DETERMINANT OF THE DIRAC OPERATOR OVER THE INTERVAL $[0, \beta]$

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In this communication we study properties of the determinant of families of boundary value problems over the interval in order to investigate the geometry of the determinant line bundle. In particular, we compare two methods, the first one involving the ζ-determinant and the second one a canonically defined determinant and we prove an identification at the level of curvatures obtained from the different methods.The one dimensional case offers a convenient toy model to review the main methods used by $[SW1]$, $[SW2]$, $[S3]$ in the higher dimensional case.

Introduction

The problem of the computation of the ζ-determinant for a family of Elliptic Boundary Value Problems over one dimensional manifolds has received a lot of attention $[BFK1, LT]$. The main objective of this communication is to show how we can use these analytical tools to understand the geometry of the determinant line bundle of a certain family of Elliptic Boundary Value problems in one dimension. In the case of a finite interval $[0, \beta]$ the analytical tools are easier to handle and the one dimensional case offers a toy model to review methods and results on odd-dimensional manifolds of higher dimension $[SW1]$, $[SW2]$, $[S3]$. Specifically, we show how the canonical and ζ-determinant for a family of Elliptic Boundary Value Problems are related to each other and how this relation is reflected in the geometry of the determinant line bundle. In theorem 1 we identify the canonical and the ζ-determinant, while in theorem 2 we identify the curvatures over a special subset U_Γ of the space that parameterizes the boundary conditions. The results of this communication were derived in the author's master thesis under supervision of Dr. Simon Scott. Detailed proofs of the main theorems can be found in $[S4]$, as well as in the the author's master thesis, $[T]$.

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1 Geometrical and analytical data

We begin by considering the interval $X = [0, \beta]$ and a trivial complex vector bundle $\mathcal{E} \cong X \times \mathbb{C}^n$ over X with a metric-compatible connection ∇. Let $\mathfrak{U}$ be the space of connections, $\mathfrak{U} \cong \Omega^1(X; \mathfrak{u}(n))$, where $\mathfrak{u}(n)$ is the Lie algebra of the unitary group U(n). We denote by $\mathfrak{g}$ the group of gauge transformations, $\mathfrak{g} \cong C^\infty(X; GL(n))$.
We consider the Dirac operator $D \doteq -i\nabla_{\frac{d}{dx}}$ that can be written locally as

$$D = -i\frac{d}{dx} + B(x), \quad B(x) \in GL(n). \tag{1}$$

Next, we take an element $P \in \mathrm{End}(\mathbb{C}^{2n})$ and define what we will call a boundary value problem.
Definition 1. *The boundary value problem (BVP) D_P, $P \in \mathrm{End}(\mathbb{C}^{2n})$ is the data,*

$$\begin{cases} D_P = D : \mathrm{dom}(D_P) \to L^2(X; \mathbb{C}^n) \\ \mathrm{dom}(D_P) = \left\{ \psi \in H^1([0,\beta], \mathbb{C}^n) : P(\psi(0), \psi(\beta)) = 0 \in \mathbb{C}^{2n} \right\} \end{cases}.$$

Since the boundary of X, $Y = \partial X = \{0\} \bigcup \{\beta\}$, the restriction of the bundle $\mathcal{E}$ to the boundary is isomorphic to $\mathbb{C}^{2n}$. A subspace ω of the space of sections of the bundle restricted to the boundary $E|_Y$ can be seen as a subspace of $(\mathbb{C}^n \oplus \mathbb{C}^n)$. In this way, we obtain that the space that parameterizes all the boundary value problems is given by the Grassmannian $Gr(\mathbb{C}^n \oplus \mathbb{C}^n)$, defined by:

$$Gr(\mathbb{C}^n \oplus \mathbb{C}^n) = \bigcup_{k \leq 2n} Gr_{k,2n}(\mathbb{C}^n \oplus \mathbb{C}^n), \tag{2}$$

where $Gr_{k,2n}(\mathbb{C}^n \oplus \mathbb{C}^n)$ denotes the space of k-dimensional subspaces of $(\mathbb{C}^n \oplus \mathbb{C}^n)$. Now, for each $k \leq 2n$, we have

