NONCOMMUTATIVE GEOMETRY AND ABSTRACT INTEGRATION THEORY

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We give an introductory survey of some recent developments in type II non commutative index theory. The Dixmier trace is extended to the case of type II_∞ spectral triples and a local formula for the Hochschild class of the Chern-Connes character is given. We also extend the Calderon formula to the equivariant and type II_∞ case and we deduce an equivariant von Neumann cyclic cohomology pairing which implements the index map associated with any equivariant von Neumann spectral triple. A brief survey of the classical Atiyah-Singer index theorem is also given.

Introduction

The Atiyah-Singer index theorem stands at the junction of many different mathematical trends and is at the origin of many recent developments. It is also involved in the understanding of innumerable results in some areas of physics. This theorem is, to the opinion of the author, certainly one of the deepest discoveries of the last fourty years.

Given a smooth closed manifold M, an elliptic differential operator D on M has finite dimensional kernel and finite dimensional cokernel. Therefore, the Fredholm index of D can be defined by

$$\mathrm{Ind}(D) = \dim(\mathrm{Ker}(D)) - \dim(\mathrm{Coker}(D)).$$

In the early fifties, Gelfand stated the problem of computation of the index of elliptic operators by topological formulae, i.e. involving topological data of the operator and of the underlying manifold. The solution of this problem has been obtained in many steps but the general result was achieved by Atiyah and Singer in 1963-65 using a new powerful tool: topological K-theory. Beyond the index formula then obtained, the methods developed by Atiyah and Singer cleared up the index problem and produced fundamental K-theory constructions.

For spin manifolds for instance, K-theory techniques enable to reduce the index problem to the computation of the index of twisted Dirac operators. In this case and for any hermitian vector bundle E over an even spin closed

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manifold M, we get the famous formula:

$$\mathrm{Ind}(\partial_E^+) =< \mathrm{Ch}(E)\hat{A}(M), [M] >,$$

where ∂_E^+ is the positive part of the self-adjoint Dirac operator twisted by the bundle E, $\mathrm{Ch}(E)$ is the Chern character of E, $\hat{A}(M)$ is the $\hat{A}$-genus of the tangent bundle TM of M and where $[M]$ is the fundamental class of M. So by Poincaré duality, this formula becomes:

$$\mathrm{Ind}(\partial_E^+) =< \mathrm{Ch}(E), \hat{A}(M) \sqcap [M] >,$$

where $\hat{A}(M) \sqcap [M] \in \mathrm{H}_*(M, \mathbb{Q})$ is the Poincaré dual of $\hat{A}(M)$. Viewing this index as a pairing between E and the operator ∂ [2], we can define the Chern character of the Dirac operator as the homology class

$$\mathrm{Ch}(\partial) := \hat{A}(M) \sqcap [M].$$

The index formula can then be reinterpreted as a homological pairing:

$$\mathrm{Ind}(\partial_E^+) =< \mathrm{Ch}(E), \mathrm{Ch}(\partial) > . \tag{0.1}$$

In other words, the Atiyah-Singer index theorem computes the so called Atiyah map:

$$K^0(M) \to \mathbb{Z} \text{ defined by } E \mapsto \mathrm{Ind}(\partial_E^+).$$

Formula (0.1) is the starting point of the Non Commutative Geometry point of view on index theory. More precisely, A. Connes has succeeded in giving an appropriate framework for index problems, when the algebra $C^\infty(M)$ of smooth functions on the manifold M is replaced by a, non commutative in general, *smooth* algebra $\mathcal{A}$. The motivating examples came from the study of singular spaces such as the space of leaves of a foliated manifold or the space of irreducible representations of a given discrete group, but many other important examples arose later from the physics of solids or quantum field theory (QFT). The concept of Dirac operator led A. Connes to his formalism of spectral triples. In the presence of any such spectral triple an index problem can be stated and the spectacular Connes-Moscovici local index theorem is a complete solution of this general index problem.

The first interesting but highly non trivial example, and that cannot be reached by classical index theory techniques, encodes the Diff-equivariant index theory. It was defined by Connes and Moscovici by using a *Diff-equivariant signature operator*. The identification of the local terms appearing in the Connes-Moscovici index formula for such triple led A. Connes and H. Moscovici to define an appropriate cyclic homology for Hopf algebras [31] where the involved characteristic numbers could live. But the story has just begun and a large program remains to be carried out even in simpler examples.

On the other hand, and since the appearance of the Atiyah-Singer index theorem, many generalizations have been obtained [1, 11, 23, 25, 26, 49, 62], encompassing many new geometric situations. Among them, some constructions involve non Fredholm operators and use in a decisive way an extra ingredient: von Neumann algebras and the Murray-von Neumann dimension theory. While the index of elliptic operators on non compact manifolds for instance, does not make much sense, some useful assumptions enable to affiliate them to appropriate type II von Neumann algebras and to give a Murray-von Neumann sense to their index. This is true for instance in the examples treated in [1, 23, 62].

We have thus taken the opportunity of the present notes to explain how von Neumann algebras can fit into the Non Commutative Geometry framework and how the formalism of spectral triples can be generalized to deal with the type II index and also with group actions. All the examples listed above become corollaries of a unifying von Neumann approach to the Connes-Moscovici index theorem. The purely non commutative consequences of such results are also interesting directions for the future.

The present notes summarize a more detailed course adressed to beginners and we have devoted the whole first chapter to an overview of the fundamental Atiyah-Singer index theorem. We point out on the other hand that some of the results of the second chapter rely on a larger investigation carried out in collaboration with T. Fack. We have chosen to expound the case of foliated flat bundles as an experimental application. In particular, we show how the Atiyah covering index theorem can be generalized to flat bundles and even to general Galois coverings of foliations.

This paper is divided in two chapters. The first one is adressed to beginners in index theory and is a tentative introduction to the now classical Atiyah-Singer index theorem. The second chapter on the other hand deals with the type II index theory in Noncommutative geometry. Experts in the domain can therefore skip the first chapter.

The contents of this paper are more precisely as follows: Section 1 summarizes the index theory for elliptic complexes on smooth closed manifolds and gathers some background material. Section 2 is devoted to the index theorem and some of its corollaries are treated in Section 3. In Section 4, we introduce Dixmier traces and residues of zeta functions in type II_∞ von Neumann algebras. Section 5 summarizes the index theory with respect to a given von Neumann algebra and explains the constructions in an instructive example. In the last Section, we give some definitions and examples of von

Neumann spectral triples with the statement of the associated index problem. The equivariant case is also treated and an equivariant Chern-Connes character is proposed.

We assume as known some background material as topological K-theory, characteristic classes, von Neumann algebras and also some classical properties of pseudodifferential operators on manifolds. However, for the convenience of the reader and because these notes are intended to be understood by a large public, we have tried as much as possible to recall some of the material or to give the appropriate references.

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Chapter I. Review of the classical index theory

Index theory has gained ground during the last thirty five years. However, we believe that the original version of the Atiyah-Singer index theorem remains the best way to introduce this active research subject to neophytes. In this first chapter, we shall review some of the basic tools needed to understand correctly the Atiyah-Singer index theorem. The detailed proof is not included in this short review and can be consulted in the literature.

1 Some preliminaries and examples

We begin this section by an overview of the index theorem for Toeplitz operators on the circle. Then we briefly recall the definitions and properties of pseudodifferential operators and state the Atiyah-Singer index theorem with a sketch of proof and some applications.

1.1 The index theorem for Toeplitz operators

Toeplitz operators on the circle are an interesting toy model and we begin by describing the meaning of the index theorem for them. We also explain how they can be generalized to higher dimensional odd manifolds.

