A DIFFERENTIABLE CALCULUS ON THE SPACE OF LOOPS AND CONNECTIONS

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In this short communication we show how a manifold carries a natural geometrical principal bundle from which any gauge theory can be represented. We also report on a natural differentiable calculus (non-local in the manifold) in this geometrical principal bundle inspired from the notion of differentiable spaces. Using the loop transform, we can apply this calculus to describe a G-gauge theory as a non-local differentiable calculus of their connections.

1 Introduction

Let us state the problem of the representation of principal bundles and connections in this way: we know from the theory of principal bundles and connections the following important lemma $[D]$:

Lemma 1.1 *Let $\pi : P \to M$ be a principal G-bundle. Let $\rho : G \to H$ be a morphism of Lie-groups, then there exists a unique principal H-bundle $\tilde{\pi} : W \to M$ and a morphism $\Phi : P \to W$ of principal bundles such that $\Phi(g.p) = \rho(g).\Phi(p)$. Moreover given a connection A on P there exist a unique connection $\tilde{A}$ on W such that $d\rho(A) = \Phi^*(\tilde{A})$.*

P is called a reduction of W via ρ and W is called an extension of P via ρ. So, with this at hand, we ask the following: Given a manifold M, is there a universal principal bundle carrying a universal connection such that any principal bundle and their connections are extensions of it?. The answer as we will see is positive, this will be the universal gauge structure of the manifold. The universal gauge group will be the *group of loops* of M and the principal bundle the *path bundle of the manifold M*. I will explain the basics of a differentiable calculus on the *path bundle* that makes this universal principal bundle similar to a differentiable principal bundle and the *group of loops* similar to a Lie group. Details can be found in $[RS]$.

2 The construction of the universal gauge structure.

In this section we introduce the universal gauge structure. The universal reduction will be the so called *path bundle of the manifold* and the universal gauge group will be the *group of loops*. It turns out that both structures are subsets of the *paths of the manifold*. We will develop a differentiable structure on the *paths of the manifold*, the differentiable structure on the *path bundle* and the *group of loops* being it restriction. So we begin by defining the *paths of the manifold*. First we define the space of parametrized paths of a manifold:

Definition 2.1 *The space of parametrized paths of a manifold M is $P_p = \{\gamma : [0,1] \to M$, piece-wise differentiable$\}$, whereby piece wise differentiable means: $\exists\ 0 \leq t_1 \leq \ldots \leq t_n \leq 1/\gamma$ is differentiable in each $[t_i, t_{i+1}]$.*

For many purposes (but for many others not) this space is big, because of the parameterization dependence of its elements. So to eliminate this redundancy we define the re-parametrization equivalence relation:

$\sim_1$: $\gamma \sim_1 \varsigma \leftrightarrow \exists\ \alpha : [0,1] \to [0,1]$, *piece $-$ wise increasing diffeomorphism* $\gamma(t) = \varsigma(\alpha(t))$.

$P_u = P_p / \sim_1$ is the space of unparametrized paths of the manifold. In this space we have a natural operation, which is simply the composition of two paths:

$$\circ : P_u \times P_u \mid_{\gamma(1)=\varsigma(0)} \to P_u \ / \ [\gamma] \circ [\varsigma] = [\gamma \circ \varsigma]$$

$$\gamma \circ \varsigma(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq 1/2 \\ \varsigma(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

We see from this that $[\gamma(t)] \circ [\gamma(1-t)]$ is not $[e(t) = \gamma(0)]$. In order to define the *group of loops* we introduce a new equivalence relation $\sim_2$.
First we define a reduction and we note it by $\varsigma < \gamma$.

$$\varsigma < \gamma \Leftrightarrow \begin{cases} \exists\ \varsigma_1, \alpha_1, \ldots, \varsigma_n, \alpha_n, \varsigma_{n+1} \ \in P_u \\ \gamma = \varsigma_1 \circ \alpha_1 \circ \alpha_1^{-1} \ldots \circ \alpha_n \circ \alpha_n^{-1} \circ \varsigma_{n+1} \\ \varsigma = \varsigma_1 \circ \ldots \circ \varsigma_n \end{cases}$$

And we define $\sim_2$ by

$$\gamma \sim_2 \varsigma \Leftrightarrow \exists\ \zeta \ / \ \gamma < \zeta \ , \ \textit{and} \ \varsigma < \zeta$$

This is a non trivial equivalence relation and serves to eliminate the "branches" of a path.
We define the *paths of the manifold* as $P = P_u / \sim_2$.
We are in a position to define the *path bundle* and the *group of loops* $[L]$. We fix a point o of the manifold.

