QUANTUM HALL CONDUCTIVITY AND TOPOLOGICAL INVARIANTS.

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A short survey of the theory of the Quantum Hall effect is given emphasizing topological aspects of the quantization of the conductivity and showing how topological invariants can be derived from the hamiltonian. We express these invariants in terms of Chern numbers and show in precise mathematical terms how this relates to the Kubo formula.

1 Introduction

Since its discovery [$vKDP$], the integer Quantum Hall effect (QHE) has been a source for interesting theoretical ideas, often providing the opportunity for an interplay between Physics and Mathematics.

One of the first works in which a theoretical justification of the effect was given involved the calculation of the conductivity in terms of the wave function of an electron in a periodic system under the influence of a magnetic field (Thouless *et.al*. [$TKNN$]). Here the conductivity was computed from the Kubo formula, a formula for the correlation function describing the response of a system to a weak electric field. Though the quantization of the Hall conductivity was explained in this approach, its stability against small perturbations could not be explained.

Later, the work by Avron *et.al*. [$ASS1$] [$ASSS$] showed how one can explain not only the quantization of the conductivity but also its stability, using topological arguments. Although these arguments used there only work for a restricted class of models, they show how the conductivity can be seen as a topological invariant, the integral of the first Chern class of a fiber bundle over the Brillouin zone (a compact manifold). In fact, the generalization to more realistic models, for which the Brillouin zone cannot be seen as a manifold anymore, still allows to express the conductivity as a topological invariant (see the work by Bellissard *et.al*. [$BvES$] and by Avron *et.al*. [$ASS1$]).

The purpose of this short communication is to review some of the abovementioned approaches to the theory of the QHE and in particular the topological aspects of the quantization of the conductivity.

The plan of the paper is as follows: in Section 2 we summarize the physics background required to describe the QHE and which motivates the mathematical problem treated in the rest of the paper. We express the Hall conductivity σ_{Hall} given by the Kubo formula in terms of the eigenfunctions of the one particle Hamiltonian for our model described below. In Section 3 we use the symmetry of the system to show how topological invariants can be obtained from the hamiltonian. In Section 4 these invariants are expressed in terms of Chern numbers, and they are shown to coincide with the result for the conductivity computed from the Kubo formula.

2 Physics Background

Let us sketch some of the basic ideas from physics motivating the mathematical problem treated in the next sections. The model we use is as follows. We consider electrons on a two dimensional lattice and a homogeneous magnetic field perpendicular to the lattice. This means that we only consider the interactions of the electrons with the lattice atoms but neglect interactions among electrons, a reasonable approximation when the gas density is small. This model allows a rather simple mathematical treatment and already contains the essential features allowing to explain the quantization and stability of the Hall conductivity. Nevertheless, we remark that in order to explain the existence of plateaus it would be necessary to take into account the effect of impurities.

The Hamiltonian for one electron in a two dimensional lattice can be written as

$$H = -\frac{\hbar^2}{2m}\nabla^2 + U(x_1, x_2), \tag{1}$$

where $U(x_1, x_2)$ (the lattice potential) is a real function periodic in the variables x_1 and x_2.

Bloch's theorem $[M]$ states that for an electron in the presence of a periodic potential the solutions to Schrödinger's equation are given in terms of plane waves modulated by periodic functions, with the same periodicity as the potential. This is a consequence of the symmetry of the system. The translation operators commute with each other and with the Hamiltonian, thus forming a complete set of commuting operators. This implies that we can label a physical state by indicating three quantum numbers: the energy band index and the eigenvalues of the translation operators. The latter are phase factors *i.e.* of the form $e^{ik_i a}$ (where a is the lattice constant) and thus are specified by numbers k_i modulo integer multiples of $2\pi/a$. The set of independent variables specifying the eigenvalues of the translation operators is therefore not a 2-plane, but a 2-torus, T^2.

If we now include the effect of a magnetic field implementing the substitution $\nabla \mapsto \nabla - i\frac{e}{\hbar c}A$ in Eq.(1), where A is the magnetic vector potential, we find that the hamiltonian is not translation invariant.

There is a way to overcome this difficulty, namely to consider, instead of the usual translation group, the so-called *magnetic translation group* $[Z]$. The operators given by the representation of this group now take the magnetic field into account. But in order to apply Bloch's theorem, one is forced to assume that the magnetic flux per unit cell, in units of $c\hbar/e$, is a rational number $\phi = p/q$. In this case, enlarging the unit cell in one direction by a factor of q, the reciprocal space, now called *magnetic* Brillouin zone, is again a two dimensional torus.

