Introduction to Quantum Invariants of 3-manifolds, Topological Quantum Field Theories and Modular Categories

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Chapter 1

LINKS, 3-MANIFOLDS AND THEIR INVARIANTS

We will work in the smooth category. For 3-dimensional manifolds, the topological and smooth classifications coincide. This means that a topological 3-manifold admits a unique C^∞ structure, and two 3-manifolds are homeomorphic if and only if they are diffeomorphic. It may be useful to consider that the manifold is equipped with a riemannian metric, e.g. if we need to consider the bundle of (oriented) orthonormal frames; whenever we do so, the choice of the metric is irrelevant.

1.1 Knots, links and diagrams

For an elementary introduction to knot theory, see [12, 25, 18, 16]. A knot in a 3-manifold is an embedding of the circle S^1 in the euclidean space $\mathbb{R}^3$ or in its one point compactification, identified with the sphere S^3; a link with m components is an embedding of a disjoint union of m copies of S^1. We are interested with knots and links up to the equivalence given by ambiant isotopy.

An immersed closed curve in the plane $\mathbb{R}^2$ is generic if and only if any multiple point is a double point with non colinear tangent vectors. A regular diagram in the plane is a closed generic immersed curve, and a choice, for each double point, of the arc which passes over. Such a diagram gives a link in $\mathbb{R}^3$ or S^3 well defined up to isotopy. Any link can be obtained this way, and the well known Reidemeister theorem describes the equivalence relation on diagrams induced by isotopy of links.

Theorem 1.1.1 (Reidemeister theorem). *Two diagrams represent isotopic links if and only if they are related by isotopy in the plane and a finite sequence of the moves in figure 1.1.*

(R1)

(R2)

(R3)

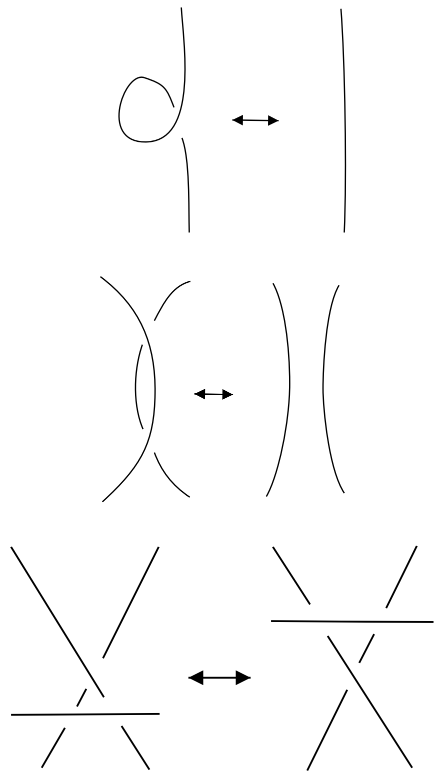


Figure 1.1: Reidemeister moves

We will mainly consider here thick knots and links: oriented embeddings of solid tori $D^2 \times S^1$, up to isotopy. (This is equivalent to considering framed knots and links.) An oriented diagram in the plane gives a thick link by using the blackboard convention. Reidemeister theorem for solid knots and links in $\mathbb{R}^3$ or S^3 holds by replacing the first Reidemeister move (R1) by (R1') in figure 1.2. In the Reidemeister moves one has to consider the various choices of orientations.

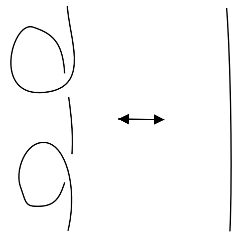


Figure 1.2: Reidemeister move (R1')

1.2 Surgery on links and Kirby calculus

Let us consider a solid knot in S^3

$$g : D^2 \times S^1 \hookrightarrow S^3 .$$

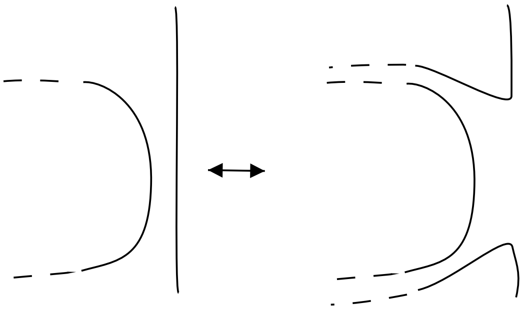
We use it to build a new 3-manifold as follows. We remove the interior of the embedded solid torus, and we glue back another solid torus $-S^1 \times D^2$ using the restriction $g_{|S^1 \times S^1}$ (the minus sign here is for reversing orientation). This process is known as surgery. The manifold only depends on the isotopy class K of the solid knot represented by g; we denote it by $S^3(K)$. More generally, we will do surgery on a solid link with m components whose isotopy class is denoted by $L = (L_1, \ldots, L_m)$, and get the manifold $S^3(L)$.

A theorem of Lickorish and Wallace asserts that any closed oriented 3-manifold is homeomorphic to $S^3(L)$ for some L. As a consequence, we have that any oriented closed 3-manifold can be represented by a diagram in the plane. A famous Kirby theorem describes the equivalence relation on diagrams induced by homeomorphism of 3-manifolds.

Theorem 1.2.1. *The equivalence relation on diagrams induced by oriented homeomorphism of oriented* 3*-manifolds is generated by isotopy in the plane, the Reidemeister moves (R1'), (R2), (R3), reversing the orientation of one component, and by the two Kirby move (K1), (K2) described in figure 1.3*

The (K1) move is called the handle slide; here the two arcs belong to distinct components of the link. The (K2) move is called stabilization.

(K1)



(K2) Add or remove

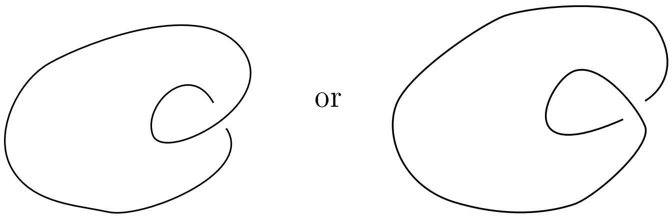


Figure 1.3: Kirby moves

1.3 Ribbon categories

The main idea in this section, is to consider invariants of links obtained by associating some algebraic quantities to small pieces of diagram, and then use gluing formulas.

1.3.1 The tangle category

We want to consider horizontal slices of solid links. Suppose that we have a solid link in $\mathbb{R}^3$ which intersects the planes $\mathbb{R}^2 \times 0$ and $\mathbb{R}^2 \times 1$ transversally. Then the intersection of the link with these planes gives a finite family of framed points in each plane. Here a framing at a point is a trivialization of the tangent bundle to the plane at this point; the orientation of this framing coincides or not with the orientation of the plane, depending if the string goes upward or downward at this point.

We define a category $\mathcal{T}$ in which objects are finite sets of framed points in the disc D^2, and morphisms are pieces of solid links in the cylinder $D^2 \times [0; 1]$, with boundary the framed points in the given source and target objects, up to isotopy which is identity on the boundary of the cylinder. Composition is given by stacking the cylinders. Such a morphism is called a tangle and $\mathcal{T}$ is called the tangle category.

In our figures the time parameter goes upward, so that the morphism gf is depicted with g lying above f; if the string at a boundary point is upward, then it is understood that the framing is the standard one in the horizontal plane, and if the string is downward then the framing is obtained from the standard one by applying $[z \mapsto -\overline{z}]$.