Definition 2.2 *The path bundle of the manifold is $\mathcal{PM} = \{\gamma \in P \ / \ \gamma(0) = o\}$.*

Definition 2.3 *The group of loops of the manifold is* $\mathcal{L} = \{\gamma \in P \ / \ \gamma(0) = \gamma(1) = o\}$.

It is easily seen that $\mathcal{L}$ is a group. The identity being $[\gamma(t) = o]$, the inverse of an element $[\gamma(t)]$ being $[\gamma(1-t)]$ and the multiplication being given by $\circ$. There are other groups of interest that are quotients of the group of loops, for example the group of hoops. A complete account of the relationship between the group of loops and the group of hoops is given in $[S]$. The path bundle is a $\mathcal{L}$ principal bundle: $\Pi : \mathcal{PM} \to M$ where Π is the canonical proyection, i.e it assigns to each path its last point.

3 The differentiable structure

We will use the notion of differentiable spaces in order to give P, $\mathcal{L}$,$\mathcal{PM}$ a differentiable structure; this notion is taken from $[C]$ where it is used in a similar context. So we begin with the definition of differentiable space and latter explain how P, $\mathcal{L}$, $\mathcal{PM}$ can be seen as differentiable spaces.

Definition 3.1 *A differentiable space is a set* A *together with a family of functions called plots* $\phi : U \subset I\!\!R^n \to A$, U *open sets, such that if* $g : V \subset I\!\!R^m \to U \subset I\!\!R^n$ *is differentiable,* $\phi \circ g$ *is also a plot.*

Differentiable manifolds offer a first example of such structures and the diffeomorphism group an infinite dimensional example.

Examples

1. $(M, \{\phi\})$. A differentiable manifold M, with plots given by all the differentiable functions $\{\phi\} = \{\phi : U \to M, \textit{differentiable}, U \textit{ open sets of } I\!\!R^n\}$.

2. $(\textit{Diff}(M), \{\phi\})$. The group of diffeomorphism of a manifold M, with plots taken as all the functions $\{\phi\} = \{\phi : U \to \textit{Diff}(M) \ / \ \phi(x)(y) = \xi(x, y),\ \textit{where } \xi : U \times M \to M,\ \textit{differentiable},\ \textit{is such that } \xi(x, .) : M \to M,\ \textit{is a diffeomorphism}\}$.

3. $(P, \{\phi\})$ where $\{\phi : U \to P \ / \ \phi(x) = [\xi(x,t)]\ ,\ \xi : U \times [0,1] \to M \ / \exists 0 \le t_1 \le \ldots \le t_n \le 1 / \phi$ is 0 in each $U \times [t_i, t_{i+1}]\}$.

For $\mathcal{PM}$ and $\mathcal{L}$ the same definition holds, except that in the first case $\xi(x, 0) = o$ and in the second $\xi(x, 0) = \xi(x, 1) = o$.

With this structure we can define the basics of differentiable calculus there, which are as we can guess straightforward generalizations of the usual one on ordinary manifolds. In what follows A will represent either P, $\mathcal{L}$ or $\mathcal{PM}$.

1. *The topology of* $P, \mathcal{L}, \mathcal{PM}$

   A set $U \in A$ is open if and only if $\phi^{-1}(U)$ is open $\forall \phi$ plot. This is the Barret Topology.

2. *The family of differentiable functions.*

   $f : A \to I\!R$ is differentiable if and only if $f.\phi$ is differentiable $\forall \phi$ plot.

3. *Tangent vectors*

   On manifolds we have two characterizations of tangent vectors. One by an equivalence class of curves $\alpha \sim \alpha', \alpha(0) = \alpha'(0) = p$ if and only if $(f.\alpha)' \mid_0 = (f.\alpha)' \mid_0 \ \forall f : M \to I\!R$, and another one by an operator satisfying the Leibniz rule $X : C^\infty \to I\!R \ / \ X_p(f.g) = X_p(f).g(p) + f(p).X_p(g)$. On differentiable spaces these definitions do not agree. We choose the second one.