Imposing a rational magnetic flux is a non-physical condition and this is a limitation of the present model. However, this allows a study of the topological aspects of the problem in rather simple mathematical terms. A further simplifying assumption is to consider only one of the following limiting cases:
(i) The lattice potential is sufficiently weak compared to $\hbar\omega_c$ (the cyclotron energy). Here the lattice potential can be treated as a perturbation of the Landau hamiltonian (which considers only one electron under the influence of a magnetic field).
(ii) The lattice potential is strong compared to $\hbar\omega_c$. In this case, approximations such as the tight-binding model $[M]$ can be considered.

The reason for this further simplification is that it can be shown that in both cases, finding the solution of Schrödinger's equation reduces to the diagonalization of a hermitian $n\times n$ matrix, whose entries are functions defined on the Brillouin zone $[HK]$,$[H]$. (If the (rational) magnetic flux is p/q, then in (i) $n = p$ and in (ii) $n = q$).

In such cases the result for the conductivity of a single band given by the Kubo formula is given by $[HK]$,$[H]$:

$$\sigma_j = -\frac{e^2/h}{2\pi i}\int_{T^2} d^2k[\nabla_k \times \vec{A}^{(j)}(k)]_z, \tag{2}$$

where

$$\vec{A}^{(j)}(k) := \sum_{m=1}^{n} \overline{\Psi^{(j)}}_m(k)\nabla_k\Psi^{(j)}_m(k). \tag{3}$$

Here $\Psi^{(j)}(k) = (\Psi^{(j)}_1(k), \ldots, \Psi^{(j)}_n(k))$ is the eigenvector corresponding to the j^{th} energy band. Finally, σ_{Hall} is given by the sum of the contributions σ_j of all filled bands.

3 Topology and Bloch Hamiltonians

As remarked in the previous section, for a periodic lattice finding the solution of the eigenvalue problem reduces (for particular values of the magnetic field) to the diagonalization of a hermitian matrix whose entries are continuous complex functions on a torus - the Brillouin zone.

Hence, in this section we consider functions from the torus into the space of hermitian matrices with non-degenerate eigenvalues. (We restrict to those matrices only, for simplicity). The topological invariants related to the conductivity arise from considering the homotopy classes of functions from the torus into this space of matrices. In this section we closely follow $[ASS1]$ and $[ASSS]$.

Let $M_n(\mathbb{C})$ denote the space of $n \times n$ complex matrices and let $H_n(\mathbb{C})$ be the subspace of $M_n(\mathbb{C})$ consisting of all hermitian matrices with non-degenerate eigenvalues.

Proposition 1 *There is a homeomorphism $H_n(\mathbb{C}) \simeq D_n \times U(n)/U(1)^n$ where $U(n)$ denotes the unitary group of order n and*

$$D_n := \{\mathrm{Diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) | \alpha_1 < \ldots < \alpha_n; \alpha_i \in \mathbb{R}\}.$$

Proof. Given $A \in H_n(\mathbb{C})$, we can find a unique diagonal matrix D_A whose components are the eigenvalues of A, organized in increasing order, since the eigenvalues of A are all real and different. $A \in H_n(\mathbb{C})$ also implies we can find $U_A \in U(n)$ such that $D_A = U_A^{-1} A U_A$. This unitary matrix is not unique, but if we consider the quotient $U(n)/U(1)^n$, then its equivalence class $[U_A]$ is uniquely defined. Thus we have a homeomorphism

$$\begin{aligned} g : H_n(\mathbb{C}) &\longrightarrow D_n \times U(n)/U(1)^n \\ A \quad &\longmapsto (D_A, [U_A]). \end{aligned}$$ □

Note that since D_n is contractible, g followed by a retraction of D_n to a fixed matrix gives rise to a homotopy equivalence between $H_n(\mathbb{C})$ and $U(n)/U(1)^n$.

Now we want to relate the homotopy groups of $U(n)$ and $U(1)$ with those of $H_n(\mathbb{C})$, since this will allow us to interpret homotopy types of maps from the torus T^2 to $H_n(\mathbb{C})$ in terms of bundles over T^2. Recall that $U(n)/U(1)^n$ has a bundle structure, so in order to compute its homotopy groups, we can make use of its corresponding homotopy sequence $[S]$:

$$\cdots \to \pi_m(U(1)^n) \overset{i_{m*}}{\to} \pi_m(U(n)) \overset{p_{m*}}{\to} \pi_m(U(n)/U(1)^n) \overset{\Delta_m}{\to} \pi_{m-1}(U(1)^n) \to \cdots \tag{4}$$

The proof of the following (standard) result is useful in order to find out the explicit form of the homomorphism i_{m*} in sequence (4) when $m = 1$. Since we are particularly interested in this homomorphism, we give the proof in detail.

