ON COHOMOGENEITY ONE RIEMANNIAN MANIFOLDS

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In this paper after giving a short introduction to the subject "Riemannian G-manifolds", we concentrate on cohomogeneity one Riemannian manifolds. We study "revolution hypersurfaces" and characterize such hypersurfaces completely. This acticle follows a previous paper [Ka] of which it uses some of the results.

0 Introduction

Let (M^n, g) be a Riemannian manifold of dimension n, let G be a closed Lie subgroup of Iso (M)=the Lie group of isometries of M. Suppose that G acts smoothly on M. In the theory of Riemannian G-manifolds one studies the topological and/or geometric properties of M, M/G (the orbit space), G and the orbits of the action of G on M with the help of each other. A good reference for G-manifold theory is [B].

Riemannian G-manifolds, which are a generalization of homogeneous spaces, form a beautiful and important subject in mathematics. They also have many applications in mathematics and physics, including Riemannian submanifold theory, especially isoparametric hypersurfaces, and submanifolds (see [CD]) invariant theory, differential equations and variational problems.

In general if G acts "nicely" on M i.e., if M admits a section (see [PT]) and one wants to study some objects defined on M, if the objects are invariant under the action of G, then one can consider M/G and study the induced objects on M/G. This is in general a (much) simpler problem. For a detailed discussion of the applications one should see [PT].

The congugacy class of a closed subgroup H of G is denoted by (H) and is called a G-orbit type. The orbit $G.x$ is said to be of type (H) if $(G_x) = (H), G_x$ is the isotropy subgroup of G at x. $M_{(H)} \subseteq M$ denotes the union of all orbits of type (H).

Among all orbit types (H) with $M_{(H)} \neq \phi$, there is a unique (U) such that for $H < G$ with $M_{(H)} \neq \phi$, there is a g in G with the property that $gUg^{-1} \subset H$. It follows that G/U has maximum dimension and (for that dimension) the maximum number of components. An orbit $G.x$ for which $(G_x) = (U)$ is called principal orbit. The nonprincipal orbits are called singular orbits. The union of principal orbits (resp. singular orbits) is denoted by M_{reg} (resp.

M_s). We have that $M = M_{reg} \cup M_s$ and that M_{reg} is an open dense subset of M.

If the dimension of a principal orbit of M^n is $n-k$, then M is called cohomogeneity k (Riemannian) manifold. In this case, if $G.x$ is a singular orbit then dim $G.x \leq n-k$; if dim $G.x = n-k$, $G.x$ is called an exceptional singular orbit.

Riemannian G-manifolds, especially cohomogeneity one Riemannian manifolds have been studied recently by many mathematicians, including Bredon [B], Palais & Terng [PT], Mostert [M], D.V& A.V. Alekseevsky [A], P. Podesta & A. Spiro [PS].

Following F. Podesta & A. Spiro, I want to concentrate on one problem in the theory of cohomogeneity one Riemannian manifolds. I want to study a family of hypersurfaces of the Eulidean space called *revolution hypersurfaces*. There is a natural family among all cohomogeneity one hypersurfaces of the Euclidean space $\mathbb{R}^{n+1}$ for which the Lie group G can be considered as a subgroup of Iso ($\mathbb{R}^{n+1}$). But one should note that not all cohomogeneity one hypersurfaces of $\mathbb{R}^{n+1}$ for which G can be considered as a subgroup of Iso ($\mathbb{R}^{n+1}$) is a revolution hypersurface. For Examples of such hypersurfaces see 2.9 of [PS] and Examples 2.2 & 2.3.

The motivation to study revolution hypersurfaces is a well-known theorem of S. Kobayashi [K] which says that:

Theorem: Any compact homogeneous Riemannian manifold M^n isometrically immersed as a hypersurface of $\mathbb{R}^{n+1}$ is congruent to a sphere S^n (i.e. there exists an $h \in$ Iso ($\mathbb{R}^{n+1}$) such that $h(M) = S^n$).

This theorem has been generalized by other authors in different directions. Nagano and Takahashi in two papers ([NT],[T]) gave a complete classification of homogeneous hypersurfaces of Riemannian space forms without the assumption of compactness. Their list is as follows: $S^m \times \mathbb{R}^{n-m}$ or H^n or S^n or $S^m \times H^{n-m}$.