   Definition 3.2 *A vector at p is an operator $X_p : C^\infty(A) \to I\!R \ / \ X_p(f.g) = (X_p f)g(p) + f(p).(X_p g)$ derived from a plot ϕ as follows:*

   $$X_p(f) = \frac{\partial^k f.\phi}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \mid_{(x_1,\ldots,x_n)}$$

   for some $\phi : U \subset I\!R \to A$ such that $\phi((x_1, \ldots, x_n)) = p$.

   Two of the basics tangent vectors are, *The Mandelstam derivative* and the *Loop derivative* [T], they are defined as:

   (a) *The Mandelstam derivative on PM*

   Definition 3.3 *The Mandelstam derivative on $\gamma \in \Pi^{-1}(x) \in \mathcal{PM}$ in the direction of $v \in T_x M$ is given by*

   $$D_v f(\gamma) = \frac{\partial(f.\phi)}{\partial x_1} \mid_{x=0}$$

   where ϕ is the plot defined by:

   $$\phi : (-\epsilon, \epsilon) \to P, \ / \ \phi(x_1) = \gamma \circ \alpha(x_1)$$

   Here $\alpha : (-\epsilon, \epsilon) \to M$ is a one parameter plot derived from a curve $\tilde{\alpha}$ such that $\tilde{\alpha}'(0) = v$ and $\tilde{\alpha}(0) = x$ and $\alpha(x_1)(t) = \tilde{\alpha}(x_1.t)$.

(b) *The Loop derivative on $\mathcal{PM}$ and $\mathcal{L}$*

Definition 3.4 *The loop derivative at $\gamma \in \Pi^{-1}(x)$ is the operator which when applied to a function $f(\gamma)$, $\gamma \in P$ is*

$$\Delta_{u,v}(\pi_o^x)f(\gamma) = \frac{\partial^2 f(\phi \circ \gamma)}{\partial x_1 \partial x_2}$$

where $\phi(x_\mu, x_\nu) = \Box x_\mu, x_\nu$ (two dimensional plot) is the parallelogram, taken in a local chart of M, with vertex at x and edges in the directions of the vectors ∂_μ and ∂_ν with lengths x_μ and x_ν respectively.

We refer the reader to figure 1.

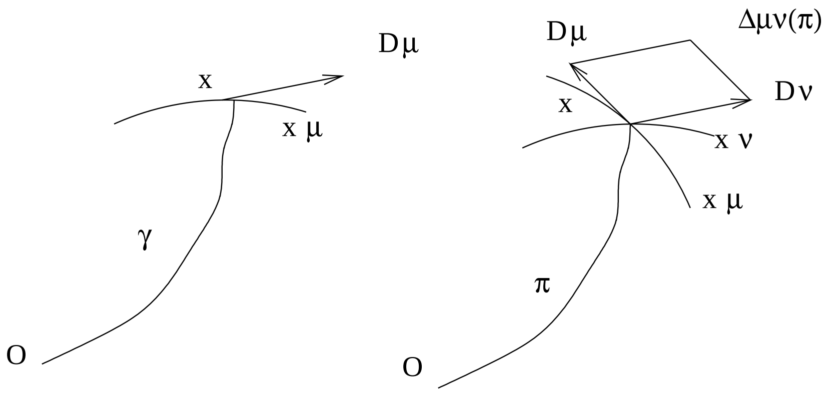

Figure 1. On the left the Mandelstam derivative and on the right the Loop Derivative.

4. *The tangent and cotangent bundles*

   Following the same idea of substituting the use of differentiable manifolds with differentiable spaces, it can be seen that both, the set of all vectors and the set of all covectors are natural differentiable spaces. Moreover they are locally trivial [RS]. So differentiable tensors of any order are defined.