Proposition 2. *For $n \in \mathbb{N}$ we have:*

(i) $\pi_1(U(n)) \simeq \pi_1(U(1)) \quad (\simeq \mathbb{Z})$,

(ii) $\pi_2(U(n)) = 0$.

Proof. The proof relies on the following facts [S] :

$\pi_2(U(1)) = 0, \quad \pi_m(S^n) = 0 \quad (\text{for} \quad m < n) \quad \text{and} \quad U(n)/U(n-1) = S^{2n-1}$.

Considering the exact homotopy sequence of the fibration $U(k)/U(k-1)$ for $k = 2, \ldots, n$, we have

$$0 = \pi_2(S^{2k-1}) \to \pi_1(U(k-1)) \stackrel{\rho_k}{\to} \pi_1(U(k)) \to \pi_1(S^{2k-1}) = 0,$$

and we see that ρ_k, the homomorphism induced by the inclusion $U(k-1) \hookrightarrow U(k)$, is an isomorphism between $\pi_1(U(k-1))$ and $\pi_1(U(k))$ $(k = 2, \ldots, n)$. Composing these isomorphisms, we obtain an isomorphism

$$\phi := \rho_n \circ \cdots \circ \rho_2 : \pi_1(U(1)) \simeq \mathbb{Z} \longrightarrow \pi_1(U(n)), \tag{5}$$

thus proving (i).

Let us now consider the same exact sequence at the π_2 level:

$$\pi_3(S^{2k-1}) \to \pi_2(U(k-1)) \to \pi_2(U(k)) \to \pi_2(S^{2k-1}),$$

for $k = 2, \ldots, n$. When $k = 2$, we obtain $\pi_2(U(2)) = 0$, since $\pi_2(U(1)) = 0 = \pi_2(S^3)$. For $3 \leq k \leq n$, we have $\pi_3(S^{2k-1}) = \pi_2(S^{2k-1}) = 0$, which yields $\pi_2(U(k-1)) = \pi_2(U(k)$, and hence proves (ii). $\square$

Proposition 3 *Up to the isomorphism $\pi_1(U(1)) \simeq \pi_1(U(n))$, the map i_{1*} in (4) is given by:*

$$i_{1*}(z_1, \ldots, z_n) = z_1 + \cdots + z_n. \tag{6}$$

Proof. Given an element $(z_1, \ldots, z_n)$ of $\pi_1(U(1)^n) = \mathbb{Z}^n$, let $f(t) = (e^{i\theta_1(t)}, \ldots, e^{i\theta_n(t)})$ be a continuous map from S^1 to $U(1)^n$ representing $(z_1, \ldots, z_n)$. Let us now define the map $\bar{f} := i \circ f : S^1 \to U(n)$, where i stands for the canonical inclusion of $U(1)^n$ into $U(n)$. If we compose $\bar{f}$ with det : $U(n) \to U(1)$, we obtain a representative of an element of $\pi_1(U(1))$ whose image under ϕ (from Eq.(5)) is:

$$\phi([\det \circ \bar{f}]) = [\mathrm{Diag}(\det \circ \bar{f}, 1, \ldots, 1)] \in \pi_1(U(n)).$$

(Here ρ_k is induced by the inclusion of $U(k-1)$ in the left upper corner of $U(k)$).

We have $i(U(1)^n) \subset SU(n)$ and $\pi_1(SU(n)) = 0$, since $S^1 \times SU(n) \simeq U(n)$. This implies that if we define

$$\begin{aligned} \gamma : S^1 &\to SU(n) \subset U(n) \\ t &\mapsto \mathrm{Diag}(e^{i(\theta_2(t)+\cdots+\theta_n(t))}, e^{-i\theta_2(t)}, \ldots, e^{-i\theta_n(t)}), \end{aligned}$$

then $[\gamma] = 0 \in \pi_1(U(n))$. Since the multiplication on any topological group induces the sum on the fundamental group, we see that $\bar{f}$ is homotopic to the map $t \mapsto \mathrm{Diag}(\det \circ \bar{f}(t)), 1, \ldots, 1)$ obtained by pointwise multiplication of $\bar{f}$ and γ. But this implies that

$$\begin{aligned} \phi^{-1} \circ i_{1*}(z_1, \ldots, z_n) &= \phi^{-1}([\bar{f}]) \\ &= [\det \circ \bar{f}] \\ &= z_1 + \cdots + z_n. \end{aligned} \tag{7}$$