$$Gr_{k,2n}(\mathbb{C}^n \oplus \mathbb{C}^n) = \left\{ P \in End(\mathbb{C}^{2n}) : P^2 = P, P^* = P \text{ and } \mathrm{Tr}\, P = k \right\}, \tag{3}$$

or equivalently, each boundary value problem is determined by an orthogonal projection of rank k in $\mathbb{C}^{2n}$. In higher dimensions there is an explicit construction of the parametrix in $[S2]$ which shows that D_ω has the analytical properties of an elliptic operator over a closed manifold. In particular, there exists a spectral resolution on the operator and the ζ-determinant exists. In dimension one we are going to define a particular subset of boundary conditions which converts the above boundary value problem into an Elliptic Boundary Value Problem $[BFK2]$. This justifies the fact that we are going to call this an Elliptic Boundary Value problem (EBVP). This special subspace of the set of boundary value problems is given by,

Definition 2. *We define U_Γ to be the subspace of $Gr_{n,2n}(\mathbb{C}^n \oplus \mathbb{C}^n)$, which consists of elements ω which are the graphs of some $T \in Hom(\mathbb{C}^n, \mathbb{C}^n)$, T invertible, i.e.*

$$\omega = \mathrm{Graph(T)} \doteq \{(\mathrm{x}, \mathrm{y}) \in \mathbb{C}^n \oplus \mathbb{C}^n | \mathrm{y} = \mathrm{Tx}\}\,.$$

We will use the notation $D_T \doteq D_{P_T}, \mathrm{Ran}(P_T) = \mathrm{Graph(T)}$ to denote the associated EBVP.

Lemma 1. *Let P_T denote the corresponding orthogonal projection associated to ω as in definition 2. Then,*

$$P_T = \begin{pmatrix} (1+T^*T)^{-1} & (1+T^*T)^{-1}T^* \\ T(1+T^*T)^{-1} & T(1+T^*T)^{-1}T^* \end{pmatrix}. \tag{4}$$

1.1 Gauge action and EVBPs

Although we are working on a trivial bundle i.e. the moduli space is trivial, the gauge group $\mathfrak{g} = C^\infty(X, GL(n))$ action on the space of connections

$$\begin{aligned} \mathfrak{U} \times \mathfrak{g} &\to \mathfrak{U} \\ (\nabla, g) &\mapsto g^{-1}\nabla g \doteq \nabla^g, \end{aligned}$$

induces a gauge action on the set of EBVP's given by

$$\begin{aligned} (\mathfrak{U} \times Gr(\mathbb{C}^n \oplus \mathbb{C}^n)) \times \mathfrak{g} &\to \mathfrak{U} \times Gr(\mathbb{C}^n \oplus \mathbb{C}^n) \\ (D_P, g) &\mapsto D^g_{[P_u]}, \end{aligned}$$

where $u \doteq \begin{pmatrix} g(0) & 0 \\ 0 & g(\beta) \end{pmatrix} \in GL(2n)$ and $[P_u] \doteq u^{-1}Pu$. In particular, over U_Γ under a gauge transformation, P_T transforms as follows:

$$P_T \mapsto P_{g(\beta)^*T(g(0))^{-1})^*}. \tag{5}$$

Finally, let us remark that $\mathrm{spec}(D_P) = \mathrm{spec}(D^g_{[P_u]})$.

2 Determinant line bundle and the canonical determinant

We know from [Q] how to build a determinant line bundle for a family of Fredholm operators $\{A_b : F_1^b \to F_2^b | b \in B, F_1^b$ and F_2^b Hilbert spaces$\}$ parameterized by a base space B . The complex line $\mathcal{L}_b$ is given by $\mathcal{L}_b \doteq (\det \mathrm{Ker} A_b)^* \otimes (\det \mathrm{Coker}\, A_b)$, where $\det V = \Lambda^{max} V$. In the particular case of the EVBP's we are considering, this gives us a holomorphic line bundle with fiber

$$\mathcal{L}_b \doteq (\det \mathrm{Ker} D_P)^* \otimes (\det \mathrm{Coker}\, D_P).$$