We denote by $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ the unit circle with its usual normalized Haar measure $d\theta/2\pi$. The Fourier transform enables to identify the Hilbert space $L^2(\mathbb{S}^1)$ of measurable square integrable functions on $\mathbb{S}^1$ (identified with $L^2(0, 2\pi)$) with the space $l^2(\mathbb{Z})$ of square summable sequences of complex numbers. This identification corresponds to the decomposition of L^2-functions with respect to the orthonormal basis $(e_n)_{n\in\mathbb{Z}}$ defined by $e_n(\theta) = e^{in\theta}$. More precisely, we set:

$$\forall f \in L^2(\mathbb{S}^1), \forall n \in \mathbb{Z}, c_n(f) := \int_{(0,2\pi)} f(\theta)e^{-in\theta} d\theta/2\pi,$$

to obtain a sequence $(c_n(f))_{n\in\mathbb{Z}}$ which belongs to $l^2(\mathbb{Z})$. The Hilbert subspace $l^2(\mathbb{N})$ of $l^2(\mathbb{Z})$ corresponds under this Fourier identification to a Hilbert subspace denoted $H^2(\mathbb{S}^1)$, of $L^2(\mathbb{S}^1)$. The Hilbert space $H^2(\mathbb{S}^1)$ is called the Hardy space and corresponds in fact to L^2-functions on the circle which admit a holomorphic extension to the unit disk. The projection $p : L^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1)$ is usually called the Szego projection. It corresponds under the Fourier identification to the quasi-trivial projection $l^2(\mathbb{Z}) \to l^2(\mathbb{N})$ and is implemented by the Poisson kernel. Let ϕ be any continuous (for simplicity) function on the circle $\mathbb{S}^1$, then pointwise multiplication by ϕ determines a bounded operator

M_ϕ on $L^2(\mathbb{S}^1)$. Moreover the map M which assigns to each ϕ the operator M_ϕ is an isometric representation of the commutative C^*-algebra $C(\mathbb{S}^1)$ of continuous functions on $\mathbb{S}^1$, in the Hilbert space $L^2(\mathbb{S}^1) \cong l^2(\mathbb{Z})$.

Lemma 1. *For any continuous function ϕ on $\mathbb{S}^1$, the commutator $[p, M_\phi]$ is a compact operator on $L^2(\mathbb{S}^1)$.*

Proof. The map $\phi \to M_\phi$ is a continuous homomorphism (in fact an isometry) and the ideal of compact operators in $L^2(\mathbb{S}^1)$ is norm-closed. Hence using the Stone-Weierstrass theorem, it remains to show that $[p, M_\phi]$ is compact whenever ϕ is a trigonometric polynomial. So we can even assume that $\phi = e_m$, with $m \in \mathbb{Z}$. But then we can easily see that $[p, M_{e_m}]$ coincides with the projection onto the finite dimensional space generated by the family $(e_j)_{Min(-m,o)<j\leq Max(-m,0)}$ and so is compact. $\square$

Definition 1. For any continuous function ϕ on $\mathbb{S}^1$, we define the Toeplitz operator T_ϕ associated with ϕ as the composite operator $p \circ M_\phi$ acting on $H^2(\mathbb{S}^1)$. In other words:

$$T_\phi : H^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1), T_\phi(f) = p(\phi f), \quad \forall f \in H^2(\mathbb{S}^1).$$

Toeplitz operators play a fundamental role in operator theory and the invariant subspace Hilbert's problem. See for instance [34]. With respect to the canonical basis $(e_n)_{n\geq 0}$ of $H^2(\mathbb{S}^1)$, such Toeplitz operators are infinite matrices with entries given by:

$$(T_\phi)_{i,j} = c_{i-j}(\phi).$$

So these matrices have constant diagonals.

Examples. (1) If $c_n(\phi) = 0, \forall n < 0$ then M_ϕ preserves the Hardy Hilbert space $H^2(\mathbb{S}^1)$ and thus $T_\phi = M_\phi|_{H^2(\mathbb{S}^1)}$.

(2) If $\phi := e_{-n}$ with $n \in \mathbb{N}^*$, then for any $m \in \mathbb{Z}$ we have:

$$T_\phi(e_m) = p(e_{m-n}).$$

Hence we obtain:

$$T_\phi(\sum_{m\geq 0} a_m e_m) = \sum_{m\geq 0} a_{m+n} e_m.$$

We can see in the example (2) that the kernel of $T_{e_{-n}}$ is finite dimensional and is the subspace of $H^2(\mathbb{S}^1)$ generated by $(e_0, \cdots, e_{n-1})$, while $T_{e_{-n}}$ is surjective. Therefore $T_{e_{-n}}$ is a very simple example of a Fredholm operator on the Hilbert space $H^2(\mathbb{S}^1)$. Its Fredholm index (say $\dim(\mathrm{Ker}) - \dim(\mathrm{Coker})$) is exactly the integer n. But there is a "topological" way to recover the integer n from $\phi = e_{-n}$, since it is exactly minus the winding number $w(e_{-n})$ of the non-vanishing smooth function e_{-n} on the circle. Recall that for any smooth nonvanishing function ϕ on the circle $\mathbb{S}^1$, the winding number $w(\phi)$ of ϕ is the Cauchy integral:

$$w(\phi) = \frac{1}{2i\pi}\int_0^{2\pi} \frac{\phi'(t)}{\phi(t)} dt.$$

The above relation between the winding number of e_{-n} and the index of the Toeplitz operator with symbol e_{-n} is a general phenomenon and we have:
Proposition 1. *Let ϕ be a smooth (for simplicity) non vanishing function on $\mathbb{S}^1$. Then the operator T_ϕ is a Fredholm operator on $H^2(\mathbb{S}^1)$. The index of the Fredholm operator T_ϕ can be computed by the formula:*

$$\mathrm{Ind}(T_\phi) = -w(\phi),$$

where $w(\phi)$ is the winding number of ϕ.

This very simple proposition is in fact the Atiyah-Singer index theorem on the simplest manifold $\mathbb{S}^1$. The idea of its proof is already in the spirit of the general proof.

Proof. We have on $L^2(\mathbb{S}^1)$:

$$p \circ M_{1/\phi} \circ p \circ M_\phi \circ p - p = p \circ [M_{1/\phi}, p] \circ M_\phi \circ p.$$

But the operator $[M_{1/\phi}, p]$ is a compact operator on $L^2(\mathbb{S}^1)$ by Lemma 1. Therefore the operator $p \circ [M_{1/\phi}, p] \circ M_\phi$ is a compact operator on $L^2(\mathbb{S}^1)$ which preserves $H^2(\mathbb{S}^1)$. Hence it is a compact operator on $H^2(\mathbb{S}^1)$. This shows that $T_{1/\phi}$ is a left inverse of T_ϕ modulo compact operators. It is easy to see that it is also a right inverse of T_ϕ modulo compact operators.

Recall that the map $\phi \mapsto w(\phi)$ classifies the homotopy class of ϕ so that two such functions that have the same winding number are necessarily homotopic. Let now ϕ be a smooth non-vanishing function on the circle and set $m = w(\phi)$. Then ϕ is homotopic to the function e_m by the above property of the winding number. Therefore, the Toeplitz operator T_ϕ is homotopic to the Toeplitz operator T_{e_m} inside Fredholm operators. Hence the Fredholm index of T_ϕ is equal to the Fredholm index of T_{e_m}. This last index is easily seen to be equal to $-m$ as we have already observed. $\square$

Hence we are able to compute the (analytic) index of the operator T_ϕ as an integral over the manifold $\mathbb{S}^1$ of a differential form given by $\frac{1}{2i\pi}\phi^{-1}d\phi$. To prove this index theorem, we have used that the space of connected components of $C^\infty(\mathbb{S}^1, \mathbb{C}^*)$ is isomorphic to $\mathbb{Z}$. This is a very simple expression of Bott periodicity. In the general proof of the Atiyah-Singer index theorem, Bott periodicity is also fundamental as we shall see.