People often call tangle category the full subcategory of $\mathcal{T}$ formed with objects whose points have standard positions along the real axis and whose framings are the standard positive or negative one as above. This subcategory is equivalent to $\mathcal{T}$; we will denote it by $\mathcal{T}^s$ and call it the strict tangle category. Morphisms in this category are represented by diagrams in $[-1, 1] \times [0; 1]$, and the appropriate Reidemeister theorem holds.

We define a tensor product on the category $\mathcal{T}$ by using

$$j = j_{-1} \amalg j_1 : D^2 \amalg D^2 \hookrightarrow D^2 \ ,$$

where, for $\epsilon = \pm 1$, $j_\epsilon : D^2 \hookrightarrow D^2$ is the embedding sending z to $\frac{\epsilon}{2} + \frac{1}{4}z$. Equipped with this tensor product, $\mathcal{T}$ is a monoidal category [19] (the associativity constraints are the obvious ones) and $\mathcal{T}^s$ is a strict monoidal category.

The strategy in the following is to use representations of the strict tangle category. We would like a monoidal functor with target a category where computations can be done, e.g. where set of morphisms are finite dimensional vector spaces.

1.3.2 Yang-Baxter operator

Let V be a free and finitely generated $\mathbf{k}$-module.

Definition 1.3.1. A Yang-Baxter operator is an isomorphism $r : V^{\otimes 2} \to V^{\otimes 2}$ such that

$$(r \otimes \mathbb{1}_V)(\mathbb{1}_V \otimes r)(r \otimes \mathbb{1}_V) = (\mathbb{1}_V \otimes r)(r \otimes \mathbb{1}_V)(\mathbb{1}_V \otimes r) \ . \tag{1.1}$$

We denote by B_n the Artin braid group. It is generated by $\sigma_1, \ldots, \sigma_{n-1}$ with relations

$$\forall i = 1 \ldots n-2, \ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\forall i,j \text{ such that } |i-j| \geq 2,\ \sigma_i\sigma_j = \sigma_j\sigma_i\ .$$

If r is a Yang-Baxter operator, then we can define a representation $B_n \to GL(V^{\otimes n})$ by sending σ_i to $\mathbb{1}_V^{\otimes i-1} \otimes r \otimes \mathbb{1}_V^{\otimes n-i-1}$. We would like an extension to the strict tangle category. In order to do that, we pick $\mu \in GL(V)$, and use the usual duality to define the homomorphisms

$$e_V : V^* \otimes V \to \mathbf{k}\ ;\ b_{V^*} : \mathbf{k} \to V^* \otimes V\ ,$$

$$b_V : \mathbf{k} \to V \otimes V^*\ ;\ e_{V^*} : V \otimes V^* \to \mathbf{k}\ ,$$

corresponding to μ.

We define $r^{\pm}_{V^*,V} \in Hom(V^*\otimes V, V\otimes V^*)$ and $r^{\pm}_{V,V^*} \in Hom(V\otimes V^*, V^*\otimes V)$ by the formulas below.

$$r^{\pm}_{V^*,V} = (e_V \otimes \mathbb{1}_{V\otimes V^*})(\mathbb{1}_{V^*} \otimes r^{\mp} \otimes \mathbb{1}_{V^*})(\mathbb{1}_{V^*\otimes V} \otimes b_V)\ ,$$

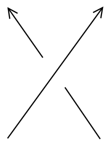
$$r^{\pm}_{V,V^*} = (\mathbb{1}_{V^*\otimes V} \otimes e_{V^*})(\mathbb{1}_{V^*} \otimes r^{\mp} \otimes \mathbb{1}_{V^*})(b_{V^*} \otimes \mathbb{1}_{V\otimes V^*})\ .$$

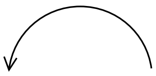
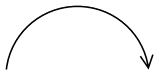
Definition 1.3.2. The triple (V, r, μ) is called an *enhanced Yang-Baxter operator* if and only if
i) $\mu \otimes \mu$ commutes with r;
ii) $(e_V \otimes \mathbb{1}_V)(\mathbb{1}_V \otimes r)(b_{V^*} \otimes \mathbb{1}_V)(e_V \otimes \mathbb{1}_V)(\mathbb{1}_V \otimes r^{-1})(b_{V^*} \otimes \mathbb{1}_V) = \mathbb{1}_V$;
iii) $r^{\pm}_{V,V^*} r^{\mp}_{V^*,V} = \mathbb{1}_{V^*\otimes V}$.

This terminology was introduced in [28] with a slightly more restrictive definition. The theorem below is established by using a presentation of the tangle category. See e.g. [13, Chapter 12]; note that our tangles here involve the framing, so that the relation (2.6) there should be replaced by the one given by the Reidemeister move (R1').

Theorem 1.3.3. *If the triple (V, r, μ) is an enhanced Yang-Baxter operator then there exists a unique monoidal functor from the strict tangle category $\mathcal{T}^s$ to the category of vector spaces such that*

$$\uparrow\ \longmapsto \mathbb{1}_V\ ,\quad \downarrow\ \longmapsto \mathbb{1}_{V^*}\ .$$

 $\longmapsto r$

 $\longmapsto e_V$,  $\longmapsto e_{V^*}$.

Exercise 1.3.1. Fix an integer $N \geq 2$. Let V be the free module over a ring $\mathbf{k}$ containing $s^{\pm\frac{1}{2}}$, with basis v_i, $i \in \{-N+1, -N+3, \dots, N-3, N-1\}$. Let $E \in End(V \otimes V)$ be defined by

$$E.e_i \otimes e_j = \sum_{k,l} E_{ij}^{kl} e_k \otimes e_l, \text{ with}$$

$$E_{ij}^{kl} = \begin{cases} s & \text{if } i < j,\ k = i \text{ and } l = j; \\ s^{-1} & \text{if } i > j,\ k = i \text{ and } l = j; \\ 1 & \text{if} i \neq j,\ k = j \text{ and } l = i; \\ 0 & \text{else.} \end{cases}$$

Let $r = s\mathbb{1}_{V\otimes V} - E$, and $\mu : V \to V$ be defined by $\mu(v_i) = s^{\frac{i}{2}} v_i$. Note for further use that $r - r^{-1} = (s - s^{-1})\mathbb{1}_{V\otimes V}$.

1. Show that $E^2 = (s + s^{-1})E$. Deduce that r is a Yang-Baxter operator.

2. Show that (V, r, μ) is an enhanced Yang-Baxter operator.

1.3.3 Ribbon category

Let $\mathcal{C}$ be a monoidal category with trivial object Θ. A (left) duality on $\mathcal{C}$ associates to any object V an object V^* and morphisms

$$e_V : V^* \otimes V \to \Theta\ ,\ b_V : \Theta \to V \otimes V^*$$

such that

$$(\mathbb{1}_V \otimes e_V)(b_V \otimes \mathbb{1}_V) = \mathbb{1}_V\ , \tag{1.2}$$

$$(e_V \otimes \mathbb{1}_{V^*})(\mathbb{1}_{V^*} \otimes b_V) = \mathbb{1}_{V^*}\ . \tag{1.3}$$

A braiding in $\mathcal{C}$ is a natural family of isomorphisms

$$c_{V,W} : V \otimes W \to W \otimes V \ , V, W \in Obj(\mathcal{C}) \ ,$$

such that

$$c_{V,W\otimes W'} = (\mathbb{1}_W \otimes c_{V,W'})(c_{V,W} \otimes \mathbb{1}_{W'}) \ , \tag{1.4}$$

$$c_{V\otimes V',W} = (c_{V,W} \otimes \mathbb{1}_{V'})(\mathbb{1}_V \otimes c_{V',W}) \ . \tag{1.5}$$

A twist in $\mathcal{C}$ compatible with the duality and the braiding is a natural family of isomorphisms

$$\theta_V : V \to V \ , V \in Obj(\mathcal{C}) \ ,$$

such that

$$(\theta_V \otimes \mathbb{1}_{V^*})b_V = (\mathbb{1}_V \otimes \theta_{V^*})b_V \ , \tag{1.6}$$

$$\theta_{V\otimes W} = c_{W,V}c_{V,W}(\theta_V \otimes \theta_W) \ . \tag{1.7}$$

Definition 1.3.4. A ribbon category is a monoidal category $\mathcal{C}$ with duality, braiding and compatible twist.