On the other hand, we are going to describe an alternative construction of the determinant line bundle due to Segal [Seg] which is better suited to deal with the boundary conditions of an EVBP. Given two Frèchet spaces V_1, V_2 and a Fredholm operator $A : V_1 \to V_2$ of order zero, the determinant line $\mathrm{Det}\, A$ is the complex line whose points are equivalence classes $[Z, \lambda]$ of pairs $(Z, \lambda), \lambda \in \mathbb{C}, Z - A$ trace-class. For $q = 1 + \text{trace-class} : V_1 \to V_2$ with non-zero Fredholm determinant $\det_F q$, the equivalence relation is

$$(Zq, \lambda) \sim (Z, \lambda \det_F q).$$

The Segal determinant line bundle of the family of EBVP's parameterized by $Gr(\mathbb{C}^n \oplus \mathbb{C}^n)$ is the line bundle $L \to Gr(\mathbb{C}^n \oplus \mathbb{C}^n)$ whose fiber over ω is $L_\omega \doteq \mathrm{Det}\,(D_\omega)$. It has the structure of a holomorphic bundle and it can be shown, see for example [$S2$][Prop. 4.2], that there exists a canonical isomorphism between Quillen's and Segal's determinant line bundles. Hence, given a EBVP D_P we can select a canonical element of its determinant line which is called the *canonical determinant*, denoted $\det_{\mathcal{C}}(D_P)$. Proper calculations [$S2, S4$] lead us to the following result,

Proposition 1. *Consider $h(x) : \mathbb{C}^n \to \mathbb{C}^n$ the solution to the paralel transport equation $Dh = 0, h(0) = \mathrm{Id}$, and let $h = h(\beta)$. For a given BVP D_T the canonical determinant is given by*

$$\det_{\mathcal{C}}(D_T) = \det\left(1 + T^* h\right).$$

Just as in the case of Quillen's determinant line bundle, we define a canonical metric on the determinant line by

$$\|\det D_\omega\|^2_{\mathcal{C}} \doteq \det_{\mathcal{C}}(D^*_\omega D_\omega). \tag{6}$$

We define a metric compatible connection and compute the curvature of this metric to find that [T], [$S4$],

Proposition 2. *Consider $U_\Gamma \subset Gr(\mathbb{C}^n \oplus \mathbb{C}^n)$ i.e the set of boundary conditions $\omega = \mathrm{Graph(T)}$, T invertible, and let $Q = (1 + T^*T)$, then the canonical curvature of the determinant line bundle is given by the Kähler form of the Grassmannian, i.e.*

$$\Omega_{\mathcal{C}} = \bar{\partial}\partial \log\det\ Q^{-1} = \bar{\partial}\partial \log\det\ (1 + T^*T)^{-1}. \tag{7}$$

3 Variational calculation of the ζ-determinant

We proceed to use the ζ-regularization method to compute the determinant of a EBVP D_P as well as the curvature of the determinant line bundle over

U_Γ . Following Seeley, for an EBVP D_ω we define the ζ function:

$$\zeta_{D_\omega}(s) = \frac{1}{2\pi i}\int_\gamma \lambda^{-s}(D_\omega - \lambda)^{-1} d\lambda,$$

where γ is an appropriate contour which does not contain the spectrum of D_ω $[Se]$ i.e. If there exists a ray $R_\theta = \left\{ r \exp^{i\theta} r \geq 0 \right\}$ such that no element of the spectrum of eigenvalues of D_ω lies on R_θ, then γ is the contour begining at ∞, transversing R_θ to a small circle around the origin, and then back along R_θ to ∞. The ζ-determinant is defined by

$$\log \det_\zeta D_\omega = exp(-\zeta'(0)).$$

where the value at zero is defined by meromorphic continuation. The operator $i(D-\lambda)^{-1}$ is a trace-class operator with kernel $[S2]$,

Lemma 2. *The Schwartz kernel K_λ of $i(D_\omega - \lambda)^{-1}$ is given by:*

$$K_\lambda(x,y;\omega) = \begin{cases} \tau_\lambda(x)(1+T^*h)^{-1}T^*\tau_\lambda(y)^{-1} & x < y \\ \tau_\lambda(x)(1+T^*h)^{-1}\tau_\lambda(y)^{-1} & x > y \end{cases} \tag{8}$$

with τ_λ the monodromy *of the operator $i(D_\omega - \lambda)^{-1}$, and $h = \tau_0(\beta)$ the parallel transport of the connection ∇.*

Let us remark that $K_\lambda(x,y,\omega)$ jumps by i at the diagonal, so by a simple computation the kernel of D_ω^{-s} is continuous and so D_ω^{-s} is trace-class, hence the associated spectral ζ function is well defined.