More generally now, if u is an invertible matrix of smooth functions on $\mathbb{S}^1$, say $u \in GL_N(C^\infty(\mathbb{S}^1))$, then we can also consider the matrix of 1-forms $\frac{1}{2i\pi}u^{-1}du$ and it is also well known that the integral of the trace of this matrix over $\mathbb{S}^1$ is always an integer (see Proposition 2 below) also called the winding number of u, i.e.

$$w(u) := \frac{1}{2i\pi}\int_{\mathbb{S}^1} \mathrm{Tr}(u^{-1}du) \in \mathbb{Z}.$$

To recover this integer as a Fredholm index, one simply considers a generalization of the Toeplitz construction. Hence, we consider the operator M_u associated with u corresponding to pointwise matrix multiplication by u in $L^2(\mathbb{S}^1)^N$ and define the Toeplitz operator

$$T_u : H^2(\mathbb{S}^1)^N \to H^2(\mathbb{S}^1)^N, T_u := (p \otimes 1_N) \circ M_u$$

Again because u is invertible, the same arguments as in Proposition 1 show that T_u is a Fredholm operator in the Hilbert space $H^2(\mathbb{S}^1)^N$. Bott periodicity now states that the group of connected components of $C^\infty(\mathbb{S}^1, GL_N(\mathbb{C}))$ is isomorphic to $\mathbb{Z}$. For the sake of simplicity, this result is admited in the following proposition:

Proposition 2.

(1) The winding number $w(u)$ is an integer and the map $u \to w(u)$ is a group homomorphism;

(2) The map $u \mapsto \mathrm{Ind}(T_u)$ is a continuous group homomorphism;

(3) We have the following index formula:

$$\mathrm{Ind}(T_u) = -w(u).$$

Proof. The matrix u defines a smooth loop in $GL_N(\mathbb{C})$ and the map $u \mapsto \mathrm{Ind}(T_u)$ is again a continuous function which only depends on the homotopy class of u. On the other hand, w is also a continuous function of u for the C^1 topology, and it is integer valued. The justification is classical: Set for $s \in [0, 2\pi]$, $F(s) = \int_0^s u^{-1}(t)u'(t)dt$, then the derivative of the $M_N(\mathbb{C})$-valued smooth function $s \mapsto u(s)e^{-F(s)}$ is everywhere zero. Thus

$$I_N = u(2\pi)^{-1}u(0) = e^{-F(2\pi)}.$$

and hence

$$e^{\mathrm{Tr}(F(2\pi))} = \det(e^{F(2\pi)}) = 1.$$

Therefore $\exists k \in \mathbb{Z}$ such that $\mathrm{Tr}(F(2\pi)) = 2ik\pi$. This shows that

$$w(u) = \frac{1}{2i\pi}\mathrm{Tr}(F(2\pi)) \in \mathbb{Z}.$$

So we have two locally constant integer valued functions w and $u \to \mathrm{Ind}(T_u)$, which are in fact group homomorphisms. This can be justified for instance as follows:

Let us consider $u, v \in C^\infty(\mathbb{S}^1, GL_N(\mathbb{C}))$ and set $x := uv \oplus 1_N \in C^\infty(\mathbb{S}^1, GL_{2N}(\mathbb{C}))$. Then the Toeplitz operator associated with x is exactly the direct sum of the Toeplitz operator associated with uv and the identity operator. Hence $\mathrm{Ind}(T_{uv}) = \mathrm{Ind}(T_x)$. In the same way, the winding number of x can be computed as follows:

$$w(x) = w(uv) + w(1_N) = w(uv).$$

But a classical argument shows that the element x is homotopic to the element $u \oplus v$. In fact we have:

$$\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$$

and we notice that:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$$

which enables to construct the allowed homotopy. Hence the two functionals are homomorphisms from the multiplicative group $C^\infty(\mathbb{S}^1, GL_N(\mathbb{C}))$ to the group $\mathbb{Z}$.

Now let us admit that for any $N \geq 1$, the homotopy classes of smooth paths in $GL_N(\mathbb{C})$ form a free abelian group isomorphic to $\mathbb{Z}$, we deduce that w is an isomorphism for any N. This is justified by the fact that w is clearly surjective. Hence if u satisfies $w(u) = m \in \mathbb{Z}$, then taking $v = diag(e_{m_1}, \cdots, e_{m_N})$ with $\sum m_i = m$, we get $w(u) = w(v)$ and hence u is homotopic to v in $C^\infty(\mathbb{S}^1, GL_N(\mathbb{C}))$. Therefore and since the index is a continuous integer valued functional, $\mathrm{Ind}(T_u) = \mathrm{Ind}(T_v)$. Finally, it remains to compute the Fredholm index of T_v, but this is easily seen to be equal to $-\sum_i m_i = -m$. $\square$

We have seen in the above proof a first appearance of K-theory. In fact, if u belongs to $C^\infty(\mathbb{S}^1, GL_N(\mathbb{C}))$ then its index (or winding number) is unchanged when u is replaced by $u \oplus 1_{N'}$ for any N'. Therefore the functionals under consideration are in fact defined on the inductive limit

$$GL_\infty(C^\infty(\mathbb{S}^1)) = C^\infty(\mathbb{S}^1, GL_\infty(\mathbb{C})) := \lim_{\to} C^\infty(\mathbb{S}^1, GL_N(\mathbb{C})),$$

with respect to the "adding 1 on diagonals" inductive system. Furthermore since these two functionals only depend on the connected components of

$C^\infty(\mathbb{S}^1, GL_\infty(\mathbb{C}))$, we deduce that these two functionals are in fact defined on the group

$$K_1(C^\infty(\mathbb{S}^1)) := \pi_0(GL_\infty(C^\infty(\mathbb{S}^1))) := \frac{GL_\infty(C^\infty(\mathbb{S}^1))}{GL_\infty^{(0)}(C^\infty(\mathbb{S}^1))}$$

where $GL_\infty^{(0)}(C^\infty(\mathbb{S}^1))$ is the connected component of the identity. So the index theorem is the claim that the following two isomorphisms are equal

$$-w, \mathrm{Ind} : K_1(C^\infty(\mathbb{S}^1)) \to \mathbb{Z}.$$

What we have precisely used in the proof is that $K_1(C^\infty(\mathbb{S}^1)) \simeq \mathbb{Z}$. This is again one of the different versions of Bott periodicity.

Let us now anticipate what will be treated in detail in the next sections. Recall that the Dirac operator on $\mathbb{S}^1$ is the usual self-adjoint operator $D = -i\frac{d}{d\theta}$. It corresponds under the Fourier decomposition to the diagonal operator

$$D(e_n) = ne_n, \forall n \in \mathbb{Z}.$$

Let now F be the Hilbert transform on $\mathbb{S}^1$, say the bounded symmetry defined by:

$$F(e_n) = sign(n)\ e_n \text{ if } n \neq 0 \text{ and } F(e_0) = e_0.$$

Then the relation $F|D| = D$ holds and F is equal to the sign of the operator D up to the rank one operator which is the projection onto the kernel of D, i.e. onto the complex line $\mathbb{C}e_0$. The Szego projection p can also be reinterpreted as the projection onto the eigenspace associated with the positive eigenvalues of the Dirac operator D and is equal to $(1+F)/2$.

Thus, given an invertible element $u \in C^\infty(\mathbb{S}^1, GL_N(\mathbb{C}))$, the Toeplitz operator can be defined as the composite operator $T_u = [\frac{1+F}{2} \otimes 1_N] \circ u \circ [\frac{1+F}{2} \otimes 1_N]$. In fact, the operator $p = \frac{1+F}{2}$ is a pseudodifferential operator with symbol given on the cosphere bundle of $\mathbb{S}^1$ by (see definitions in the next subsection):

$$\sigma(p)(z,1) = 1 \text{ and } \sigma(p)(z,-1) = 0, \quad \forall z \in \mathbb{S}^1.$$

Now extending the Toeplitz operator T_u to $L^2(\mathbb{S}^1)$ by setting $A_u := T_u + (1-p) \otimes I_N$ we can easily see that the principal symbol of the pseudodifferential operator A_u is exactly pointwise multiplication by u on $\mathbb{S}^1 \times \{1\}$ and is the identity on $\mathbb{S}^1 \times \{-1\}$. The Chern character of this principal symbol is therefore given by:

$$\mathrm{Ch}(\sigma(A_u)) = -\,\mathrm{Tr}(\frac{u^{-1}du}{2i\pi}),$$

see for instance [9]. Thus the above index formula can be reinterpreted as the homological pairing:

$$\mathrm{Ind}(A_u) =< \mathrm{Ch}(\sigma(A_u)), [\mathbb{S}^1] > .$$

It is the goal of the Atiyah-Singer index theorem to produce such formula on each smooth closed manifold computing the analytic index of an elliptic pseudodifferential operator as the integral over the manifold of a local topological data of the operator and of the manifold.