A. Ros proved in [R] that any compact hypersurface of constant scalar curvature is congruent to a sphere.

Recently F. Podesta & A.Spiro inspired by the subject of G-manifolds want to generalize Kobayashi's theorem by weakening the homogeneity assumption and consider compact cohomogeneity one Riemannian hypersurfaces of $\mathbb{R}^{n+1}$, they characterize the family of revolution hypersurfaces among all compact cohomogeneity one hypersurfaces.

I want to consider revolution hypersurfaces of $\mathbb{R}^{n+1}$ without the assumption of compactness. The main result of the paper is Theorem 2.1 and Examples 2.2 & 2.3.

1 Preliminaries

Definition 1.1. Let (M^n, g) and $(\overline{M}^m, h)$ be two Riemannian manifolds, a smooth function $f : (M, g) \longrightarrow (\overline{M}, h)$ is called an isometric immersion if $h_{f(x)}(T_x f(v), T_x f(w)) = g_x(v, w) \ \ \forall x \in M, \forall v, w \in T_x M$.

In this case $f(M)$ is called an isometrically immersed Riemannian submanifold of $\overline{M}$.

$f(M)$ is locally a submanifold of $\overline{M}$, thus as far as local notions are concerned one can consider $f(M)$ with the induced metric, as a Riemannian submanifold of $\overline{M}$.

In differential geometry it is common to make no distinction between $M, f(M)$ and consider M (itself) as a (local) submanifold of $\overline{M}$. So the extrinsic geometry of $f(M)$ in $\overline{M}$ is attributed to $f(M)$ or M or even f equally.

Definition 1.2. Let $M^n \subset \overline{M}^m$ be a Riemannian submanifold, let NM be the normal bundle of M, let $\xi \in \Gamma(NM)$ be a (local) normal vector field, let $\overline{D}$ be the Levi-Civita connection of $\overline{M}$, let $X \in \chi(M)$ be a (local) tangent vector field, then A_ξ defined by $A_\xi X = -(\overline{D}_X \xi)^T = -$ (tangent component of $\overline{D}_X \xi$) is called the *shape operator* tensor of M along ξ.

Remark 1.3. $A_\xi X$ is a tensor field on M, so for each $x \in M$, we have a map $A : TM \longrightarrow TM$, defined by $A(v) = -(\overline{D}_X \xi)^T \ \forall X \in \chi(M), X(x) = v$. A is a self-adjoint linear map called the shape operator of M along ξ.

Definition 1.4. If $m - n = 1$, then the rank of A at $x \in M$ is called the type number of the hypersurface M at x. If $f : (M^n, g) \longrightarrow (\overline{M}^{n+1}, h)$ is an isometric immersion, then $rk(f)_x$ is the rank of the shape operator of $f(M)$ in $\overline{M}$ at x.

Definition 1.5. An (immersed) hypersurface M^n of $\mathbb{R}^{n+1}$ is of *revolution (or a revolution hypersurface)* if there exists a line $l \subset \mathbb{R}^{n+1}$ such that the rotation group $G_l \cong So(n)$, of rotations around the line l, leaves M invariant.

Definition 1.6. A cohomogeneity one Riemannian manifold M^n is called *umbilical* if each principal orbit is *totally umbilic* in M, i.e., the shape operators of the principal orbits are of the form λId_{n-1}, where the λ's are smooth functions on the orbits.

When M is a cohomogeneity one manifold under the action of G, then it is well-known (see [A]) that $M/G \cong \mathbb{R}, S^1$, $[0, \infty)$ or $[0, 1]$. If $k : M \longrightarrow M/G = \Omega$ is the projection, $G.x =$ the orbit of x is called *principal* (resp. *singular*) if $k(x) \in \Omega^0$ (the interior of Ω), (resp. $k(x)$ is in the boundary of Ω). Note that these notions are in agreement with the above definitions. The point x is called regular (resp. singular). So when the orbit space M/G is $\mathbb{R}$ or S^1 then all orbits are regular, if M/G is the interval $[0,\infty)$ there is just one singular orbit and if M/G is $[0,1]$, there are two singular orbits. We set M_{reg} to be the set of all regular points of M, M_{reg} is an open dense subset of M. If $x \in M$, there exists a neighborhood U of x such that (U,g) is locally isometric to $(G/H \times I, g_t + dt^2)$. Here H is the isotropy subgroup of G at x, I is an open interval of $\mathbb{R}, g_t$ is a family of G-invariant metrics on G/H depending smoothly on $t \in I, dt^2$ is the standard metric of $\mathbb{R}$.