The first theorem can be proved by a contour integral computation $[S2]$, but here we give a variational argument which is of some use for the understanding of the higher dimensional case and is valid not only for unitary conditions i.e. $T^* = T^{-1}$, the case treated in $[S2]$, but for more general invertible boundary conditions . Specifically, we take $T_0, T_1 \in GL(n)$ such that the EBVP's D_{P_0}, D_{P_1} are invertible. Let T_r, $0 \leq r \leq 1$ be a path in $GL(n)$ connecting such a pair of EBVP's such that $D_{P_r} \doteq D_{P_{T_R}}$ is invertible. In order to compute the ζ-variation we transfer the r-dependence into the operator via a gauge transformation from $C^\infty(X, GL(n))$. Associated to the path we define the path $g_r = (T_r^*)^{-1}h^* \in GL(n)$. Next, fix a smooth non-decreasing function,

$$f : [0,\beta] \to [0,1],$$

such that $f(x) = 0$ for $x < \beta/4$ and $f(x) = 1$ for $x > 3\beta/4$. Hence, we have a smooth path of gauge transformations $g_{rf}(x) \doteq g_{rf(x)}$. Under the gauge action we have that D_{P_r} and $D^r_{[u_r^{-1}Pu_r]}$ are equivalent, where $D^r \doteq g_{rf}^{-1} D g_{rf}$

and

$$u_r = \begin{pmatrix} g_0 & 0 \\ 0 & g_r \end{pmatrix}.$$

It is easily shown that under the choice $T_1 = h$, $D^r_{[u_r^{-1} P u_r]} = D^r_{P_h}$. Let us note that the gauge equivalence implies that $\det_\zeta D_{P_r} = \det_\zeta D^r_{P_h}$ and $\det_{\mathcal{C}} D_{P_r} = \det_{\mathcal{C}} D^r_{P_h}$.
One can now find the suitable Schwartz kernel $K_r(x, y, P_h)$ of $(D^r_{P_h})^{-1}$, which is given by

$$K_r(x,y,P_h) = \begin{cases} -i g^{-1}_{rf(x)} \tau(x)(1 + h^{-1}(T^*_r)^{-1})^{-1} \tau(y)^{-1} g_{rf(y)} & x < y \\ g^{-1}_{rf(x)} \tau(x)(1 +^{-1} T^* h)^{-1} \tau(y)^{-1} g_{rf(x)} & x > y \end{cases}. \tag{9}$$

Using the previous results one can show that
Proposition 3.

$$\frac{d}{dr} \log \det\nolimits_\zeta D_{P_r} = \int_X \operatorname{tr}\,(K_r(x,x)\dot{D}^r)dx, \tag{10}$$

where

$$K_r(x,y) = -i g^{-1}_{rf(x)} h(x)(1 + T^*_r)h)^{-1} T^*_r h h(y)^{-1} g_{rf(y)}, \ \textit{and}\ \dot{D}^r = \frac{d}{dr} D^r.$$

Before giving a proof of the first theorem, we reformulate the results concerning the canonical determinant for the path of EBVP's $D^r_{P_h}$.
Lemma 3. *With the same notations as before,*

1. $\det_{\mathcal{C}}(D^r_{P_h}) = \det(I + T^*_r h)$.
2. $\frac{d}{dr} \log \det_{\mathcal{C}} D^r_{P_h} = \operatorname{Tr}\,(I + T^*_r h)^{-1} \frac{d}{dr} T^*_r h$.