Theorem 1.3.5. *Let $\mathcal{C}$ be a strict ribbon category, and V an object in $\mathcal{C}$, then there exists a unique ribbon functor from the strict tangle category $\mathcal{T}^s$ to $\mathcal{C}$, which sends the standard object with one point, denoted by 1, to the object V.*

For a proof see e.g. [29]. Note that a link is an endomorphism of the trivial object, hence we have that this construction provides an invariant of links with values in $End_{\mathcal{C}}(\Theta)$.

The functor in the above theorem has various extensions; in view of our construction of TQFT in the next chapters, we will introduce here a slightly modified version of the category of colored ribbon graphs considered in [29], which we call solid graphs. Roughtly speaking, a solid graph is a 3-dimensional thickening of a finite graph.

A *solid graph* is a 3-dimensional manifold with boundary, which is decomposed into parametrized copies of cylinders $D^2 \times [0,1]$, and solid tori $D^2 \times S^1$. The cylinders are decomposed in two parts which are called respectively the (solid) edges, and the (solid) vertices. The tori are called the closed components. The bottom and top of each solid vertex have (positively or negatively) framed points and so are objects of the category $\mathcal{T}$, and each of these points is glued to the end of a solid edge, using the framing.

A coloring of a solid graph over a ribbon category $\mathcal{C}$, is a labeling of the closed components and the solid edges by objects in $\mathcal{C}$, and of the solid vertices by morphisms in $\mathcal{C}$, which is compatible with the gluings.

We define the category of $\mathcal{C}$-colored solid graphs similarly as we did for tangles; here objects are finite sets of framed points in the disc, labeled with objects of $\mathcal{C}$, and morphisms are embedded colored solid graphs whose free ends of edges are conveniently glued to the source and target points. We denote by $\mathcal{G}_\mathcal{C}$ this category, and by $\mathcal{G}^s_\mathcal{C}$ the strict subcategory (i.e. the full subcategory formed with objects whose points have standard position along the real axis and have standard positive or negative framing).

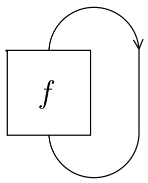
Here also, morphisms in this category are represented by diagrams. Edges are represented by arcs; solid vertices are represented by boxes, and for each of them top and bottom are given by the diagram. A colored solid vertex will be called a *coupon*.

The following theorem is a reformulation of [29, Theorem I.2.5].

Theorem 1.3.6. *Let $\mathcal{C}$ be a strict ribbon category, then there exists a unique monoidal functor from the strict $\mathcal{C}$-colored solid graphs category $\mathcal{G}^s_\mathcal{C}$ to $\mathcal{C}$, which respects duality, braiding and twist, and sends a colored positively framed standard point to its label, and a coupon to the morphism given by its label.*

Note that this theorem allows to use the graphical calculus for morphisms in the ribbon category. Whenever we do so, the functor is implicit.

In a ribbon category there is a notion of trace of morphisms and dimension of objects. The trace of a morphism f is denoted by $\langle f \rangle$.

$\langle f \rangle =$ 

The dimension of an object V is the trace of $\mathbb{1}_V$; we will use the notation $\langle V \rangle$ as well as $\langle \mathbb{1}_V \rangle$. We often say *quantum* trace and dimension to distinguish from the usual trace and dimension in vector spaces.

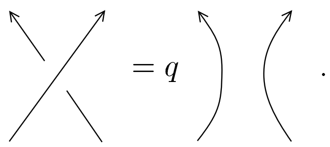
Exercise 1.3.2. Associate a ribbon category to an enhanced Yang-Baxter operator.

1.3.4 The skein method

The tangle category $\mathcal{T}$ is a ribbon category. It will be useful to switch to a category in which the objects are the same, but morphisms are linear combinations of tangles with coefficients in some ring $\mathbf{k}$. This category is denoted by $\mathbf{k}[\mathcal{T}]$; note that composition and tensor product are bilinear.

The skein method builds ribbon categories by quotienting the $\mathbf{k}$-additive tangle category by local relations; the local relations generate a tensor ideal, and the quotient is a ribbon category.

Example 1.3.1 (Simplest example). The ring is $\mathbb{Z}[q^{\pm 1}]$, the local relation is given below.



Example 1.3.2 (Homfly skein theory). Here $\mathbf{k}$ is a ring which contains $a^{\pm 1}$, $v^{\pm 1}$, $s^{\pm 1}$, $\frac{v^{-1}-v}{s-s^{-1}}$. The local relations are given in figure 1.4.

This example with specific specializations of the parameters will play a key role in our constructions. The ribbon category defined with these relations is called the Hecke category; we denote it by H. The trivial object is denoted by 0, the standard object with n points is denoted by n, the module $End_H(n)$ is denoted by H_n.

Exercise 1.3.3. We suppose that $N \geq 2$, $v = s^{-N}$ and $a = 1$ in $\mathbf{k}$. Show that the functor in exercise 1.3.1 induces a functor from the category H to the category of $\mathbf{k}$-modules.

Theorem 1.3.7. *H_0 is isomorphic to $\mathbf{k}$.*

This theorem gives an invariant of link $\langle L \rangle \in \mathbf{k}$, which is the Homfly polynomial (framed version); here we fix the isomorphism with the normalization $\langle \emptyset \rangle = 1$.

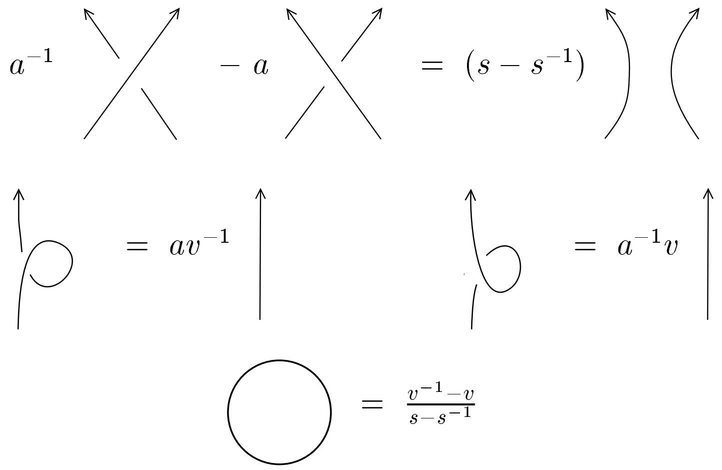


Figure 1.4: Homfly relations

Theorem 1.3.8. *The algebra H_n is isomorphic to the quotient of the Artin braid group algebra $\mathbf{k}[B_n]$ by the quadratic relations $a^{-1}\sigma_i - a\sigma_i^{-1} = s - s^{-1}$.*

This algebra is the Hecke algebra; it is a deformation of the symmetric group algebra.

1.3.5 Representation theory of quantum groups

We saw in exercise 1.3.1 that some $r \in End(V \otimes V)$ satisfies the Yang-Baxter equation. This can be explained by the fact that V is a module over the quantum group $U_q sl(N)$. We give an outline of this approach.