1.2 Some properties of pseudodifferential operators

We shall very briefly recall some properties of classical pseudodifferential operators. For more details we refer to the lecture notes of S. Paycha [59] in the same volume. Let X be a smooth closed manifold of dimension n. Let E^0, E^1 be two smooth hermitian vector bundles over X and denote by $\mathcal{P}^m(X; E^0, E^1)$ the space of classical pseudodifferential operators of order m over X acting from the smooth sections of E^0 to the smooth sections of E^1. The formula for the change of coordinates shows that the top component of the locally defined symbol of P yields a global section $\sigma(P) := p_m$ of the vector bundle $\mathrm{Hom}(E^0, E^1)$ lifted to the cotangent bundle of X. This section is called the principal symbol of P. More precisely, if $\pi : T^*X \setminus X \to X$ is the projection, then $\sigma(P) \in C^\infty(T^*X \setminus X, \mathrm{Hom}(\pi^*E^0, \pi^*E^1))$, and we get a linear morphism:

$$\sigma : \mathcal{P}^m(X; E^0, E^1) \to C^\infty(T^*X \setminus X, \mathrm{Hom}(\pi^*E^0, \pi^*E^1)).$$

The range of this symbol map lies in the m-positively homogeneous sections and therefore is completely determined by its restriction to the cosphere bundle $\mathbb{S}^*X$, i.e. the quotient bundle of $T^*X \setminus X$ under the radial action of the multiplicative group $\mathbb{R}_+^*$. In other words, m-homogeneity is carried by a line bundle L_m and we have

$$\sigma : \mathcal{P}^m(X; E^0, E^1) \to C^\infty(\mathbb{S}^*X, L_m \otimes \mathrm{Hom}(\pi^*E^0, \pi^*E^1)).$$

We denote by $H^s(X, E^i)$ the s-Sobolev Hilbert space of (X, E^i). The pseudodifferential operator P of order m extends for any $s \in \mathbb{R}$ to a bounded operator

$$P_s : H^s(X, E^0) \to H^{s-m}(X, E^1).$$

Moreover the Rellich theorem states that for $t < s$, the inclusion $H^s(X, E^i) \hookrightarrow H^t(X, E^i)$ is a compact operator. Therefore if $m < 0$ then P induces a bounded operator P_0 between the Hilbert spaces of L^2-sections which is in addition compact. To get similar results in the case of non compact manifolds, one needs to use local Sobolev spaces, this is described for instance in [63]. The proof of these classical assertions can be found for instance in [42]. Moreover, we have a more precise relation between the order $m < 0$ and the ideal of compact operators where $P = P_0$ lies.

Lemma 2. *Let E be a hermitian vector bundle over the closed Riemannian n-manifold X. If $P \in \mathcal{P}^m(X; E)$ with $m \in \mathbb{Z}^*_-$, then the extended bounded operator $P : L^2(X, E) \to L^2(X, E)$ satisfies:*

(1) If $m < -n$ then P belongs to the ideal $L^1(L^2(X, E))$ of trace-class operators;

(2) If $-n \leq m \leq -1$ then P belongs to the Dixmier ideal $L^{-n/m,\infty}(L^2(X, E))$. In particular, if $P \in \mathcal{P}^{-n}(X; E)$ then P is in the Dixmier ideal $L^{1,\infty}(L^2(X, E))$.

The Dixmier ideal $L^{1,\infty}(L^2(X, E))$ is the space of compact operators T whose singular numbers

$$\|T\| = \mu_0(T) \geq \mu_1(T) \geq \cdots \geq \mu_k(T) \geq \cdots$$

satisfy $\sum_{k=0}^{N} \mu_k(T) = O(\mathrm{Log}(N + 1))$, see [29]. The Dixmier ideal $L^{r,\infty}(L^2(X, E))$ for $r \geq 1$ is here the space of compact operators T such that $(T^*T)^{r/2}$ belongs to $L^{1,\infty}(L^2(X, E))$, where T^* is the adjoint operator of T. For a positive operator $T \in L^{1,\infty}(H)$, the Dixmier trace $\mathrm{Tr}_\omega(T)$ of T was defined [29] as a renormalized limit:

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\mathrm{Log}(N)} \sum_{n=0}^{N} \mu_n(T),$$

where $a = (a_n)_{n\geq 0} \mapsto \omega(a) = \lim_\omega(a_n)$ denotes a state on $l^\infty(\mathbb{N})$ which vanishes on $c_0(\mathbb{N})$ and satisfies $\lim_\omega a_n = \lim_\omega a_{2n}$. The above lemma shows that the Dixmier trace of any order $-n$ pseudodifferential operator makes sense.

We can be more precise in the above lemma: If μ_k is the decreasing sequence of eigenvalues of the compact operator $|P|$, then we have [62]:

$$\mu_k \sim k^{m/n}.$$

Assume that P is a pseudodifferential operator on the closed n-manifold X with coefficients in the vector bundle E as before but such that the order of P is exactly $-n$. Then A. Connes has proved the following formula for the Dixmier trace $\mathrm{Tr}_\omega(P)$ of P [24]:

$$\mathrm{Tr}_\omega(P) = \frac{1}{n(2\pi)^n} \int_{\mathbb{S}^*X} \mathrm{Tr}(\sigma(P)(x, \xi)) dx d\xi = \frac{1}{n} \times \lim_{z\to 0^+} x\, \mathrm{Tr}(P \circ \Delta^{-z}), \quad (1.1)$$

where Δ is a Laplace operator and $\sigma(P)$ is the principal symbol of P, so it is homogeneous of degree $-n$. This integral is taken with respect to the Liouville measure on $\mathbb{S}^*X$, i.e. the interior product of the symplectic measure on the cotangent bundle T^*X by the outgoing radial vector field $\mathcal{R}$ on each fibre of $\mathbb{S}^*X$. Hence $\mathrm{Tr}_\omega(P)$ is independent of the mean ω used to define it and is given by the integral over the cosphere bundle of a local residue. The Dixmier trace is therefore a local data of the operator P.

Denoting by $\mathcal{P}^\infty(X,E)$ the set of classical pseudodifferential operators of integral order on the smooth closed manifold X with coefficients in E, we obtain a filtered algebra for the composition of operators [42]. In addition, the formal adjoint of a pseudodifferential operator is also pseudodifferential. We denote by $\mathcal{P}^{-\infty}(X,E)$ the ideal of regularizing operators defined by:

$$\mathcal{P}^{-\infty}(X,E) := \cap_{m\in\mathbb{Z}}\mathcal{P}^m(X,E).$$

Hence we have an exact sequence of algebras:

$$0 \to \mathcal{P}^{-\infty}(X,E) \to \mathcal{P}^\infty(X,E) \to \mathcal{A}(X;E) \to 0,$$

The quotient algebra $\mathcal{A}(X;E)$ is called the algebra of complete symbols on X with coefficients in E.

To each pseudodifferential operator P we can associate, using the Schwartz theorem, a distributional kernel $k(x,y) \in \mathrm{Hom}(E_y, E_x)$ so that for any $u \in C^\infty(X,E)$,

$$P(u)(x) = \int_X k(x,y)u(y)dy.$$

This kernel has the additional property that it is smooth outside the diagonal and is more and more regular as the order of P decreases. For instance when the order is $< -n$, this kernel is continuous and the trace of P can then be computed by the formula:

$$\mathrm{Tr}(P) = \int_X \mathrm{Tr}(k(x,x))dx.$$

Operators in $\mathcal{P}^{-\infty}(X,E)$ have smooth Schwartz kernel and thus are regularizing operators, i.e. They send Sobolev distributional sections to smooth sections. Up to regularizing operators, we can always assume that the Schwartz kernel of the pseudodifferential operator under study is supported in a given neighborhood of the diagonal.