5. *Differential forms and exterior derivative*

   As a particular tensor, a p-differential form is an alternating p- tensor with vectors as entries. We denote them by $W(p)(.,\ldots,.)$. What is rather nontrivial is the existence of exterior derivative. We define it by:

$$d\omega(X_1,\ldots,X_{n+1}) = \frac{1}{n+1}\left[\sum_{i=1}^{n+1}(-1)^{i+1}X_i(\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{n+1})) + \right.$$
$$\left. + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{n+1})\right].$$

$X_1,\ldots,X_{n+1}$ are vectors fields which at p take the values $X_1(p),\ldots,X_{n+1}(p)$. Although this is a well defined formula it is not clear that it depends only of $X_1(p),\ldots,X_{n+1}(p)$. In $[RS]$ it is shown to be true, thus exterior derivative is a well defined alternated tensor.

6. *The "Lie" algebra of $\mathcal{L}$*

   As in Lie groups, the "Lie" algebra $(T_e\mathcal{L},[\ ,\ ])$ is the set of right invariant vectors fields, and the bracket simply the bracket of the fields.

4 Universal connection and curvature

$\Pi : \mathcal{PM} \to M$, $\Pi([\gamma]) = \gamma(1)$ verifies all the properties of a principal $\mathcal{L}$-bundle except differentiability. A principal bundle is defined by local trivialization and transition functions, a construction we want to extend to $\mathcal{PM}$. Let us see how they are. Suppose $\{U_\alpha\}$ is a covering of M by contractible open sets. Then a local section over U_α around x is a plot $\phi_\alpha : U_\alpha \to \mathcal{PM}$ such that $\phi_\alpha(x)(1) = x$. This local section defines a trivialization over $\Pi^{-1}(U_\alpha)$, $\tilde{\phi}_\alpha : \Pi^{-1}(U_\alpha) \to U_\alpha \times \mathcal{L}$ by $\tilde{\phi}_\alpha(\gamma) = (\gamma(1), \gamma \circ (\phi(\gamma(1)))^{-1}) \in \mathcal{L}$. So the transition functions are $\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1} : U_\alpha \cap U_\beta \times \mathcal{L} \to U_\alpha \cap U_\beta \times \mathcal{L}$, $\circ$ being the left action.

We have now the universal principal bundle which is $\mathcal{PM}$ and its gauge group $\mathcal{L}$. It remains to define the universal connection. Let us define parallel transport.

Definition 4.1 *The parallel transport of γ, and element in the fiber over x, along a curve α joining x with y is $T_{x\to y}\gamma = \gamma \circ [\alpha]$.*

This parallel transport defines a set of horizontal n-planes, which in fact define a *universal connection one form*. This is a common connection in the sense that it is a one form in $\mathcal{PM}$ with values in $T_e\mathcal{L}$ that is equivariant under $\mathcal{L}$ and the identity over the vertical subspaces. It is defined as follow.

Given $D \in T_\pi\mathcal{PM}$ and $f \in C^\infty(\mathcal{L})$ we first transform f into a function on an open subset of $\mathcal{PM}$ given by: $g(\gamma) = f(\gamma \circ v(\gamma) \circ \pi^{-1})$, where $v(\gamma)$ is the segment that joins $\Pi(\gamma)$ (the final point of γ) with $\Pi(\pi)$ (the final point of π)

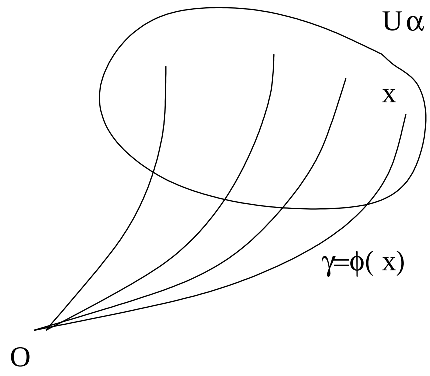

Figure 2. A trivialization over U_α

in a chart $h : U \to M$ with U a convex neighborhood of 0 and $h(0) = \Pi(\pi)$. We define $\delta_\pi(D)(f) = D(g)$.

