form $expTr^Q\mathrm{Log}$. The expression of this obstruction confirms that there is a quadratic non linearity hidden in the definition of the ζ-determinant, as M.Kontsevich and S.Vishik pointed out in [KV1].

Corollary 2 [D]: *Let $A, B \in Ell^{*,adm}(M,E)$, of resp. orders $a > 0$, $b > 0$, such that their principal symbols are sufficiently close to positive definite self-adjoint ones. The multiplicative anomaly of the ζ-determinant is given by:*

$$\begin{aligned} logF_\zeta(A,B) &= \frac{1}{2a}res\Big((LogA - \frac{a}{a+b}Log(AB))^2\Big) \\ &+ \frac{1}{2b}res\Big((LogB - \frac{b}{a+b}Log(AB))^2\Big) \\ &+ Tr^{AB}\Big(Log(AB) - LogA - LogB\Big). \end{aligned} \tag{1}$$

In particular, if A and B commute, then we have:

$$logF_\zeta(A,B) = \frac{1}{2ab(a+b)}res\Big(\mathrm{Log}(A^bB^{-a}))^2\Big). \tag{2}$$

Proof. We write $\mathrm{Log}F_\zeta(A,B) = Tr^{AB}(\mathrm{Log}(AB)) - Tr^A(\mathrm{Log}A) - Tr^B(\mathrm{Log}B) = Tr^{AB}(\mathrm{Log}(AB) - \mathrm{Log}A - \mathrm{Log}B) + Tr^{AB}(\mathrm{Log}A) - Tr^A(\mathrm{Log}A) +$

$Tr^{AB}(\mathrm{Log}B) - Tr^{B}(\mathrm{Log}B)$. Applying Proposition 7 to the two last differences leads to (1). If $[A,B]=0$, then one can check that $\mathrm{Log}(AB)=\mathrm{Log}A+\mathrm{Log}B$. Hence we deduce (2) from (1) by a direct calculus.

In [O1], K.Okikiolu established a Campbell-Hausdorff formula for operators with scalar leading symbols: if $A, B \in Ell^{*,adm}_{ord\geq 0}(M,E)$ admit scalar leading symbols and satisfy some technical assumptions then $\mathrm{Log}(AB) - \sum_{k=1}^{k=N} C^{(k)}(\mathrm{Log}A, \mathrm{Log}B)$ has order $-N$, where $C^{(k)}(\mathrm{Log}A, \mathrm{Log}B)$ is a linear combination of Lie monomials of degree k in $\mathrm{Log}A$ and $\mathrm{Log}B$. Using this formula, she gives an expression of $F_\zeta(A,B)$ in terms of a Wodzicki Residue. If we apply her Campbell-Hausdorff formula to $Tr^{AB}(\mathrm{Log}(AB)-\mathrm{Log}A-\mathrm{Log}B)$, we recover her expression of $F_\zeta(A,B)$.
In the case of commuting operators, the second formula has been established by M.Wodzicki (see [K] and [EVZ] for a proof).

Appendix

In this appendix, we give a brief presentation of classical and odd-class classical PDOs on an open subset of $\mathbb{R}^d$. For details, we refer the reader to [Sh], [T1], [Di7].

• *Classical symbols*

Let U be an open subset of $\mathbb{R}^d$. Given $m \in \mathbb{R}$, let us denote by $S^m(U)$ the set of symbols of order m i.e. the set of complex valued smooth function

$$\begin{aligned} U \times \mathbb{R}^d &\to \mathbb{C} \\ (x,\xi) &\to \sigma(x,\xi) \end{aligned}$$

such that given any compact subset K of U and any two multiindices $\gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{N}^d$, $\delta = (\delta_1, \cdots, \delta_d) \in \mathbb{N}^d$, there exists a constant $C_{\gamma,\delta}(K)$ satisfying the following requirement

$$|D_x^\gamma D_\xi^\delta \sigma(x,\xi)| \leq C_{\gamma,\delta}(K)(1+|\xi|)^{m-|\delta|} \quad \forall x \in K, \forall \xi \in \mathbb{R}^d.$$

We shall denote $S(U) := \cup_{m\in\mathbb{R}} S^\alpha(U)$, $S^{-\infty}(U) := \cap_{m\in\mathbb{R}} S^\alpha(U)$. The relation: $\sigma_1 \simeq \sigma_2 \Leftrightarrow \sigma_1 - \sigma_2 \in S^{-\infty}(U)$ defines an equivalence relation on $S(U)$.

A symbol σ of order m (which can be a complex number) is called a *classical symbol* if there exists an asymptotic expansion

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \psi(\xi)\sigma_{m-j}(x,\xi) \qquad (A1)$$

where ψ is a cut-off function i.e. $\psi \in C^\infty(\mathbb{R}^d)$, $\psi(\xi) = 0$ for $|\xi| \leq 1/2$, $\psi(\xi) = 1$ for $|\xi| \geq 1$, and $\sigma_{m-j}(x, \xi)$ is a positively homogeneous function on $U \times \mathbb{R}^d$ of degree $Re\alpha - j$ in ξ, i.e. $\sigma_{\alpha-j}(x, t\xi) = t^{m-j}\sigma_{\alpha-j}(x, \xi) \quad \forall t \in \mathbb{R}^+$.
In (A1), the symbol $\sim$ means that, for every integer $N > 0$, $\sigma - \sum_{j=0}^{N-1} \psi(\xi)\sigma_{m-j}(x, \xi) \in S^{m-N}(U)$. We shall call $\sum_{j=0}^{\infty} \sigma_{m-j}$ a formal expansion of σ and we shall note: $\sigma \sim \sum_{j=0}^{\infty} \sigma_{m-j}$.