Recall that a trace is by definition a cyclic 0-cocycle [25]. As we have observed, any trace on the algebra $\mathcal{P}^\infty(X,E)$ which is trivial on the ideal $\mathcal{P}^{-\infty}(X,E)$ furnishes a local tool for index theory. Moreover, any cyclic cocycle on $\mathcal{P}^\infty(X,E)$ which vanishes on $\mathcal{P}^{-\infty}(X,E)$ also yields a local ingredient.

These remarks show the relevance of the computation of the cyclic cohomology of the quotient algebra $\mathcal{A}(X,E)$. For the basic definitions about cyclic (co)homologies, we refer for instance to [25, 29, 48, 53].

Proposition 3. *[21] Let $\mathcal{A}(X,E)$ be the algebra of complete symbols on the smooth closed manifold X with coefficients in the vector bundle E. Then the cyclic cohomology $\mathrm{HC}^*(\mathcal{A}(X,E))$ of $\mathcal{A}(X,E)$ is given by:*

$$\mathrm{HC}^k(\mathcal{A}(X,E)) \simeq \bigoplus_{j\geq 0} \mathrm{H}^{k-2j}(\mathbb{S}^*X \times \mathbb{S}^1),$$

where $\mathbb{S}^*X$ is the cosphere bundle of the manifold X. By direct inspection we deduce that the periodic cyclic cohomology $\mathrm{HP}^*(\mathcal{A}(X,E))$ is given by:

$$\mathrm{HP}^k(\mathcal{A}(X,E)) \simeq \bigoplus_{j\in\mathbb{Z}} \mathrm{H}^{k-2j}(\mathbb{S}^*X \times \mathbb{S}^1).$$

We refer the reader to [29, 53] for these basic concepts of NCG.

Remark 1. The easy consequence of this computation is that the space of traces $\mathrm{HC}^0(\mathcal{A}(X,E))$ is one dimensional. We also deduce that the only trace, called the residual trace and denoted RTr, is up to constant given by:

$$\mathrm{RTr}(P) = \int_{\mathbb{S}^*X} \mathrm{Tr}(\sigma_{-n}(P)(x,\xi))dxd\xi,$$

where $\sigma_{-n}(P)$ is the component of order $-n$ of the complete symbol of P. This trace is the famous Wodzicki residue trace [67].

The above homology computation can be generalized to other geometric situations. A good tool to reach interesting geometric situations is then to use groupoids. This generalization will not be taken up in these notes and we refer the interested reader to [18] for the computation in a general class of groupoids including for instance manifolds with corners as well as foliations. Here we only give the result. Let then $\mathcal{G}$ be any longitudinally smooth groupoid with corners and let $A\mathcal{G}$ be the Lie algebroid of $\mathcal{G}$ and $A^*\mathcal{G}$ the Lie coalgebroid of $\mathcal{G}$. The sphere bundle of $A^*\mathcal{G}$ will be denoted by $\mathbb{S}^*\mathcal{G}$. The algebra $\psi^\infty(\mathcal{G})$ of classical (scalar for simplicity) pseudodifferential operators on $\mathcal{G}$ was defined in [57] and it recovers all the interesting situations. We consider the algebra $\mathcal{A}_\mathcal{L}(\mathcal{G})$ of complete symbols thus obtained but which have rational expansion near the corners. Then the periodic cyclic homology of $\mathcal{A}_\mathcal{L}(\mathcal{G})$ is given by:

$$\mathrm{HP}^j(\mathcal{A}_\mathcal{L}(\mathcal{G})) \simeq \bigoplus_{j\in\mathbb{Z}} \mathrm{H}_\mathcal{L}^{k-2j}(\mathbb{S}^*\mathcal{G} \times \mathbb{S}^1),$$

where $\mathrm{H}_\mathcal{L}$ denotes rational de Rham cohomology. See [18] for more details.

1.3 The index of elliptic complexes

Definition 2. A pseudodifferential operator $P \in \mathcal{P}^m(X; E^0, E^1)$ is elliptic if its principal symbol $\sigma(P)$ is a section over the cosphere bundle $\mathbb{S}^*X$ of isomorphisms from E^0 to E^1, i.e.

$$\forall (x,\xi) \in \mathbb{S}^*X, \sigma(x,\xi) : E^0_x \to E^1_x \text{ is an isomorphism.}$$

Theorem 1. *Let $P \in \mathcal{P}^m(X; E^0, E^1)$ be an elliptic pseudodifferential operator over the smooth closed manifold X. Then there exists a pseudodifferential operator $Q \in \mathcal{P}^{-m}(X; E^1, E^0)$ such that:*

$$1 - QP \in \mathcal{P}^{-\infty}(X; E^0, E^0) \text{ and } 1 - PQ \in \mathcal{P}^{-\infty}(X; E^1, E^1).$$

For the proof of Theorem 1, see for instance [63].

Lemma 3. *If $P \in \mathcal{P}^m(X; E^0, E^1)$ is elliptic then its nullspace* $\mathrm{Ker}(P)$ *in the Sobolev Hilbert space $H^s(X, E^0)$ does not depend on s and is built up of smooth sections of E^0.*

Proof. Let $1 - QP = R$ be regularizing. Let $u \in \mathrm{Ker}(P)$ then we have $R(u) = u$ and therefore u is a smooth section of E^0 since R is regularizing. □

Definition 3. Let $P : C^\infty(X, E^0) \to C^\infty(X, E^1)$ be an elliptic pseudodifferential operator. Then we define the analytic index $\mathrm{Ind}(P)$ of P as:

$$\mathrm{Ind}(P) := \dim(\mathrm{Ker}(P)) - \dim(\mathrm{Coker}(P)).$$

In the above definition, we have used that the kernel of the extension of P to Sobolev spaces is the same as the kernel of P acting on smooth sections and is finite dimensional. The cokernel of P can also be defined either as the kernel of the adjoint of the extension to the Sobolev spaces or as the kernel of the formal adjoint acting on the space of smooth sections, the resulting space is the same.

In the same way we consider, more generally, complexes of pseudodifferential operators. A pseudodifferential complex $(E, d) = (E^i, d^i)_{0 \leq i \leq k}$ is a sequence

$$0 \to C^\infty(X, E^0) \xrightarrow{d^0} C^\infty(X, E^1) \xrightarrow{d^1} \cdots \xrightarrow{d^{k-1}} C^\infty(X, E^k) \to 0,$$

of pseudodifferential operators such that $d^{j+1} \circ d^j = 0, \forall j$. Such a complex is elliptic if the corresponding sequence of principal symbols

$$0 \to \pi^*(E^0) \xrightarrow{\sigma(d^0)} \pi^*(E^1) \xrightarrow{\sigma(d^1)} \cdots \xrightarrow{\sigma(d^{k-1})} \pi^*(E^k) \to 0$$

is exact over the cosphere bundle $\pi : \mathbb{S}^*X \to X$.

The case $k = 1$ corresponds of course to the previous definition of elliptic operators. The link between complexes and operators is made explicit in the following proposition:

Proposition 4. *Let (E, d) be a pseudodifferential complex as above and fix as before hermitien metrics on the vector bundles $(E^j)_{0\le j\le k}$. Denote by $(d^j)^*$ the formal adjoint of d^j. We define for any j the Laplacian operator $\Delta^j = (d^j)^* \circ d^j + d^{j-1} \circ (d^{j-1})^*$ and set:*

$$D : C^\infty(X, \oplus_j E^{2j}) \to C^\infty(X, \oplus_j E^{2j+1})$$
$$D(u_0, u_2, \cdots) := (d^0(u_0) + (d^1)^*(u_2), d^2(u_2) + (d^3)^*(u_4), \cdots).$$

Then we have:

(i) $D^* \circ D = \oplus\Delta^{2j}$ and $D \circ D^* = \oplus\Delta^{2j+1}$;

(ii) The pseudodifferential complex (E, d) is elliptic iff the operator D is elliptic;

(iii) The pseudodifferential complex (E, d) is elliptic iff the operators Δ^j are elliptic;

(iv) If the pseudodifferential complex (E, d) is elliptic then the cohomology spaces $\mathrm{H}^j(E, d) := \mathrm{Ker}(d^j)/Im(d^{j-1})$ of the complex (E, d) are finite dimensional vector spaces, and we have:

$$\mathrm{Ind}(D) = \sum_j (-1)^j \dim \mathrm{H}^j(E, d).$$

Proof. (i) is immediate.