Sketch of the proof. The first result is obtained using the rules of transformation of the paralel transport under a gauge transformation as well as the rules for the transformation of T_r. The second assertion is obtained from the first one and using the usual formula $\frac{d}{dr} \log \det A = \operatorname{Tr} A^{-1} \frac{d}{dr} A$ valid for operators A acting on finite dimensional spaces. □

The following theorem gives a variational identification of the ζ and the canonical approach to the computation of the determinant of an EBVP:
Theorem 1. *Let D^r be the one parameter family of EBVPs defined as above, then*

$$\frac{d}{dr} \log \det\nolimits_\zeta (D_{P_{T_r}}) = \frac{d}{dr} \log \det\nolimits_{\mathcal{C}} (D_{P_{T_r}}). \tag{11}$$

Proof. We compute that

$$\dot{D}^r = g_{rf(x)}^{-1}\left\{i\frac{d}{dx}G_{rf(x)} + [B(x), G_{rf(x)}]\right\} g_{rf(x)}, \tag{12}$$

where $G_{rf(x)} = \frac{d}{dr}(g_{rf(x)})g_{rf(x)}^{-1}$. From (10) and using the symmetry of the trace

$$\begin{aligned}\frac{d}{dr}\log\det{}_\zeta(D_{P_r}) = -\int_0^\beta \mathrm{tr}\,\left\{h(x)(I+T_r^*h)^{-1}T_r^*hh(x)^{-1}\frac{d}{dx}G_{rf(x)}\right\}\,dx \\ +i\int_0^\beta \mathrm{tr}\,\left\{h(x)(I+T_r^*h)^{-1}T_r^*hh(x)^{-1}[B(x),G_{rf(x)}]\right\}\,dx.\end{aligned} \tag{13}$$

Integrating by parts, the first term of (13) reads

$$\begin{aligned}-\int_0^\beta \mathrm{tr}\left\{h(x)(I+T_r^*h)^{-1}T_r^*hh(x)^{-1}\frac{d}{dx}G_{rf(x)}\right\}\,dx = \\ -\left[\mathrm{tr}\,\left\{h(x)(I+T_r^*h)^{-1}T_r^*hh(x)^{-1}G_{rf(x)}\right\}\right]_{x=0}^\beta \\ +\int_0^\beta \mathrm{tr}\,\left\{\frac{d}{dx}\left(h(x)(I+T_r^*h)^{-1}T_r^*hh(x)^{-1}\right)G_{rf(x)}\right\}\,dx.\end{aligned} \tag{14}$$

The first term of (14) reads

$$\begin{aligned}-\left[\mathrm{tr}\,\left\{h(x)(I+T_r^*h)^{-1}\quad T_r^*hh(x)^{-1}G_{rf(x)}\right\}\right]_{x=0}^\beta = \\ -\,\mathrm{tr}\,\left\{h(I+h^{-1}(I+T_r^*h)^{-1}T_r^*G_r\right\} + \\ \mathrm{Tr}\,\left\{(I+T_r^*h)^{-1}T_r^*hG_0\right\}.\end{aligned} \tag{15}$$

We have $G_r = \frac{d}{dr}(g_r)g_r^{-1}$ and $g_r = (T_r^*)^{-1}h^*$, and hence $G_r = -(T_r^*)^{-1}\frac{d}{dr}T_r^*$. Since T_0 is independent of r, the second term in (15) vanishes, while the first term is equal to

$$\mathrm{tr}\,\left\{(I+T_r^*h)^{-1}\frac{d}{dr}T_r^*h\right\} = \frac{d}{dr}\log\det{}_{\mathcal{C}}(D_{P_{T_r}})\ . \tag{16}$$

On the other hand, the second term of (14) is equal to

$$\begin{aligned}\int_0^\beta \mathrm{tr}\,\Bigg\{\Bigg(\frac{d}{dx}h(x).(I+T_r^*h)^{-1}T_r^*hh(x)^{-1} - \\ h(x).(I+T_r^*h)^{-1}T_r^*hh(x)^{-1}\frac{d}{dx}h(x).h(x)^{-1}\Bigg)G_{rf(x)}\Bigg\}\,dx.\end{aligned}$$

$$= -i\int_0^\beta \mathrm{tr}\,\left\{h(x).(I+T_r^*h)^{-1}T_r^*hh(x)^{-1}[B(x),G_{rf(x)}]\right\}\,dx, \tag{17}$$

since $Dh(x) = 0$ implies $\frac{d}{dx}h(x) = iB(x)h(x)$, which cancels the second term in (13). Combining (13),(14),(15),(16) and (17) completes the proof of the theorem. $\square$

4 Calculation of Quillen's metric

We now proceed to the computation of the ζ-curvature of the determinant line bundle. According to Quillen, we need to make the computation of the ζ-determinant of the associated Laplacian Δ_ω of a EBVP D_ω, $\Delta_\omega \doteq D_\omega^* D_\omega$. Since it is a second order operator, in order to compute its determinant we use a method which essentially reduces the problem to the computation of the determinant of an associated first order operator to which we can apply the methods described in the preceding sections. Much of this section is based on results from $[LT]$ and we refer to it for complete proofs.