□

Proposition 4. *The first two homotopy groups of $H_n(\mathbb{C})$ are given by:*

$$\begin{aligned} &\pi_1(H_n(\mathbb{C})) \simeq 0 \\ &\pi_2(H_n(\mathbb{C})) \simeq \ker i_{1*} \simeq \mathbb{Z}^{n-1}. \end{aligned}$$

Proof. Proposition 1 implies $\pi_k(U(n)/U(1)^n) \simeq \pi_k(H_n(\mathbb{C}))$. From Proposition 3 we know that i_{1*} is surjective. Using sequence (4) we obtain $\ker p_{1*} = \mathrm{Im} i_{1*} = \pi_1(U(n))$, from which $\mathrm{Im} p_{1*} = 0$ follows. Hence Δ_1 is injective, since $\ker \Delta_1 = \mathrm{Im} p_{1*}$, so $\pi_1(U(n)/U(1)^n) \simeq \mathrm{Im}\Delta_1 = 0$.

On the other hand from Proposition 2, we have $\pi_2(U(n)) = 0$. This implies $\mathrm{Im} p_{2*} = 0$, but then $\ker \Delta_2 = 0$. It follows that $\pi_2(U(n)/U(1)^n) \simeq \mathrm{Im}\Delta_2 = \ker i_{1*}$. Finally, from Proposition 3, we know that $\ker i_{1*} \simeq \mathbb{Z}^{n-1}$. □

Let us now establish a relation between the homotopy groups of a sphere and those of a torus.

We recall that given a pointed space X with base point x_0, the *cone* of X, denoted $C(X)$, is defined as the topological space obtained from $X \times I$ by regarding $X \times \{0\} \cup \{x_0\} \times I$ as a single point, that is:

$$C(X) = X \times I/(X \times \{0\} \cup \{x_0\} \times I).$$

Similarly,the *suspension* of X, denoted $S(X)$, is defined to be the quotient space of $X \times I$ in which the space $(X \times \{0\}) \cup (\{x_0\} \times I) \cup (X \times \{1\})$ is identified with a single point.

Let $f : X \to Y$ be a continuous map. There is an inclusion of X into $X \times I$ given by the map $x \mapsto x \times \{0\}$. This inclusion is preserved when going to $C(X)$ so that X can be considered as a subset of $C(X)$, via inclusion. Consider the disjoint union of $C(X)$ and Y and define the equivalence relation given (for $x \in X \hookrightarrow C(X)$ and $y \in Y$) by $x \sim y \Leftrightarrow y = f(x)$. The quotient space obtained from this equivalence relation is called the *mapping cone* of f and is denoted by $C(f)$. It can be shown that $C(f)/Y = S(X)$. For $f : X \to Y$ there is a natural inclusion $j : Y \hookrightarrow C(f)$. We can construct $C(j)$ and consider the inclusion $k : C(f) \hookrightarrow C(j)$. The procedure can be iterated to obtain a sequence of maps:

$$X \xrightarrow{f} Y \xhookrightarrow{j} C(f) \xhookrightarrow{k} C(j) \hookrightarrow C(k) \hookrightarrow \cdots$$

Since $C(j)$ can be shown to have the same homotopy type as $S(X)$, and similarly $C(k)$ to have the same homotopy type as $S(Y)$, given another space W, we have the following sequence, called the *track group sequence* [W],[Sp]:

$$\begin{gathered}\cdots \to [S^{n+1}(X), W] \to [S^n(C(f)), W] \to [S^n(Y), W] \to [S^n(X), W] \to \cdots \\ \cdots \to [S(Y), W] \to [S(X), W] \to [C(f), W] \to [Y, W] \to [X, W],\end{gathered} \tag{8}$$

where $S^n(X)$ denotes the space resulting from iterating (n times) the suspension operation to the space X.

Denoting the set of homotopy classes of functions from the torus T^2 into the space $H_n(\mathbb{C})$ by $[T^2, H_n(\mathbb{C})]$, we have the following

Proposition 5. *There is a one to one correspondence between* $[T^2, H_n(\mathbb{C})]$ *and* $\pi_2(H_n(\mathbb{C}))$.