The proof of (ii) and (iii) uses the fact that the principal symbol of Δ^j is given by:

$$\sigma(\Delta^j) = \sigma(d^{j-1}) \circ \sigma(d^{j-1})^* + \sigma(d^j)^* \circ \sigma(d^j).$$

(iv) It is clear that

$$\mathrm{Ker}(D) = \mathrm{Ker}(D^*D) = \oplus_j \mathrm{Ker}(\Delta^{2j}) \text{ and } \mathrm{Ker}(D^*) = \mathrm{Ker}(DD^*) = \oplus_j \mathrm{Ker}(\Delta^{2j+1}).$$

Therefore it remains to show that $\mathrm{Ker}(\Delta^j)$ is isomorphic to the j^{th} cohomology space of the elliptic complex. Now working on corresponding Sobolev Hilbert spaces, we have:

$$\mathrm{Ker}(d^{j-1})^* \simeq Im(d^{j-1})^\perp,$$

and hence from the ellipticity of Δ^j, we see that $\mathrm{Ker}(\Delta^j) \simeq \mathrm{Ker}(d^j) \cap \mathrm{Ker}(d^{j-1})^*$ is built up of smooth sections and is isomorphic to the orthogonal of $Im(d^{j-1})$ in $\mathrm{Ker}(d^j)$ ($d^j \circ d^{j-1} = 0$). This shows that this last orthogonal is finite dimensional and hence that $Im(d^{j-1})$ is closed and also that $\mathrm{Ker}(d^j)/Im(d^{j-1})$ is isomorphic to $\mathrm{Ker}(\Delta^j)$. □

This proposition encompasses for instance the de Rham complex as well as the Dolbeault complexes and enables to treat them as elliptic operators.
Definition 4. The Euler-Poincaré characteristic (or analytic index) of an elliptic pseudodifferential complex (E, d) on a smooth closed manifold X is the alternate sum:

$$\chi(E, d) := \sum_{j \geq 0} (-1)^j \dim(\mathrm{H}^j(E, d)).$$

The computation of the Euler-Poincaré characteristic of (E, d) is then equivalent to the computation of the index of one elliptic operator constructed using hermitian metrics on the vector bundles involved.

1.4 Some geometric operators

The first complex one meets in differential topology is the de Rham complex. In this case $E^j = \Lambda^j_{\mathbb{C}}(T^*X)$ is the complexified j^{th} exterior power of the cotangent bundle T^*X of X. The smooth sections of E^j are differential j-forms on X. The de Rham differential $d : C^\infty(X, E^j) \to C^\infty(X, E^{j+1})$ enables to form an elliptic complex. The principal symbol of d is given by

$$\sigma(d)(x, \xi) : \Lambda^j(T^*_x X) \to \Lambda^{j+1}(T^*_x X), \sigma(d)(x, \xi) = \sqrt{-1} ext(\xi),$$

where $ext(\xi)$ is exterior multiplication by the covector ξ. The Euler-Poincaré characteristic of the manifold X is by definition the Euler-Poincaré characteristic of the de Rham complex on X. This characteristic is a topological invariant of the manifold and it can be computed in many ways, using triangulations on the manifold, vector fields, Morse theory, $\cdots$ etc. The well known Gauss-Bonnet theorem enables to compute the Euler-Poincaré characteristic of X in terms of the Euler class $e(X)$. This is the answer to the index problem in the case of the de Rham complex. The Euler-Poincaré characteristic of odd manifolds is always zero, as a consequence of the index theorem (see below).

In the classification of oriented manifolds up to oriented homotopy, we need yet further invariants of the manifold. The most important one is the signature (and its higher versions), it is a relevant observation of Atiyah and Hirzebruch that such invariant is the index of an elliptic operator. The index theorem is then a good tool to compute this oriented homotopy invariant. Moreover we shall see in Section 3 that the signature is, thanks to the index theorem, a rational combination of the Pontryagin numbers of the manifold. Therefore, the index formula for the signature operator gives a partial answer to the following problem:

" Find all the rational combinations of the Pontryagin numbers which are oriented homotopy invariants."

Pontryagin numbers are not homotopy invariants. Other conjectured homotopy invariants are constructed by using the fundamental group of the manifold. More precisely, if $\Gamma = \pi_1(X)$ is the fundamental group of the smooth closed manifold X, then there exists (up to homotopy) a continuous map $f : X \to B\Gamma$, where $B\Gamma$ is the classifying space of Γ, which classifies the universal cover. If now we denote by $L(X)$ the Hirzebruch polynomial in the Pontryagin classes of X, then the push-forward $f_*(L(X) \sqcap [X]) \in H_*(B\Gamma, \mathbb{Q})$ is suspected to be an oriented homotopy invariant of the manifold. This is the famous Novikov conjecture. The pairing of this class with the 0-dimensional cohomology class corresponding to the 1 function on X is in fact equal to the signature of the manifold by the index theorem as we shall see and is therefore an oriented homotopy invariant. Let us now explain more precisely the Atiyah-Hirzebruch construction, which plays a fundamental role in the construction of the Diff-equivariant spectral triple.

Let X be a smooth oriented closed manifold with even dimension $n = 2m$. Then we have a well defined bilinear form on $\mathrm{H}^m(X, \mathbb{Z})$ given by cup product. More precisely,

$$< \alpha, \beta >:= \int_X \alpha \sqcup \beta.$$

If $m = 2k+1$ so that the dimension of X is $n = 4k+2$ then the above bilinear form on $\mathrm{H}^{2k+1}(X, \mathbb{Z})$ is antisymmetric and thus its signature does not make much sense. In this case the convention is to take this signature to be zero. In the case $m = 2k$, the signature of the quadratic form associated with the above symmetric bilinear form is by definition the signature of the manifold X and is denoted by $\mathrm{Sgn}(X)$. Recall that the signature of a symmetric non degenerate bilinear form is the difference between the number of positive eigenvalues and the number of negative eigenvalues of the associated invertible symmetric matrix.

Definition 5. Let X and Y be two oriented smooth closed manifolds. We say that X and Y are (strongly) homotopy equivalent if there exist two orientation preserving smooth maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim id_Y$ and $g \circ f \sim id_X$.

In the above definition $h_0 \sim h_1 : Z \to Z'$ means homotopy equivalence of maps in the sense that there exists a smooth map $h : Z \times [0,1] \to Z'$ such that $h|_{Z\times\{0\}} = h_0$ and $h|_{Z\times\{1\}} = h_1$. If X and Y are homotopy equivalent even oriented closed smooth manifolds, then it is clear that $\mathrm{Sgn}(X) = \mathrm{Sgn}(Y)$.