2 Main results

Theorem 2.1 ([Ka]): Let $M^n (n \geq 4)$ be a complete Riemannian manifold with nonnegative curvature and of cohomogeneity one under the action of a compact connected Lie subgroup of Iso (M) and let $f : M \longrightarrow \mathbb{R}^{n+1}$ be an isometric immersion such that f is $1-1$ on the set of points where $rk(f) \leq 1$. Then M is a revolution hypersurface iff M is an umbilical cohomogeneity one Riemannian manifold.

Proof. See [Ka] for the complete proof of the theorem.

Example 2.2 ([Ka]). Let $M = \mathbb{R}^n \# S^n \subset \mathbb{R}^{n+1}$ (the smooth connected sum of $\mathbb{R}^n, S^n$), the imbedding $\phi : M \longrightarrow \mathbb{R}^{n+1}$ is defined by

$$\phi([x]) = \begin{cases} x \text{ if the class } [x] \text{ has just one point,} \\ (x,0) \text{ if the class } [x] \text{ has two points } x \in \mathbb{R}^n, x' \in S^n \end{cases}$$

Note that we can assume in the second case that $x \in \partial(\mathbb{R}^n - \overset{\circ}{D}{}^n), x' \in \partial(S^n - \overset{\circ}{E})$ and x (identified with $(x,0)$), x' are the same vector as points of $\mathbb{R}^{n+1}$. Here $\overset{\circ}{D}{}^n$ is the interior of the disk $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $\overset{\circ}{E}$ is the interior of the closed lower hemisphere i.e., $E = \{x \in S^n : x = (x_1, \dots, x_{n+1}), x_{n+1} \leq 0\}$, ∂ stands for the boundary. Now we consider M with the induced metric of $\mathbb{R}^{n+1}$. G_l=the rotations in $\mathbb{R}^{n+1}$ around the line l passing through the center of S^n and orthogonal to $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$, so $G_l \cong So(n)$. $M \subset \mathbb{R}^{n+1}$ is a revolution hypersurface, by definition 1.5, since $G_l(M) \subseteq M$. If $[x] \in M$ either $[x] \cong x \in \mathbb{R}^n$ then the orbit $G_l.x$ is $S^{n-1}(\|x\|)$ or $[x] \cong x \in S^n$, then $G_l.x = S^{n-1}_{(0,x_{n+1})}(\sqrt{1-x^2_{n+1}})$, i.e., the $(n-1)$-sphere with center $(0, x_{n+1}) \in \mathbb{R}^{n+1}$ where $x = (x_1, \dots, x_{n+1})$ and radius $\sqrt{1-x^2_{n+1}}$. Both $S^{n-1}(\|x\|)$ and $S^{n-1}_{(0,x_{n+1})}(\sqrt{1-x^2_{n+1}})$ are totally umbilic in M, so again by theorem 2.1, M is a revolution hypersurface.

Example 2.3 ([Ka]). Let $M = S^n \times \mathbb{R}^m \subset \mathbb{R}^{n+m+1}$, $n \geq 1, m \geq 2$, $G = So(n+1) \times So(m)$ with the action $(h_1, h_2)(x, y) = (h_1 x, h_2 y)$ $\forall (h_1, h_2) \in G$, $\forall (x, y) \in M$. M is a cohomogeneity one hypersurface that is not of revolution since it does not satisfy definition 1.5. To see this, take any line $l \subset \mathbb{R}^{n+m+1}$ and consider G_l, take any point $(x, y) \in M - l$, then $G_l \cong So(n+m)$ leaves the $(n+m)$-dimensional plane P through (x, y) and orthogonal to l invariant and it acts transitively on the $(n+m-1)$-dimensional sphere S in P with center $P \cap l$. So certainly there is a point $(x', y') \in S$ with $\|x'\| \neq 1$ and there is a $g \in G_l$ s.t $g(x, y) = (x', y')$, but $(x', y') \notin M$, so M is not invariant under G_l.

If $(x, y) \in M$ is an arbitrary point, its orbit $G.(x, y)$ is $S^n \times S^{m-1}(\|y\|)$, so the shape operator of $G.(x, y)$ in M at (x, y) is $\left[\begin{array}{c|c} 0 & \\ \hline & -\frac{1}{\|y\|} Id_{m-1} \end{array}\right]$ for $\|y\| \neq 0$ and the shape operator is identically 0 for $(x, 0) \in M$. Hence the principal orbits are not totally umbilic in M. Thus theorem 2.1 shows as well that M is not a revolution hypersurface.

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