Having now the *universal connection one form* the curvature is:

$$\Omega = d\delta + \frac{1}{2}[\delta, \delta]$$

In a chart around a point x of M, we have the vectors fields ∂_μ which can be raised to $\Pi^{-1}(U)$ to the horizontal vector field $D_\mu(\gamma),\ \gamma \in \Pi^{-1}(U)$. Now

$$\Omega(D_\mu, D_\nu) = \frac{1}{2}\Big[D_\mu\delta(D_\nu) - D_\nu\delta(D_\mu) - \delta([D_\mu, D_\nu]) + [\delta(D_\mu), \delta(D_\nu)]\Big] =$$
$$= -\frac{1}{2}\delta([D_\mu, D_\nu]) = -\frac{1}{2}[D_\mu, D_\nu] = -\frac{1}{2}\Delta_{\mu,\nu}(\gamma),$$

This shows that the curvature is $-1/2$ times the loop derivative. The Bianchi identity holds:

$$d\Omega = [\delta, \Omega]$$

5 Holonomy, representation and reconstruction

Let us state clearly what a universal reduction is.

Definition 5.1 *A differentiable space principal bundle (PU, GU, Π, M) and connection δU is a universal reduction if and only if all differentiable principal G- bundles and connections are extensions of (PU, GU, Π, M) for some C^∞ morphism $\rho : GU \to G$.*

We can now address the main point announced in the introduction, namely the fact that $(\mathcal{PM}, \mathcal{L}, \Pi, M)$ and δ form the universal reduction. We first recall the concept of holonomy. Supose (P, G, π, M) is a principal G-bundle with a connection A, both C^∞. Given a piece-wise differentiable path γ in $\mathcal{PM}$ and a point p in $\pi^{-1}(0)$, there is a unique curve $\tilde{\gamma}$ in P, such that $\pi(\tilde{\gamma}) = \gamma$ whose tangent vectors are horizontal at each point.

Definition 5.2 *If $\gamma \in \mathcal{L}$ then the holonomy is the element of G, $H(\gamma)$, defined by $\tilde{\gamma}(1) = H(\gamma)(\tilde{\gamma}(0))$.*

Theorem 5.3 $(\mathcal{PM}, \mathcal{L}, \Pi, M)$ *is a universal reduction.*

Proof:

Let us sketch the proof of that.

Suppose (P, G, π, M) is a principal G-bundle with a connection A. Construct the C^∞ holonomy morphism H and take it as ρ. Over the convex covering U_α define the principal G-bundle by transition functions $\tilde{\phi}_{\alpha\beta}$: $U_\alpha \cap U_\beta \times G \to U_\alpha \cap U_\beta \times G, \tilde{\phi}_{\alpha\beta} = H(\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1})$. Because $\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1}$ are the transition functions for $\mathcal{PM}$, $\tilde{\phi}_{\alpha\beta}$ are transition functions defining a principal G-bundle. To prove that this principal bundle does indeed coincide with P it is enough to see that these are the transition function for P over $\{U_\alpha\}$ induced by sections in P, $s_\alpha : U_\alpha \to P$, $s_\alpha(x) = \Phi(\phi_\alpha(x))$. The connection $A_\alpha(\partial_\mu) = dH(\partial_\mu \phi_\alpha)$ is $s_\alpha^* A(\partial_\mu)$ so the connection induced by H indeed coincides with the given A.$\Box$

Two morphisms H and H' are conjugate equivalent if and only if there exists $g \in G$ such that $H = gH'g^{-1}$, so C^∞ principal bundles and connections are in one to one correspondence with conjugacy classes of C^∞ morphisms. This is because of the p dependence of the holonomy.

6 A differentiable calculus for connections

The fact that conjugated holonomies $H : \mathcal{L} \to G$ are in one to one correspondence with principal G-bundles and connections suggest to think loops and connections as dual to each other. The fundamental idea for the loop approach to quantum gauge theories is to use this kind of non linear duality to go from functions of connections to functions of loops (in fact you have to use the group of hoops[S]) via the loop transform that uses the trace of the holonomy as a kernel

$$\Psi(\gamma) = \int TrH_A(\gamma)\tilde{\Psi}(A)DA.$$

The problem with this integral is the measure DA. It has been successfully defined $[A]$ as a Gaussian measure for the $U(1)$ electromagnetic free gauge theory, and gives a reasonable interpretation of loops as lines of electric flux. With this transform at hand we wish to apply the differentiable loop calculus as a differentiable calculus for functions of connections. But a lot yet remains to be done to achieve this goal.

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