1.3.6 Hopf algebra

Definition 1.3.9. A bialgebra is an algebra A equipped with two algebra morphisms

$$\Delta : A \to A \otimes A$$

$$\epsilon : A \to \mathbf{k}$$

such that Δ (the comultiplication) is coassociative, and ϵ (the augmentation) is a counit.

A Hopf algebra (A, Δ, ϵ, S) is a bialgebra with an antipode S, i.e. an endomorphism S of A such that (here $m : A \otimes A \to A$ denote the multiplication).

$$\forall x \in A \ (m \circ (S \otimes Id_A) \circ \Delta)(x) = (m \circ (Id_A \otimes S) \circ \Delta)(x) = \epsilon(x) 1_A \ .$$

As a fundamental example, we have the universal envelopping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$.

Consider the category of finite dimensional representations of a Hopf algebra, denoted by $Rep(A)$. This category has a tensor product, defined by using the coproduct on A, making $Rep(A)$ into a monoidal category. Moreover, by using the antipode, we can provide $Rep(A)$ with a duality. To be more precise, denote the coproduct Δ by

$$\Delta(a) = \sum_{(a)} a' \otimes a'' \ .$$

Then the A-module structure on a tensor product and on the dual are defined by

$$a.(v \otimes w) = \sum_{(a)} a'.v \otimes a''.w \ ,$$

$$(a.f)(v) = \sum_{(a)} a'.f(S(a'').w \ ,$$

A universal R-matrix on A is an invertible element in $A \otimes A$ such that

$$\forall a \in A \ \Delta^{\mathrm{op}}(a) = R \Delta(a) R^{-1} \tag{1.8}$$

$$(Id_A \otimes \Delta)(R) = R_{13} R_{12} \tag{1.9}$$

$$(\Delta \otimes Id_A)(R) = R_{13} R_{23} \ . \tag{1.10}$$

The element v in the center of A is then a universal twist if and only if

$$\Delta(v) = P_A(R) R (v \otimes v) \ , \ \epsilon(v) = 1 \ , \ S(v) = v \ . \tag{1.11}$$

Here $P_A : A \otimes A \to A \otimes A$ is the flip, $\Delta^{\mathrm{op}}(a) = P_A \Delta(a)$ and

$$R_{12} = R \otimes 1_A \ , \ R_{23} = 1_A \otimes R \ , \ R_{13} = (Id_A \otimes P_A)(R_{12}) \ .$$

Definition 1.3.10. A ribbon Hopf is a Hopf algebra with universal R-matrix and twist.

Proposition 1.3.11. *If A is a ribbon Hopf algebra, then $Rep(A)$ is a ribbon category.*

The universal envelopping algebra $U\mathfrak{g}$ of a finite dimensional simple Lie algebra admits a deformation $U_q\mathfrak{g}$ which is almost a ribbon Hopf algebra. Here the *almost* means that the universal R-matrix lives in a suitable completion. If q is a generic parameter, the representation category (which is a ribbon category) is very closed to the classical case, and type 1 irreducible representations are indexed by highest weights. In particular, there exists an invariant for a link whose components are colored with highest weights. For $\mathfrak{g} = sl(N)$, we recover the invariant obtained in exercise 1.3.1 by coloring every component with the fundamental representation.

Chapter 2

TOPOLOGICAL QUANTUM FIELD THEORY

In this chapter, we introduce the axiomatic definition of a TQFT in dimension 3. This concept first appeared in [32, 4]. For basic references, see [29, 5].

2.1 Axiomatic definition

A TQFT in dimension 3 will provide the following assignments.

> $\Sigma \mapsto \mathbf{V}(\Sigma)$, where Σ is a surface (possibly with additional structure) and $\mathbf{V}(\Sigma)$ is a finite dimensional vector space;
> $M \mapsto \mathbf{V}(M) \in Hom(\mathbf{V}(\Sigma), \mathbf{V}(\Sigma'))$, where M is a 3-dimensional cobordism between Σ and Σ', i.e. a 3-manifold with boundary $-\Sigma \amalg \Sigma'$.

This has to satisfy axioms which are conveniently expressed in the categorical language.

The cobordism category C^2 is defined as follows. Objects are closed oriented surfaces; morphisms are cobordisms up to equivalence. The appropriate notion of equivalence on the cobordisms is orientation preserving diffeomorphism rel. boundary (the diffeomorphism is identity on the boundary).

Disjoint union makes this category into a monoidal category. The surface with opposite orientation is a left and right dual; the duality morphisms are given by the product of the interval $[0;1]$ with the surface; the evaluation e_Σ is equal to $[0;1]\times\Sigma$, viewed as a cobordism from $-\Sigma \amalg \Sigma$ to $\emptyset$, and the coevaluation b_Σ is equal to $[0;1]\times\Sigma$, viewed as a cobordism from $\emptyset$ to $\Sigma \amalg -\Sigma$. We get a (symmetric) ribbon category.

Definition 2.1.1. A TQFT in dimension 3 is a monoidal functor $\mathbf{V}$ from C^2 to the category of finite dimensional vector spaces.

From the definition, we have that $\mathbf{V}(\emptyset)=\mathbf{k}$, hence for a closed 3-manifold M we have an invariant $\mathbf{V}(M)\in End(\mathbf{V}(\emptyset))=\mathbf{k}$. If $\partial M=\Sigma$, then we have a vector $\mathbf{V}(M)\in Hom(\mathbf{k},\mathbf{V}(\Sigma))\simeq\mathbf{V}(\Sigma)$.

Exercise 2.1.1. Use the the duality cobordisms e_Σ and b_Σ to construct inverse isomorphisms between $\mathbf{V}(-\Sigma)$ and $\mathbf{V}(\Sigma)^*$.

2.2 Structures on manifolds

In most cases, we have to consider a cobordism category where surfaces and bordisms are equipped with additionnal structure. Many kinds of structure may be considered; see [26] for an extensive exposition on cobordism with structure. We quote that gluing structures along a surface and changing orientation of the surface have to be well defined. Here are some examples.

2.2.1 Framed manifolds

The surfaces are equipped with a trivialization of the oriented stable tangent bundle; cobordisms are equipped with an extension of the given trivialization on the boundary, up to homotopy rel. boundary. Here one has to fix the identification on the boundary. Our convention is 'outgoing normal vector first' for the oriented boundary. The discussion of existence of trivialization of an oriented bundle, and parametrization up to homotopy, is a piece of obstruction theory (see e.g. [20]). The first homotopy groups of $SO(n)$ are: $\Pi_1(SO(n))=\mathbb{Z}/2$ for $n\geq(3)$, $\Pi_2(SO(n))=0$ for all n and $\Pi_3(SO(n))=\mathbb{Z}$ for $n\geq 5$. In the dimensions we are interested with (up to dimension 4), obstruction theory and parametrization up to homotopy will rest on the Stiefel-Whitney class w_2, and the Pontryagin class p_1. If we do so, we have to consider spin structures [10]; the theory has interesting features, but for a basic exposition it is better to have a simpler context.

2.2.2 Complex framing

It appears that in most cases, because of a so called *framing anomaly* which will be explained further, some kind of framing should be considered. A weak version of framing was proposed by Atiyah to solve this question; this was called 2-framing. The key point is that obstruction theory in dimension up to 4 rests exclusively on the Pontryagin class p_1, or equivalently on signatures of 4-manifold, by Hirzebruch formula. An equivalent notion called p_1-structure was considered in [9]. We would like to give here a simple geometric definition.