Let us consider again the de Rham differential $d : \Omega^*(X) \to \Omega^*(X)$. The operator associated with the de Rham complex and whose index is exactly the Euler-Poincaré characteristic is then exactly

$$d + d^* : \Omega^{even}(X) \to \Omega^{odd}(X).$$

The Euler characteristic thus corresponds to the splitting of $d+d^*$ with respect to even and odd forms. In order to take into account the Riemannian structure of X we shall use another splitting of $d + d^*$, i.e. another grading involution of $\Omega^*(X)$ which anticommutes with $d+d^*$. So fix a metric on X, this induces metrics on the bundles $\Lambda^j(X)$. On each fibre, the inner product can be defined for every fixed j-form β, and by duality, as a wedge product with a form of degree $2m-j$ denoted $*\beta_x$, so that:

$$< \alpha_x, \beta_x > dx^1 \wedge \cdots \wedge dx^n = \alpha_x \wedge *(\beta_x).$$

The operator $*$ is the usual Riemannian Hodge operator. It enables to write:

$$< \alpha, \beta > := \int_X \alpha \wedge *\beta.$$

We shall need the following equality, which can be easily checked:

$$* \circ * = (-1)^j \text{ on } \Omega^j(X).$$

Lemma 4. *The Riemannian adjoint d^* of d is given by:*

$$d^* = - * \circ d \circ *.$$

Proof. From the symmetry of the scalar product defined above we deduce that for any $\omega \in \Omega^j(X)$ and any $\delta \in \Omega^{2m-j}(X)$ we have:

$$\int_X *\omega \wedge *\delta = (-1)^j \int_X \delta \wedge \omega.$$

If now $(\alpha, \beta) \in \Omega^j(X) \times \Omega^{j-1}(X)$, then:

$$\begin{aligned} < d^*\alpha, \beta > &= < \alpha, d\beta > \\ &= \int_X d\beta \wedge *\alpha \\ &= (-1)^j \int_X \beta \wedge d(*\alpha) \\ &= (-1)^j \int_X \beta \wedge *((-1)^{2m-j+1}(* \circ d \circ *)(\alpha)) \\ &= < \beta, -(* \circ d \circ *)(\alpha) > . \end{aligned}$$

□

We set now $\tau = \sqrt{-1}^{j(j-1)+m} *$ on $\Omega^j(X) \otimes \mathbb{C}$. Then when m is odd, τ is imaginary and it is real only when m is even. In addition we have:

$$\tau \circ \tau = id.$$

Hence τ enables to split $\Omega^*(X) \otimes \mathbb{C}$ into the eigenspaces associated with the $+1$ and -1 eigenvalues. We denote these subspaces respectively by $\Omega_+(X)$ and $\Omega_-(X)$. Using the expression of d^* in Lemma 4, we deduce that the operator $D = d + d^*$ anticommutes with τ and hence can be written in the following way:

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

where D_- is the formal adjoint of D_+ with respect to the above inner product. The operator D^2 is the Laplace operator $d \circ d^* + d^* \circ d$ and is thus a differential operator of order 2 whose principal symbol is given by:

$$\sigma(D^2)(x, \xi) = \pm |\xi|^2 \times id.$$

Therefore the operators D_+ and D_- are elliptic operators of order 1. We set:

$$H_+ = \mathrm{Ker}(D_+) \text{ and } H_- = \mathrm{Ker}(D_-).$$

These spaces are exactly the intersections of the kernel of the Laplace operator with $\Omega_+(X)$ and $\Omega_-(X)$ respectively and are therefore composed of harmonic forms. We set

$$h_+ := \dim(H_+) \text{ and } h_- := \dim(H_-).$$

Proposition 5. *The index* $\mathrm{Ind}(D_+) = h_+ - h_-$ *is equal to the signature of the closed smooth oriented manifold* X.

Proof. [10] If $0 \leq j < m$, we denote $A_j = \Omega^j_{\mathbb{C}}(X) \oplus \Omega^{2m-j}_{\mathbb{C}}(X)$. Then A_j is stable by τ so that it splits into:

$$A_j = A_j^+ \oplus A_j^-$$

with respect to the involution τ. On the other hand when $j < m$, $\tau(A_j^+) = A_j^-$, more precisely we have for any $(\omega_1, \omega_2) \in \Omega^j_{\mathbb{C}}(X) \oplus \Omega^{2m-j}_{\mathbb{C}}(X)$:

$$\tau(\omega_1, \omega_2) = \pm(\omega_1, \omega_2) \Longleftrightarrow \omega_2 = \pm\tau(\omega_1).$$

Therefore $\dim(H^j_+) = \dim(H^j_-)$. Thus we obtain

$$\mathrm{Ind}(D_+) = h^m_+ - h^m_-,$$

and it remains to treat the case $j = m$.

If m is odd, then $\tau = \sqrt{-1}\sigma$ with σ real and $\sigma^2 = -1$. The eigenspaces associated with the eigenvalues $\pm\sqrt{-1}$ of σ are then conjugated and

$$\mathrm{Ind}(D_+) = 0.$$

Since in this case the signature of our manifold X is trivial by definition, the proof is complete.

If now m is even, say $m = 2k$, then $\tau|_{\Omega^{2k}(X)}$ is nothing but the Hodge operator $*$. Hence on $\mathrm{H}^{2k}_+(X, \mathbb{R})$, the quadratic form giving the signature coincides with the norm:

$$\int \omega \wedge \omega = \int \omega \wedge *\omega =< \omega, \omega > \quad > 0 \text{ if } \omega \neq 0.$$

On $\mathrm{H}^{2k}_-(X, \mathbb{R})$ on the other hand, we obtain exactly the opposite of the norm. This shows that h^+_{2k} is exactly the number of positive eigenvalues of the matrix associated with the quadratic form $\omega \mapsto \int \omega \wedge *\omega$ while h^-_{2k} corresponds to the number of negative eigenvalues. The conclusion follows. $\square$

Another important example in Non Commutative Geometry is the Dirac operator, which broadly speaking, corresponds to a “root” of a Laplacian. Let us recall now the definitions and main properties. Recall that if E is a euclidien vector bundle over X then the Clifford bundle of E, $\mathrm{Cliff}(E)$ is a bundle of algebras in which we have the relation $\xi * \xi - |\xi|^2 1 = 0$, for any $\xi \in E \subset \mathrm{Cliff}(E)$.

The group $\mathrm{Spin}(k)$ is the non trivial twofold cover of $\mathrm{SO}(k)$, i.e. the connected twofold cover of $\mathrm{SO}(k)$. It can be realized using Clifford algebras as follows:

$$\mathrm{Spin}(k) := \{w = v_1 * \cdots * v_{2j}, v_l \in \mathbb{R}^k \text{ and } |v_l| = 1\},$$

where $*$ is Clifford multiplication. We then define a group morphism ϱ by:

$$\varrho : \mathrm{Spin}(k) \to \mathrm{SO}(k), \quad \varrho(w)(x) = w * x * w^t,$$

where $(v_1 * \cdots * v_{2j})^t = v_{2j} * \cdots * v_1$. It is easy to show that ϱ is a surjective homomorphism which is not the trivial cover, see [52].

To study the representations of $\mathrm{Spin}(2r)$ we can make it act naturally on $\mathrm{Cliff}(\mathbb{R}^{2r})$ by Clifford multiplications on the left. It turns out that this representation enables to construct the Dirac operator we are looking for. More precisely denote by $L : \mathrm{Spin}(2r) \to \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2r})$ this representation and by $\tilde{L}$ the extended representation to $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2r})$. Then it is well known [42] that $\tilde{L}$ splits into 2^r equivalent subrepresentations $\tilde{\Delta}$ of $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2r})$ which are irreducible. The restriction Δ of $\tilde{\Delta}$ to the spin group $\mathrm{Spin}(2r)$ is called the spin representation and it also splits into 2 inequivalent irreducible subrepresentations $\Delta \simeq \Delta^+ \oplus \Delta^-$ obtained by using the Δ-preserving involution $(\sqrt{-1})^r e_1 * \cdots * e_{2r}$ of the Clifford algebra.