Proof. Regarding the unit circle S^1 as a pointed subspace of $\mathbb{C}$ with 1 as base point, we consider the wedge product of S^1 with itself:

$$S^1 \vee S^1 = (S^1 \times \{1\}) \cup (\{1\} \times S^1).$$

Define a map $f : S^1 \to S^1 \vee S^1$ as follows:

$$f(e^{2i\pi t}) = \begin{cases} (e^{8i\pi t}, 1), & t \in [0, 1/4] \\ (1, e^{8i\pi t}), & t \in [1/4, 1/2] \\ (e^{-8i\pi t}, 1), & t \in [1/2, 3/4] \\ (1, e^{-8i\pi t}), & t \in [3/4, 1]. \end{cases} \tag{9}$$

If we set $X = S^1$, $Y = S^1 \vee S^1$ and $W = H_n(\mathbb{C})$, we obtain $S(X) = S^2$ and $C(f) = T^2$. We also have $S^2 = T^2/(S^1 \vee S^1)$. Denote by $q : T^2 \to S^2$ the corresponding quotient map. In this case, the track group sequence (8) reads:

$$\cdots \to [S^2 \vee S^2, H_n(\mathbb{C})] \to \pi_2(H_n(\mathbb{C})) \xrightarrow{q^*} [T^2, H_n(\mathbb{C})] \to [S^1 \vee S^1, H_n(\mathbb{C})] \to \cdots$$

Since $\pi_1(H_n(\mathbb{C}))$ is trivial, q^* is surjective. The map $[S^2 \vee S^2, H_n(\mathbb{C})] \to \pi_2(H_n(\mathbb{C}))$ is the null map since it is induced by the suspension of f, which is of degree zero (because f is of degree zero on each circle of the wedge, as can be read from Eq. (9)). Therefore q^* is also injective and thus an element of $[T^2, H_n(\mathbb{C})]$ can be completely specified by an element of $\pi_2(H_n(\mathbb{C}))$. $\square$

Summarizing, we have:

$$\begin{aligned} [T^2, H_n(\mathbb{C})] &\simeq \pi_2(H_n(\mathbb{C})) \simeq \ker i_{1*} \\ &= \{(z_1, \ldots, z_n) \in \pi_1(U(1))^n \mid z_1 + \cdots + z_n = 0\}. \end{aligned} \tag{10}$$

But $\pi_1(U(1)) \simeq \pi_2(\mathcal{B}U(1))$, where $\mathcal{B}U(1)$ denotes the classifying space for $U(1)-$bundles, that is to say an element of $\pi_1(U(1))$ determines a unique $U(1)-$bundle over S^2, up to bundle isomorphism [MS], [S]. Since we can equivalently consider its associated line bundle, in this case we have n line bundles subject to a restriction. We can now use the projection q to obtain n induced line bundles over T^2. The interpretation is that the sum of this bundles is a trivial bundle. We elaborate this point in the next section.

4 Kubo formula and Chern numbers

We have seen how to obtain topological invariants from particular hamiltonians, implying that they are stable against 'small' perturbations of the latter. The purpose of this section is to give an expression for these invariants in terms of Chern classes and compare it with the expressions in Eq. (2).

Consider, as in the previous section, the set $[T^2, H_n(\mathbb{C})]$. For a given $\alpha \in [T^2, H_n(\mathbb{C})]$ we can certainly choose a smooth representative $f \in \alpha$ and for $x \in T^2$, $f(x)$ is a hermitian matrix all of whose eigenvalues are different. It follows that $p_i(x)$, the idempotent matrix defined by the projection into the i^{th} eigenspace of $f(x)$ in $\mathbb{C}^n$, has trace equal to 1 ($i = 1, \ldots, n;\ x \in T^2$). Furthermore, the matrix valued map $x \mapsto p_i(x)$ is smooth and naturally defines

$$\begin{aligned} \varphi_i : T^2 &\longrightarrow \mathbb{C}P^{n-1} \\ x &\longmapsto \varphi_i(x) := p_i(x)(\mathbb{C}^n), \end{aligned} \tag{11}$$

Since tr $p_i(x) = 1$ and $p_i(x)^2 = p_i(x)$, $p_i(x)(\mathbb{C}^n)$ is the complex line $\{y \in \mathbb{C}^n : p_i(x)y = y\}$, considered here as an element of $\mathbb{C}P^{n-1}$.