Definition 2.2.1. A complex framing of a manifold is a trivialization of the complexified stable tangent bundle.

For an oriented manifold, the first Chern class of this complexified bundle vanishes, and the first obstruction is the second Chern class $c_2 = -p_1$. We deduce the following.

Proposition 2.2.2. *a) An oriented surface Σ admits a complex framing which is unique up to homotopy.*
b) A closed oriented 3-manifold M admits a complex framing and homotopy classes are affinely isomorphic to $H^3(M)$.
c) A closed oriented 4-manifold W admits a complex framing if and only if its signature is zero and homotopy classes are affinely isomorphic to $H^3(W)$.

We may also consider the extension problem. If we fix a complex framing on the boundary of an oriented manifold, we are interested with extensions of it to the whole manifold up to homotopy rel. boundary.

Proposition 2.2.3. *a) For an oriented 3-manifold M, a complex framing of the boundary extends, and relative homotopy classes of extensions are affinely isomorphic to $H^3(M, \partial M)$.*

b) For an oriented 4-manifold W, a complex framing α on the boundary $\partial W = M$ extends, if and only if some relative obstruction $c_2(\alpha) = -p_1(\alpha) \in H^4(W, \partial W)$ vanishes; moreover another complex framing α' on M is homotopic to α if and only if $p_1(\alpha) = p_1(\alpha')$.

The lemma below will be the key point for further use.

Lemma 2.2.4. *Let M be a connected closed oriented 3-manifold. There exists a bijection $\alpha \mapsto \sigma(\alpha)$ between complex framings on M up to homotopy and $\mathbb{Z}$, such that for any oriented W with $\partial W = M$ one has $\sigma(\alpha) = 3signature(W) - \langle p_1(\alpha), [W] \rangle$.*

The integer $\sigma(\alpha)$ will be called the *sigma invariant* of α.

2.2.3 Links and colored solid graphs

Let $\mathcal{C}$ be a ribbon category. We have already considered $\mathcal{C}$-colored solid graphs. It will be useful in the construction to extend the cobordism category with embedded $\mathcal{C}$-colored solid graphs.

The cobordism category $\mathrm{Cob}(\mathcal{C})$ is defined as follows. Objects are oriented surfaces with a trivialization of the complexified stable tangent bundle, and a finite set of points inside with trivialized neighbourhood labeled with objects in $\mathcal{C}$; in brief such an object will be called a $\mathcal{C}$-marked surface.

Morphisms are cobordisms with embedded $\mathcal{C}$-colored solid graphs, and extension of the given complex framings, up to the equivalence given by homeomorphism rel. boundary, isotopy of the graph rel. boundary and homotopy of the complex framing rel. boundary.

2.3 The universal construction

In this section we fix a ribbon category $\mathcal{C}$, and we work in the cobordism category $\mathrm{Cob}(\mathcal{C})$. The whole TQFT can be derived from a convenient invariant of closed 3-manifolds. Our method follows the construction in [9].

Here a closed 3-manifold $M = (M, \alpha, G)$ is an oriented closed 3-manifold M with a complex framing α ang a $\mathcal{C}$-colored solid graph G. Recall that the complex framing α can be defined by its sigma invariant $\sigma(\alpha)$; we denote simply by S^3 and $S^2 \times S^1$ the corresponding 3-manifold equipped with the complex framing α_0 whose σ-invariant is zero, and empty graph.

Let Z be an invariant of closed 3-manifolds, with value in a field $\mathbf{k}$. We say that Z satisfies the surgery axioms provided (S0), (S1) and (S2) below are satisfied.

In (S1) and (S2), the surgery gives a four dimensional cobordism between the two closed 3-manifolds M and M'. We fix the (homotopy class of the) complex framing on M' from the given one on M, by extending to this cobordism.

(S0) $Z(S^3) \neq 0$.
(S1) There exists a scalar $\mathcal{D}$ such that, if M' is obtained from M by an index one surgery, then $Z(M') = \mathcal{D}Z(M)$.
(S2) There exists a linear combination of objects of $\mathcal{C}$, ω, such that, if M' is obtained from M by an index two surgery on a solid knot K, then $Z(M') = Z(M, K(\omega))$.

Here the component K is colored by a linear combination of objects in the ribbon category $\mathcal{C}$; the corresponding invariant $Z(M, K(\omega))$ is obtained by expanding linearly.

Exercise 2.3.1. Suppose that Z satisfies the surgery axioms, then show that $\mathcal{D}Z(S^3) = 1$ and $Z(S^2 \times S^1) = 1$.

Given an invariant Z which satisfies the surgery axioms, then we associate to objects Σ of the category $\mathrm{Cob}(\mathcal{C})$ vector spaces as follows. The vector space $\mathcal{V}(\Sigma)$ is freely generated by morphisms with boundary Σ, i.e. by $Hom_{\mathrm{Cob}(\mathcal{C})}(\emptyset, \Sigma)$. The vector space $\mathbf{V}(\Sigma)$ is the quotient of $\mathcal{V}(\Sigma)$ by the left kernel of the natural bilinear map

$$\mathcal{V}(\Sigma) \otimes \mathcal{V}(-\Sigma) \to \mathbf{k} \ .$$

From the construction we have that $\mathbf{V}$ is a functor from the cobordism category $\mathrm{Cob}(\mathcal{C})$ to the category of vector spaces. However, we will need additionnal hypothesis in order to have finite dimension and multiplicativity. The role of the surgery axioms is given by the following important lemma.

Lemma 2.3.1 (Connexion lemma). *Let N (resp. N') be a connected oriented manifold with boundary the surface underlying the object Σ (resp. $-\Sigma$), then*
a) $\mathbf{V}(\Sigma)$ is generated by bordisms M with underlying manifold equal to N.
b) A linear combination $\sum \lambda_i M_i$ of bordisms is zero in $\mathbf{V}(\Sigma)$ if and only if, for any bordism M' with underlying manifold N', one has $\sum \lambda_i Z(M_i \cup_\Sigma M') = 0$.

Chapter 3

MODULAR CATEGORIES

3.1 Pre-modular and modular categories

Let $\mathbf{k}$ be a field. A monoidal category $\mathcal{C}$ is said to be $\mathbf{k}$-additive if the Hom sets are $\mathbf{k}$-vector spaces, composition and tensor product are bilinear, and $End(\text{trivial object}) = \mathbf{k}$. An object X of a $\mathbf{k}$-additive monoidal category is said to be *simple* if the map $u \mapsto u\mathbb{1}_X$ from $\mathbf{k} = End(\text{trivial object})$ to $End(X)$ is an isomorphism. Note that this simplicity condition is stronger that the usual one in module theory.

Definition 3.1.1. A *modular category* [29], over the field $\mathbf{k}$, is a $\mathbf{k}$-additive ribbon category $\mathcal{C}$ in which there exists a finite family Γ of simple objects λ satisfying the four axioms below.

1. (Normalization axiom) The trivial object is in Γ.
2. (Duality axiom) For any object $\lambda \in \Gamma$, its dual λ^* is isomorphic to an object in Γ.
3. (Domination axiom) For any object X of the category there exists a finite decomposition $\mathbb{1}_X = \sum_i f_i \mathbb{1}_{\lambda_i} g_i$, with $\lambda_i \in \Gamma$ for every i.
4. (Non-degeneracy axiom) The following matrix is invertible.

$$S = (S_{\lambda\mu})_{\lambda,\mu\in\Gamma} \ ,$$

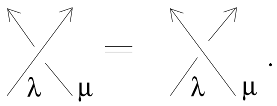
where $S_{\lambda\mu} \in \mathbf{k}$ is the endomorphism of the trivial object associated with the (λ, μ)-colored, 0-framed Hopf link with linking $+1$.