Notice that Property (A1) implies that for $|\xi| \geq 1$, $\sigma_{m-j}(x, \xi) \in S^{m-j}(U)$ ([Di7], p114). Furthermore, if another classical symbol σ' is given by the same homogeneous components as σ, then $\sigma \simeq \sigma'$.

On the set of symbols, there is a composition law denoted by $*$ and defined by:

$$\sigma_1 * \sigma_2(x, \xi) \simeq \sum_\alpha i^{-|\alpha|} \frac{1}{\alpha!} D_\xi^\alpha \sigma_1(x, \xi) D_x^\alpha \sigma_2(x, \xi).$$

In particular, if a, b are two classical symbols with formal expansion: $a \sim \sum_{j=0}^{\infty} a_{m-j}$, $b \sim \sum_{j=0}^{\infty} b_{n-j}$, then $a * b$ is also a classical symbol and the homogeneous component of $a * b$ of degree $m + n - j$ is given by:

$$(a * b)_{m+n-j} = a_m b_{n-j} + \sum_{k+l+|\alpha|=j,\, l<j} i^{-|\alpha|} \frac{1}{\alpha!} D_\xi^\alpha a_{m-k} D_x^\alpha b_{n-l}.$$

• *Odd-class classical symbols*

Following Kontsevich and Vishik [KV1], we shall say that a classical symbol lies in the *odd-class* if the positively homogeneous components σ_{m-j} are moreover homogeneous i.e.: $\sigma_{m-j}(x, t\xi) = t^{m-j}\sigma_{m-j}(x, \xi) \quad \forall t \in \mathbb{R}$.
From the composition law formula, it follows that the composition of two odd-class classical symbols is also an odd-class classical symbol.

• *Classical pseudo-differential operators*

To a symbol $\sigma \in S(U)$ we can associate an operator $Op(\sigma)$ defined by:

$$\begin{aligned} Op(\sigma) : C_c^\infty(U) &\to \mathbb{C}^\infty(U) \\ u &\to \left(x \to Op(\sigma)u(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma(x, \xi) \hat{u}(\xi) d\xi \right). \end{aligned}$$

where $C^\infty(U)$, resp. $C_c^\infty(U)$ denotes the space of complex valued smooth functions, resp. with compact support, in U.
As usual, we shall say that an operator is regularizing if it admits a C^∞ kernel.

If $\sigma \in S^{\infty}(U)$ then $Op(\sigma)$ is a regularizing operator on U and the converse is true (see [T1], p35).

A *pseudo-differential operator* A on U is an operator which can be written in the form:

$$A = Op(\sigma) + R$$

where R is a regularizing operator and σ belongs to $S(U)$.

The symbol σ is called a symbol of A and the order of σ is called the order of A. We denote a symbol of A by $\sigma(A)$ and the order of A by $o(A)$. Notice that $\sigma(A)$ is defined modulo $S^{-\infty}(U)$. If $\sigma(A)$ is classical then A is called a *classical PDO* . An *odd-class classical PDO* is a classical PDO such that its symbol lies in the odd class. Notice that ordinary differential operators provide example of classical PDOs in the odd class.

We shall say that an operator A is a matricial PDO of order m if A is a matrix with all terms given by PDO of order m.

Let us recall classical results on the composition of PDOs.

A PDO A on U is called *properly supported* if for every compact $K \subset U$, the set $\{(x, y) \in SuppK_A : x \in K$ or $y \in K\}$ is compact. A properly supported PDO A on U is defined on $C^{\infty}(U)$ and it admits a unique symbol given by: $\sigma(A)(x, \xi) = e_{-\xi}(x) A e_{\xi}(x)$, where $e_{\xi}(x) := e^{ix.\xi}$.

PDOs and properly supported PDOs are related as follows: Let A be a PDO. Then A can be written as a sum: $A = A_0 + A_1$ where A_0 is a properly supported PDO and A_1 is a regularizing PDO. In general, an operator of the form $Op(\sigma)$ is not properly supported.

Now, if A, B are two properly supported PDOs on U, then the composition AB is a properly supported PDO and $\sigma(AB) = \sigma(A) * \sigma(B)$. Furthermore, if A is a PDO and R is a regularizing operator, then AR and RA are both regularizing operators. From this result and the composition law formula, it follows that the set of PDOs, resp. classical, resp. odd-class classical PDOs, on U are algebras.

Finally let us recall the classical result on change of coordinates:
Let $\chi : U \to \chi(U)$ be a diffeomorphism and let A be a PDO on U. Let A_1 be the operator on $C_c^{\infty}(\chi(U))$ defined by:

$$A_1 u_1 := [A(u_1 o \chi)] o \chi^{-1}.$$

Then A_1 is a PDO and a symbol of A_1 is given by:

$$\sigma(A_1)(y, \eta)|_{y=\chi(x)} \sim \sum_{\alpha} \frac{1}{\alpha!} \sigma(A)^{(\alpha)}(x, {}^t\chi'(x)\eta).D_z^{\alpha} exp(i\chi_x''(z).\eta|_{z=x},$$

where $\sigma(A)^{(\alpha)}(x,\xi) := D_\xi^\alpha \sigma(A)(x,\xi)$ *and* $\chi_x''(z) := \chi(z) - \chi(x) - \chi'(x)(z-x)$.

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