Let $\mathcal{L}_{(n-1)}$ denote the canonical line bundle over $\mathbb{C}P^{n-1}$ and let i and π denote respectively the inclusion $\mathcal{L}_{(n-1)} \hookrightarrow \varepsilon^n$ and the projection $\varepsilon^n \to \mathcal{L}_{(n-1)}$, where $\varepsilon^n = \mathbb{C}P^{n-1} \times \mathbb{C}^n$. The maps i and π induce the following maps between spaces of sections:

$$i_* : \Gamma(\mathcal{L}_{(n-1)}) \to \Gamma(\varepsilon^n) \quad \text{and} \quad \pi_* : \Gamma(\varepsilon^n) \to \Gamma(\mathcal{L}_{(n-1)}). \tag{12}$$

The trivial connection ∇_0 on ε^n induces a connection on $\mathcal{L}_{(n-1)}$ defined by:

$$\nabla := (\mathrm{id} \otimes \pi_*) \circ \nabla_0 \circ i_*, \tag{13}$$

where id is the identity map on $\Omega^1(\mathbb{C}P^{n-1})$, the space of one forms on the projective space. If $s \in \Gamma(\mathcal{L}_{(n-1)})$, it follows that $\nabla(s) = A \otimes s$, where A is the connection 1 form defined as $[KN]$,$[MT]$:

$$\begin{aligned} A : \quad \mathbb{C}P^{n-1} &\longrightarrow \Omega^1(\mathbb{C}P^{n-1}) \\ [z_1, \ldots, z_n] &\longmapsto \frac{\bar{z}_1 \mathrm{d}z_1 + \cdots + \bar{z}_n \mathrm{d}z_n}{|z_1|^2 + \cdots + |z_n|^2}. \end{aligned} \tag{14}$$

Locally, we may write $\varphi_i(x) = [z_1(x), \ldots, z_n(x)]$ for some smooth functions $z_i : T^2 \to \mathbb{C}$. Hence we have

$$\varphi_i^*(A)(x) = \frac{\bar{z}_1(x)\mathrm{d}z_1(x) + \cdots + \bar{z}_n(x)\mathrm{d}z_n(x)}{|z_1(x)|^2 + \cdots + |z_n(x)|^2}. \tag{15}$$

Now, the map φ_i induces a bundle $\xi_i = \varphi_i^*(\mathcal{L}_{(n-1)})$ over T^2. The first Chern class of the (line) bundle ξ_i, denoted by $c_1(\xi_i)$, is a cohomology class depending only on the isomorphism class of ξ_i ($[MS]$). From the properties of Chern classes, it follows that

$$c_1(\xi_i) = \varphi_i^* c_1(\mathcal{L}_{(n-1)}) = \frac{-1}{2\pi i} \varphi_i^* \mathrm{d}A. \tag{16}$$

But, noting that $(z_1(x), \ldots, z_n(x))$ is the i^{th} eigenvector of $f(x)$ and comparing Eqs. (15) and (16) with Eqs. (3) and (2), we see that σ_i as in Eq.(2) can be expressed as

$$\sigma_i = \frac{e^2}{h} \int_{T^2} c_1(\xi_i)$$

if we take f to be the hamiltonian.

At this stage we have n line bundles $\xi_1, \ldots, \xi_n$ over T^2. To each bundle corresponds an integer number, namely the integral over T^2 of $\frac{-1}{2\pi i} c_1(\xi_i)$. The sum of these integers must be zero, in accordance with Eq. (10). This can be shown using the fact that all the Chern classes of a trivial bundle vanish and that the property

$$c_1(\xi_1 \oplus \cdots \oplus \xi_n) = c_1(\xi_1) + \cdots + c_1(\xi_n)$$

holds for sums of bundles.

To see why the bundle $\xi_1 \oplus \cdots \oplus \xi_n$ is trivial, we recall that each ξ_i was obtained from the map $x \mapsto p_i(x)$. In fact, the following isomorphism of finitely generated projective $C^\infty(M)-$modules holds $[MT]$:

$$\Gamma(\xi_i) \simeq p_i C^\infty(T^2)^n.$$

From the fact that $\Gamma(\xi_1) \oplus \cdots \oplus \Gamma(\xi_n) \simeq \Gamma(\xi_1 \oplus \cdots \oplus \xi_n)$ and the fact that $p_1 \oplus \cdots \oplus p_n = \mathrm{id}$, it follows that $\xi_1 \oplus \cdots \oplus \xi_n$ must be a trivial bundle $[MT]$.

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