It follows that Γ is a representative set of isomorphism classes of simple objects. If we remove the last axiom, we get the definition of a *pre-modular* category.

If the category $\mathcal{C}$ has direct sums and a zero object, then it is pre-modular if and only if it has finitely many isomorphisms classes of simple objects, and any object splits as a direct sum of simple objects.

Note. Direct sums may be added formally. By using the domination property, one can show that any pre-modular category with direct sums is abelian.

Definition 3.1.2. An object λ of a pre-modular category A is called *transparent*, if for any object μ in A



Such an object is also called a central object. It is enough to have the above equality for any μ in a representative set of simple objects. Note that a category containing a nontrivial transparent simple object cannot be modular, simply because the row in the S-matrix corresponding to this transparent object is colinear to the row of the trivial one. In the next section we show that the absence of nontrivial transparent simple objects implies (under a mild assumption) that the category is modular.

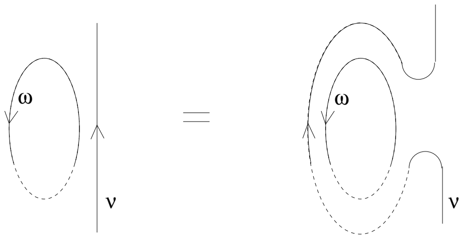
3.2 Properties of pre-modular categories

We will first give some general facts about pre-modular categories. Let $\mathcal{C}$ be a pre-modular category over the field $\mathbf{k}$ and let $\Gamma(\mathcal{C})$ be a representative set of isomorphism classes of its simple objects.

A morphism $f \in Hom_{\mathcal{C}}(X, Y)$ is called negligible if for any $g \in Hom_{\mathcal{C}}(Y, X)$ $\langle fg \rangle = 0$. We suppose that $\mathcal{C}$ has no nontrivial negligible morphisms (we quotient out by negligible morphisms if necessary).

We denote by Ω the *Kirby color*, i.e. $\Omega = \sum_{\lambda \in \Gamma(\mathcal{C})} \langle \lambda \rangle \lambda$. Here the notation $\langle \lambda \rangle$ is the dimension of λ.

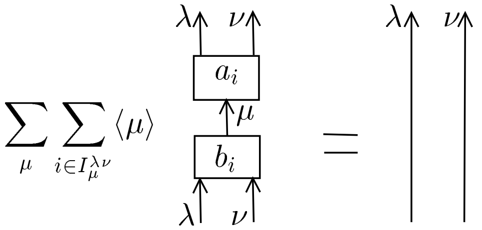
Proposition 3.2.1. *(Sliding property) Let ν be an object in the premodular category $\mathcal{C}$, then the two colored solid graphs depicted below define the same morphism in $End_{\mathcal{C}}(\nu)$.*



Here the dashed line represents a part of the closed component colored by Ω. This part can be knotted or linked with other components of a colored solid graph representing the morphism. Note that the morphism is unchanged if we reverse the orientation of this closed component.

Proof. For $\lambda, \mu, \nu \in \Gamma(\mathcal{C})$ we have a canonical isomorphism $Hom_{\mathcal{C}}(\lambda \otimes \nu, \mu) \simeq Hom_{\mathcal{C}}(\mu, \lambda \otimes \nu)^*$. (Note that the obvious pairing is non singular, because we have no negligible morphisms.) We denote by a_i, $i \in I_\mu^{\lambda\nu}$, a basis in $Hom_{\mathcal{C}}(\mu, \lambda \otimes \nu)$, and by b_i the dual basis in $Hom_{\mathcal{C}}(\lambda \otimes \nu, \mu)$. By the domination property, we get that a basis in $Hom_{\mathcal{C}}(\lambda \otimes \nu, \lambda \otimes \nu)$, is given by the $b_j a_i$, $i, j \in I_\mu^{\lambda\nu}$, $\mu \in \Gamma(\mathcal{C})$.

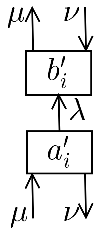
By writing the identity of $\lambda \otimes \nu$ in this basis, we get the following decomposition formula (*fusion* formula)



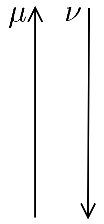
(3.1)

We denote by $a'_i \in Hom_{\mathcal{C}}(\mu \otimes \nu^*, \lambda)$ (resp. $b'_i \in Hom_{\mathcal{C}}(\lambda, \mu \otimes \nu^*)$) the element corresponding to a_i (resp. to b_i) via the isomorphism given by duality. We also have a fusion formula.

$$\sum_{\mu} \sum_{i\in I_{\mu}^{\lambda\nu}} \langle\mu\rangle$$



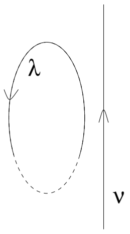
$$=$$



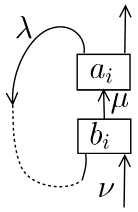
(3.2)

The calculations below establish the sliding property.

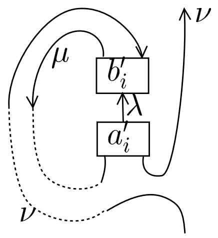
$$\sum_{\lambda\in\Gamma(\mathcal{C})} \langle\lambda\rangle$$



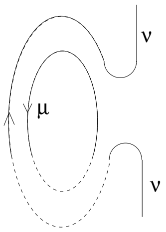
$$= \sum_{\lambda,\mu} \sum_{i\in I_{\mu}^{\lambda\nu}} \langle\lambda\rangle\langle\mu\rangle$$



$$= \sum_{\lambda,\mu} \sum_{i\in I_{\mu}^{\lambda\nu}} \langle\lambda\rangle\langle\mu\rangle$$



$$= \sum_{\mu\in\Gamma(\mathcal{C})} \langle\mu\rangle$$

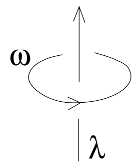


In the first and third equalities we use the fusion formula, the second equality holds by isotopy.

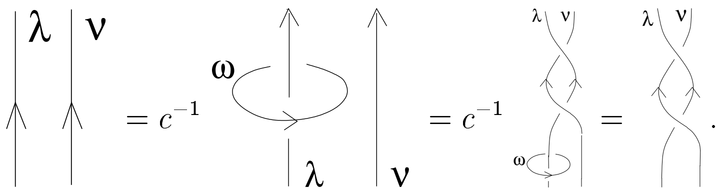
□

Lemma 3.2.2. *(Killing property) Suppose that $\langle\Omega\rangle$ is nonzero. Let $\lambda \in \Gamma(\mathcal{C})$, then the following morphism is nonzero in $\mathcal{C}$ if and only if λ is trans-*

parent.