4.1 The Laplacian of D_ω

When dealing with the Laplacian we are faced with a different boundary value problem, because the range of D_ω does not coincide with the domain of the adjoint boundary problem D_ω^*. First we define the Laplacian of the BVP D_ω as follows,

Definition 3. *The Laplacian Δ_P of the boundary value problem D_ω, is defined by:*

$$\begin{cases} \Delta_P = D_\omega^* D_\omega \\ \mathrm{dom}\,(\Delta_P) = \\ \left\{\psi \in H^2(X;\mathbb{C}^n) P(\psi(0),\psi(\beta)) = 0, P^*(D\psi(0), D\psi(\beta)) = 0\right\}. \end{cases} \tag{18}$$

Remark 1. Over U_Γ, P^*, the adjoint boundary condition, has an appropriate expression in terms of T given by:

$$P_T^* = \begin{pmatrix} Q^{-1} & Q^{-1}T^* \\ TQ^{-1} & TQ^{-1}T^* \end{pmatrix}, Q = 1 + T^*T. \tag{19}$$

More generally we define for an elliptic operator A of order r,

$$A \doteq \sum_{k=0}^{r} a_k \frac{d^k}{dx^k}, \tag{20}$$

the Elliptic Boundary Value Problem, A_{P_A}, with $P_A \in Gr(\mathbb{C}^{nr} \oplus \mathbb{C}^{nr})$, given by

Definition 4.

$$\begin{cases} A_{P_A} = A \\ \mathrm{dom}\,(A_{P_A}) = \\ \left\{\psi \in H^r(X;\mathbb{C}^n) : P(\psi(0)\cdots\psi^{(r-1)}(0), \psi(\beta)\cdots\psi^{(r-1)}(\beta)) = 0\right\}. \end{cases} \tag{21}$$

Furthermore, we define a boundary condition ω_A, which will play an important role in the subsequent discussion:

$$\omega_A = \left\{ \begin{pmatrix} v_0 \\ v_\beta \end{pmatrix} \in \mathbb{C}^{nr} \oplus \mathbb{C}^{nr} : R_0 v_0 + R_\beta v_\beta = 0 \right\}. \tag{22}$$

In order to compute of the determinant of this EBVP, we define an equivalent EBVP $\tilde{A}$ for an operator of order 1 defined on $C^\infty(X;\mathbb{C}^{nr})$, which is obtained in terms of the coefficients of A as:

$$\tilde{A} \doteq \frac{d}{dx} - \begin{pmatrix} 0 & 1 & 0\cdots & 0 \\ 0 & 0 & 1\cdots & 0 \\ 0 & & \cdots & 0 \\ -a_r a_0 & -a_r a_1 & \cdots & -a_r a_{r-1} \end{pmatrix}. \tag{23}$$

The meaning of this equivalence is expressed in the next result,

Proposition 4. *There exists between the BVP A_{P_A} and $\tilde{A}_{\omega_A}$ a canonical isomorphism of the determinant line bundles:*

$$\mathrm{Det}\, A_{P_A} \cong \mathrm{Det}\, \tilde{A}_{\omega_A}. \tag{24}$$

Remark 2. 1. To see the reason for this isomorphism, let us note that given $\psi \in H^r(X,\mathbb{C}^n)$:

$$\tilde{A}\tilde{\psi} = \begin{pmatrix} 0 \\ A\psi \end{pmatrix},$$

where $\tilde{\psi} \doteq \left(\psi, \psi^{(1)}\cdots, \psi^{(r-1)}\right)$. Hence, one can see that this relation gives a canonical identification $\mathrm{Ker}\tilde{\mathrm{A}} \cong \mathrm{KerA}$ $(\mathrm{Coker}\tilde{\mathrm{A}} \cong \mathrm{CokerA})$.

2. This last result assures us that the curvature over the determinant line bundle is once again the preceding Kähler form.