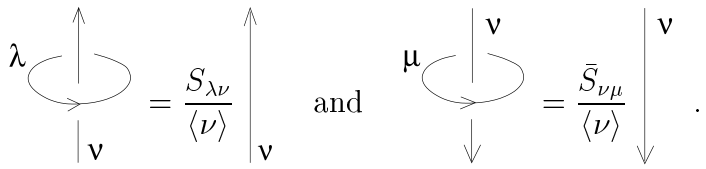
Proof. If λ is transparent, then this morphism is equal to $\langle\Omega\rangle \mathbb{1}_\lambda$, which is nonzero. Conversely, if this morphism is nonzero, it is equal to $c\mathbb{1}_\lambda$ for some $0 \neq c \in \mathbf{k}$. Then, for any $\nu \in \Gamma(\mathcal{C})$, we have



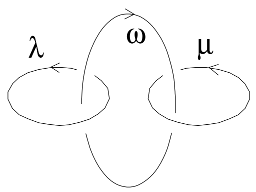
The second equality holds by the sliding lemma. □

Proposition 3.2.3. *A pre-modular category $\mathcal{C}$ with $\langle\Omega\rangle \neq 0$ which has no non-trivial transparent simple object is modular.*

Proof. We have to check the non-degeneracy axiom. Let us denote by $\bar{S}$ the matrix whose (λ, μ) entry is equal to the value of the 0-framed Hopf link with linking -1 and coloring of the components λ, μ. Then we have that



We deduce that the (λ, μ) entry of the matrix $S\bar{S}$ is equal to the invariant of the colored link depicted below.



By using (3.1) and the killing property we obtain the formula

$$S\bar{S} = \langle\Omega\rangle I \ ,$$

where I is the identity matrix, which proves the invertibility of the S matrix.

□

3.3 TQFT associated with a modular category

3.3.1 3-manifold invariant

For $\epsilon = \pm 1$, we denote by U_ϵ the unknot with framing epsilon. If $L = (L_1, \ldots, L_m)$ is a link in S^3, let $B_L = (b_{ij})$ be its linking matrix; for $i \neq j$, b_{ij} is the linking number $lk(L_i, L_j)$ and b_{ii} is the framing coefficient. We denote by b_+ (resp. b_-) the number of positive (resp. negative) eigenvalues of B_L. The signature $sign(L) = b_+ - b_-$ is the signature of the 4-manifold given by the surgery on L, and whose boundary is $S^3(L)$. From the sliding property we get the following theorem (here Ω is the Kirby color).

Theorem 3.3.1. *Let $\mathcal{C}$ be a premodular category over a field $\mathbf{k}$ such that $\langle U_1(\Omega)\rangle\langle U_{-1}(\Omega)\rangle$ is non zero. Then there exists an invariant $\theta_{\mathcal{C}}(M)$ of closed oriented 3-manifolds such that for any link $L = (L_1, \ldots, L_m)$ in S^3, one has*

$$\theta_{\mathcal{C}}(S^3(L)) = \frac{\langle L_1(\Omega), \ldots, L_m(\Omega)\rangle}{\langle U_1(\Omega)\rangle^{b_+}\langle U_{-1}(\Omega)\rangle^{b_-}}$$

This invariant can be extended to closed 3-manifolds with a $\mathcal{C}$-colored graph K inside by the formula

$$\theta_{\mathcal{C}}(S^3(L), K) = \frac{\langle L_1(\Omega), \ldots, L_m(\Omega), K\rangle}{\langle U_1(\Omega)\rangle^{b_+}\langle U_{-1}(\Omega)\rangle^{b_-}} \tag{3.3}$$

In view of the index 2 surgery axiom (S2), we note that a surgery on U_ϵ, $\epsilon = \pm 1$ does not change the 3-manifold. We would like to represent this surgery by inserting some $U_\epsilon(\omega)$. A renormalization $\omega = \zeta\Omega$ should be a good candidate for representing this surgery, but will contributes by a multiplicative coefficient $U_\epsilon(\omega)$; In most cases we cannot choose ζ in such a way that the coefficient is 1 for both values of ϵ. This is a manifestation of the so-called *framing anomaly.* We will not have the required surgery formula unless we consider an additionnal structure.

Suppose from now on that $\mathcal{C}$ is a modular category.

Exercise 3.3.1. Show that $\langle U_1(\Omega)\rangle\langle U_{-1}(\Omega)\rangle = \langle\Omega\rangle \neq 0$.

We extend $\mathbf{k}$ if necessary, and fix κ such that $\kappa^6 = \frac{\langle U_1(\Omega)\rangle}{\langle U_{-1}(\Omega)\rangle}$. Let $\mathcal{D} = \kappa^{-3}\langle U_1(\Omega)\rangle = \kappa^3\langle U_{-1}(\Omega)\rangle$; note that $\mathcal{D}^2 = \langle\Omega\rangle$.

We define the normalized invariant of connected closed 3-manifolds M equipped with complex framing α and colored graph K by

$$Z_{\mathcal{C}}(M, \alpha, K) = \mathcal{D}^{-1}\kappa^{\sigma(\alpha)}\theta_{\mathcal{C}}(M, K) \tag{3.4}$$

We extend Z to non connected closed 3-manifold mutiplicatively.

Proposition 3.3.2. *The invariant $Z_{\mathcal{C}}$ satisfies the surgery axioms (S0), (S1) and (S2) with $\omega = \mathcal{D}^{-1}\Omega$.*

3.3.2 The universal construction

As explained in the preceeding chapter, we use the invariant Z to construct a functor $\mathbf{V}$ from the category $\mathrm{Cob}(\mathcal{C})$ to the category of vector spaces.

Exercise 3.3.2. Show that:

1. $\mathbf{V}(S^2) = \mathbf{k}$.
2. If λ is a non trivial simple object, then $\mathbf{V}(S^2, \lambda) = 0$.
3. If λ, ν are simple objects, then $\mathbf{V}(S^2, \lambda, \nu^*)$ is equal to $\mathbf{k}$ if λ and ν are isomorphic and zero else.
4. If λ, μ, ν are simple objects, then $Hom_{\mathcal{C}}(\lambda \otimes \mu, \nu) = \mathbf{V}(S^2, \lambda, \mu, \nu^*)$ and $Hom_{\mathcal{C}}(\lambda, \mu \otimes \nu) = \mathbf{V}(S^2, \lambda, \mu^*, \nu^*)$.

Theorem 3.3.3. *If $\mathcal{C}$ is a modular category, then the functor $\mathbf{V}$ is multiplicative and for any $\mathcal{C}$-marked surface Σ the vector space $\mathbf{V}(\Sigma)$ has finite dimension.*

Corollary 3.3.4. *The functor $\mathbf{V}$ defines a Topological Quantum Field Theory.*

In the theorem above, multiplicativity is shown by using the surgery axioms. The description of the vector space $\mathbf{V}(\Sigma)$ reduces to those in exercise 3.3.2 by using the splitting formula below.

Lemma 3.3.5 (Splitting formula). *Let γ be a simple closed curve on a connected surface Σ such that Σ cut along γ has two connected components Σ' and Σ'', then one has an isomorphism*

$$\sum_{\lambda\in\Gamma} \mathbf{V}(\Sigma'_\lambda) \otimes \mathbf{V}({}_\lambda\Sigma'') \approx \mathbf{V}(\Sigma) \ ,$$

where Σ'_λ is obtained from Σ' by closing with a standard disc whose origin is colored by λ, and ${}_\lambda\Sigma''$ is obtained by closing Σ'' with a copy of the same disc with reversed orientation.

Exercise 3.3.3. Let γ be a simple closed curve on a connected surface Σ such that Σ cut along γ is a connected surface Σ'. Show that there exists an isomorphism

$$\sum_{\lambda\in\Gamma} \mathbf{V}({}_\lambda\Sigma'_\lambda) \approx \mathbf{V}(\Sigma) \ ,$$

where ${}_\lambda\Sigma'_\lambda$ is obtained by closing as above.

3.3.3 Verlinde formula

The Verlinde formula gives the dimension of the TQFT modules.

Proposition 3.3.6. *Let $\mathcal{C}$ be a modular category with Γ as a representative set of simple objects, then for a closed genus g surface Σ_g without marked point, the dimension of the TQFT module $\mathbf{V}(\Sigma_g)$ is*

$$d_\mathcal{C}(g) = \mathcal{D}^{2g-2} \sum_{\lambda\in\Gamma} \langle\lambda\rangle^{2-2g} \ .$$

The formula follows from the computation of the trace of the identity, which is equal to $Z(\Sigma_g \times S^1)$.

Chapter 4

EXAMPLES

4.1 The skein method

We give a sketchy description of the whole procedure. This was first applied to the Kauffman bracket skein theory, which corresponds to the A_2 case in the classification corresponding to the classical series. The whole A series is obtained by using the Homfly skein theory [8]. The skein theory associated with the 2 variables Kauffman polynomial corresponds to the BCD series [30, 7].

Step 1 Consider a multiplicative (for disjoint union) invariant of (framed) links in the sphere with values in some field $\mathbf{k}$.

Step 2 Define a $\mathbf{k}$-linear ribbon category by considering linear combinations of framed tangles, and quotienting out by negligible morphisms rel. to the invariant given in step 1. Show that the set of morphisms are finite dimensional vector spaces.

Step 3 Add the idempotents of the category as objects and show the (strong) semisimplicity.

Step 4 Study simple objects. If isomorphism classes of simple objects form a finite set, and there is no non trivial transparent object, then the category is modular. Else, by considering transparent simple objects apply a modularization criterion and procedure developed by Bruguières [11]; see also [23].

Exercise 4.1.1. Construct a TQFT based on the simplest skein theory in exercise 1.3.1 with q a N-th root of unity if N is odd and a $2N$-th root of unity if N is even (cf [22]).

Exercise 4.1.2. We consider the Homfly skein theory as defined in exercise 1.3.2.

1. Suppose that the parameters are generic and $\mathbf{k}$ is a field. Show that there exists a unique idempotent $f_n \in H_n$ such that for $i = 1, \ldots, n-1$, $\sigma_i f_n = f_n \sigma_i = as f_n$, and a unique idempotent $g_n \in H_n$ such that $i = 1, \ldots, n-1$, $\sigma_i g_n = g_n \sigma_i = -as^{-1} g_n$.

2. Construct a TQFT based on Homfly skein theory with parameters $s = a^{-2}$, $v = s^{-2}$, with a l-th root of unity if l is odd and a a $2l$-th root of unity if N is even.

4.2 The quantum group machinery

Let $\mathfrak{g}$ be a simple Lie algebra, then its universal enveloping algebra admits a deformation $U_q\mathfrak{g}$ which is called a quantum group. It is a Hopf algebra, and we are interested with the category of its finite dimensional modules. In the generic case, this category is a semisimple ribbon category, but has infinitely many simple objects, indexed as in the classical case by the fundamental weights. If q is a root of unity, the representation category is much more complicated, however one can obtain the following facts, for an appropriate version of the quantum group (see [29, 15, 5]).

1. It is a ribbon category.

2. There exists a finite set of simple objects (indexed by the affine Weyl alcove) which, after quotienting by negligible morphisms, is closed under decomposition of tensor products and generate a modular category.

The required decomposition of tensor products is established in [2, 3]. The construction was first done in [24]. See [30, 21, 14, 17] for further developments.

Bibliography

[1] A.K. Aiston, H.R. Morton, *Idempotents of Hecke algebras of type A*, Journal of Knot Theory and Ram. 7 (1998), 463–487.

[2] H. Andersen, *Tensor product of quantized tilting modules*, Comm. in Math. Physics 149 (1991), 149–159.

[3] H. Andersen, Paradowski J. *Fusion categories arising from semisimple Lie algebras*, Comm. in Math. Physics 169 (1995), 563–588.

[4] M. Atiyah, *Topological quantum field theories*, Publ. Math. IHES 68 (1989),175–186.

[5] B. Bakalov, A. Kirillov, *Lectures on Tensor Categories and Modular Functors*, Univ. Lect. Series Vol. 21, AMS.

[6] A. Beliakova, C. Blanchet, *Skein construction of idempotents in the Birman-Murakami-Wenzl algebras*, to appear in Math. Ann. , math.QA/0006143 .

[7] A. Beliakova, C. Blanchet, *Modular categories of types B,C and D*, to appear in Comm. Math. Helv. , math.QA/0006227 .

[8] C. Blanchet, *Hecke algebras, modular categories and* 3*-manifolds quantum invariants*, Topology, Vol. 39 (2000), 193–223.

[9] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, *Topological Quantum Field Theories derived from the Kauffman bracket*, Topology 34, No 4 (1995), 883–927.

[10] C. Blanchet, G. Masbaum, *Topological Quantum Field Theories for Surfaces with Spin Structure*, Duke Math. Journal, Vol. 82, No. 2, 229–267.

[11] A. Bruguières, *Catégories prémodulaires, modularisations et invariants des variétés de dimension 3*, Math. Ann., Vol. 316 (2000), No. 2, 215–236.

[12] G. Burde, H. Zieschang, *Knots*, De Gruyter Studies in Math. 5, 1985.

[13] C. Kassel, Quantum groups, Graduate Texts in Math. 155, Springer-Verlag.

[14] A. Kirillov, *On inner product in modular tensor categories*, Journal AMS 9 (1996), 1135–1170.

[15] C. Kassel, M. Rosso, V. Turaev, *Quantum groups and knots invariants*, Panoramas et Synthèses No 5, SMF (1997).

[16] L. H. Kauffman, Knots and Physics, World Scientific (1991).

[17] T. Le, *Quantum invariants of* 3*-manifolds: integrality, splitting, and perturbative expansion*, math.QA/0004099 .

[18] W. B. R. Lickorish, *An introduction to Knot Theory*, Springer-Verlag (1997).

[19] S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag (1971).

[20] J. Milnor, J. Stasheff, *Characteristic classes*, Annals of Math. Studies 76, Princeton Univ. Press.

[21] G. Masbaum, H. Wenzl, *Integral modular categories and integrality of quantum invariants at root of prime order*, Journal fur die Reine Ang. Math. 505 (1998) 209–235.

[22] H. Murakami, T. Othsuki, and M. Okada, *Invariants of* 3*-manifolds derived from linking matrices of framed links*, Osaka J. Math. 29 (1992), 545-572.

[23] M. Müger, *Galois theory for braided tensor categories and the modular closure*, Advances in Math. 150 (2000), 151–201.

[24] N. Reshetikhin, V. Turaev, *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Inv. Math. 103 (1991), 547–597.

[25] D. Rolfsen, *Knots and Links*, Publish and Perish (1976).

[26] R. Stong, *Notes on cobordism theory*, Princeton Univ. Press (1968).

[27] V. Turaev, *The Yang-Baxter equation and invariants of links.*, Invent. Math. 92 (1988), 527–533.

[28] V. Turaev, *Operator invariants of tangles and R-matrices*, Math. USSR Izvestia Vol. 35 (1990), 411–443

[29] V. Turaev, *Quantum invariants of knots and* 3*-manifolds*, De Gruyter Studies in Math. 18, 1994.

[30] V. Turaev, H. Wenzl, *Quantum invariants of* 3*-manifolds associated with classical simple Lie algebras*, Int. J. of Math., Vol. 4, No. 2 (1993) 323–358.

[31] V. Turaev, H. Wenzl, *Semisimple and modular categories from link invariants*, Math. Ann., Vol. 309 (1997), 411–461.

[32] E. Witten, *Topological Quantum Field Theory*, Comm. Math. Phys. 117 (1988), 353–386.