Now, A_{P_A} has an inverse $(A_{P_A})^{-1}$ which has a Schwartz kernel that can be found explicitly [$S4$].

Lemma 4. *Let us denote $R = R_0 + R_\beta h_{\tilde{A}}$, then A_{P_A} is invertible iff R has an inverse. In this case $A_{P_A}^{-1}$ is an integral operator ,*

$$A_{P_A}^{-1} : L^2(I;\mathbb{C}^n) \to \mathrm{dom}(A_{P_A}),$$

and the Schwartz kernel is given by given by

$$k(x,y)=\begin{cases} -\left[h_{\tilde{A}}(x)(R^{-1}R_\beta h_{\tilde{A}})h_{\tilde{A}}^{-1}(y)\right]_{[1,r]} a_r^{-1}(y) & x<y \\ \left[h_{\tilde{A}}(x)(\mathrm{I}-R^{-1}R_\beta h_{\tilde{A}})h_{\tilde{A}}^{-1}(y)\right]_{[1,r]} a_r^{-1}(y) & x>y \end{cases}, \tag{25}$$

with $h_{\tilde{A}}(x)$ the parallel transport from 0 to x, of $\tilde{A}$, and $h_{\tilde{A}}=h_{\tilde{A}}(\beta)$. The notation $D_{[1,r]}$ means the matrix $r\times r$ located in the top left block of D.

4.2 *The ζ-function metric of the determinant line bundle*

The essential result that we need in order to apply the above discussion to the Laplacian is [*LT*] :

Proposition 5. *Given a BVP A_{Pz}, we have:*

$$\det\nolimits_\zeta A_{P(z)}=\exp(-\operatorname{LIM}_{z\to\infty}\log\det R(z))\det R, \tag{26}$$

where $R(z)=R_0+R_\beta h_{\tilde{A}}(z)$, $R=R(\beta)$ and where LIM *means the* regularized limit*, in the sense of the quoted paper* [*LT*].

Then, we find an explicit expression for the matrices R_0 and R_β for the case of the Laplacian $\Delta_P=-\frac{d^2}{dx^2}$ and its associated first order operator $\tilde{A}=\begin{pmatrix}\frac{d}{dx} & 1\\ 0 & \frac{d}{dx}\end{pmatrix}$.

Lemma 5. *The matrices R_0 and R_β are given by:*

$$R_0=\begin{pmatrix}Q^{-1} & -T^*\tilde{Q}^{-1}T\\ TQ^{-1} & \tilde{Q}{-1}T\end{pmatrix}, R_\beta=\begin{pmatrix}Q^{-1}T^* & -T^*\tilde{Q}^{-1}\\ TQ^{-1}T^* & \tilde{Q}{-1}\end{pmatrix}. \tag{27}$$

Hence, using the results in [*LT*], we have for $R=R_0+R_\beta h_\beta$, that the $\det R$ of (26) is equal to:

Proposition 6.

$$\det R=\det Q^{-1}\det(1+T)\det(1+T^*). \tag{28}$$

On the other hand, we have for the limit of the same expression, the result:

Proposition 7.

$$\exp\operatorname{LIM}_{z\to\infty}\log\det R(z)=2^n.$$

Finally, combining these results we arrive at

Theorem 2.

$$\begin{aligned}\det\nolimits_\zeta\Delta_{P_T}&=2^n\log\det Q^{-1}\det(1+h^*T)\det(1+T^*h)\\ &=2^n e^{-k(T)}\left|\det\nolimits_\zeta D_{P_T}\right|^2,\end{aligned} \tag{29}$$

where we have used the expression $e^{-k(T)} = \log\det Q^{-1}$ with $Q = 1 + T^*T$ to express the Kähler potential.

Now we are ready to assert the relation between the canonical and ζ curvature of the determinant line bundle announced in the introduction.

Corollary 1. *The ζ-curvature of the determinant line bundle is given by the Kähler form of the Grassmannian.*

5 Final remark

As stated in the introduction it is remarkable that there exists such an identification between two such different ways of computing determinants of boundary value problems. Now, it is possible to obtain more general results for the 1-dimensional case by using the notion of relative determinant, $[F]$, which allows a very different proof of the same results $[S4]$. Additionally, the same method has been used by Scott for the computation of the curvature of the determinant line bundle in higher dimensions $[S3]$.

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