Given a vector bundle E with even dimension $2r$ and with a spin structure, the spin frame bundle $P(E)$ of E, a principal $\mathrm{Spin}(2r)$-bundle, then enables to form the spin bundle

$$S := P(E) \times_{\mathrm{Spin}(2r)} \Delta.$$

This bundle is in evidence $\mathbb{Z}_2$-graded and its fibres are isomorphic to the typical fibre $\Delta = \Delta^+ \oplus \Delta^-$. So S is a vector bundle of irreducible representations of the complexified Clifford algebra bundle of E and we have more precisely

$$\mathrm{End}(S) \simeq \mathrm{Cliff}_{\mathbb{C}}(E).$$

Clifford multiplication by sections of E interchanges $S^+ := P(E) \times_{\mathrm{Spin}(2r)} \Delta^+$ and $S^- := P(E) \times_{\mathrm{Spin}(2r)} \Delta^-$ and we denote this section by:

$$c : E \to \mathrm{End}(\pi^*(S)), \text{ where } \pi : E \to X \text{ is the projection.}$$

Let us now take $E = TX$, the tangent bundle to the even closed spin manifold X and denote again by S the resulting spin bundle. Let ∇^S be a connection on the bundle S associated with the metric on X. This means that the connection ∇^S on S is spin, i.e. it satisfies the Clifford compatibility with the Levi-Civita connection ∇ on TX:

$$\nabla_j^S(c(u)(\omega)) = c(\nabla_j(u))(\omega) + c(u)(\nabla_j^S(\omega)),$$
$$\forall u \in C^\infty(X, TX) \text{ and } \forall \omega \in C^\infty(X, S).$$

The construction of the connection ∇^S is given for instance in [35].

From the definition of a connection we also know that $\nabla^S : C^\infty(S) \to C^\infty(S \otimes T^*X)$ satisfies the Leibniz rule:

$$\nabla^S(fu) = f\nabla^S(u) + u \otimes df, \quad \forall f \in C^\infty(X) \text{ and } \forall u \in C^\infty(X, S).$$

Definition 6. The Dirac operator on X is the first order differential operator which is the composite of the above differential operator ∇^S and the zero order differential operator

$$\tilde{c} : C^\infty(X, S \otimes T^*X) \to C^\infty(X, S),$$

given by $\tilde{c}(u \otimes \omega)(x) = c(\omega(x))(u(x))$.

In a local frame $(e^1, \cdots, e^{2r})$ of T^*X around a point $x \in X$, we can easily show that:

$$D = \sum_{j=1}^{2r} c(e^j)\nabla_j^S \text{ where } \nabla_j^S(u) =< \nabla^S(u), e_j > .$$

In the above formula e_j is the dual basis of e^j.

The symbol of the Dirac operator D does not depend on the base variable x and is given by Clifford multiplication:

$$\sigma(D)(x,\xi) = c(\xi) : S_x \to S_x.$$

Hence D is elliptic. Since each $c(e^j)$ interchanges the fibres S^+ and S^-, the Dirac operator anticommutes with the grading $S = S^+ \oplus S^-$. We obtain a formally selfadjoint operator which extends to a unique essentially selfadjoint operator still denoted D on the L^2 sections of S. We denote by D_+ the Dirac operator acting from $C^\infty(X, S^+)$ to $C^\infty(X, S^-)$.

Remark 2. We deduce immediately from the Leibnitz rule that

$$[D, M_f] := D \circ M_f - M_f \circ D = c(df), \quad \forall f \in C^\infty(X).$$

The operator M_f is pointwise multiplication by f and the operator $c(df)$ is pointwise Clifford multiplication by the covector d_xf. It turns out that the norm of the bounded operator $c(df)$ acting in $L^2(X, S)$ is exactly the Supremum of the norm of d_xf when x runs over X. Therefore we can recover the geodesic distance in X by the following Connes' mean formula [24]:

$$d(x,y) = \sup\{|f(x) - f(y)|, \|[D,f]\| \leq 1\}.$$

If now E is a further hermitian vector bundle over X, then we consider the bundle $S \otimes E$ and use a hermitian connection on E to define out of the spin connection on S a connection ∇_E on $S \otimes E$. Therefore we can define similarily the Dirac operator with coefficients in E that we denote by $D \otimes E$ hoping that no confusion can occur. So:

$$D \otimes E : C^\infty(X, S \otimes E) \to C^\infty(X, S \otimes E).$$

The positive part $(D \otimes E)_+$ of $D \otimes E$ acting from $C^\infty(X, S^+ \otimes E)$ to $C^\infty(X, S^- \otimes E)$ is then an elliptic operator whose index is important for the purpose of Poincaré duality. From the definition of topological K-theory [9], we deduce the following

Proposition 6. *The map $E \mapsto \mathrm{Ind}[(D \otimes E)_+]$ induces a well defined morphism from $K(X)$ to $\mathbb{Z}$ called the Atiyah-Chern character of the Dirac operator D.*

To work with a Dirac operator, it suffices to assume the existence of the bundle S of irreducible representations of the Clifford bundle. This assumption is weaker and is called K-orientation. It corresponds to the existence of a spin^c structure on X, see [52]. The following Thom isomorphism is valid for K-oriented manifolds and is Poincaré duality (in K-theory).

Theorem 2. *(The Thom isomorphism) The map $K(X) \to K(T^*X)$ which assigns to a vector bundle E the class of the symbol of a Dirac operator with coefficients in E, is an isomorphism. In other words, $K(T^*X)$ is a free $K(X)$-module which is generated by the K-theory class of the symbol of the Dirac operator.*

This theorem is proved in [6], see also [28, 49]. Indeed, working with the dual theory of K-homology, the Dirac operator determines a K-homology class in X, say an element of the K-homology group $K_0(X)$. The above theorem then states exactly that

$$K_0(X) \simeq K^0(X) \text{ whenever } X \text{ is spin (or even spin}^c\text{)}.$$

We point out that the Chern character enables to read these properties in the usual de Rham homology and cohomology. In the NonCommutative case, the existence of Poincaré duality or rather some weaker version is one of the axioms assumed on (non commutative) manifolds in [27].

2 The Atiyah-Singer index theorem

The first general proof of the index formula was given by Atiyah and Singer in [9] and heavily rests on topological K-theory. We have already seen that any elliptic complex admits an analytic index also called the Euler-Poincaré characteristic of the elliptic complex. This analytic index only depends on the K-theory class of the principal symbol of the complex and yields a group morphism

$$\mathrm{Ind}_a : K(T^*X) \to \mathbb{Z}.$$

The study of the properties of this map will completely identify it and prove the index theorem. The content of this section can be generalized to deal with other situations.

2.1 Review of the splitting principle

The splitting principle enables to handle characteristic classes. The idea that governs this principle is that one can always assume that the complex vector bundles under study are Whitney sums of line bundles. For precise definitions and more details about characteristic classes, we refer to [46, 55].

Proposition 7. *Let E be a complex vector bundle over the manifold X. Then there exists a smooth manifold BE and a smooth fibration $\pi : BE \to X$ over X such that $\pi^* : \mathrm{H}^*(X) \to \mathrm{H}^*(BE)$ is injective and the pull-back vector bundle $\pi^* E$ splits into line bundles*

$$\pi^* E \simeq l_1 \oplus \cdots \oplus l_k,$$

over BE.

A similar result holds for oriented real vector bundles which split into real plane bundles when pulled back to the right manifold. When the vector bundle E is the complexification of an oriented real vector bundle F, we obtain a splitting of $\pi^* E = \pi^*(F \otimes \mathbb{C})$ into

$$\pi^*(F \otimes \mathbb{C}) \simeq (l_1 \oplus \bar{l}_1) \oplus \cdots \oplus (l_k \oplus \bar{l}_k),$$

where $\bar{l}_j$ is the conjugate bundle of l_j. More precisely, the splitting $F \simeq F_1 \oplus \cdots \oplus F_k$ of F into oriented plane bundles F_j enables to describe $F_j \otimes \mathbb{C} \simeq l_j \oplus \bar{l}_j$. These results are now classical and the proof can be found for instance in [52][page 225].

To sum up and in the complex case for instance, the total Chern class $c(E)$ of a given complex vector bundle E can always -using the injectivity- be formally decomposed into a product

$$c(E) = \Pi_{j=1}^{k}(1 + x_j),$$

where the x_j are the Chern classes of the bundles l_j that appear in the above splitting principle. In other words, the j^{th} Chern class $c_j(E) \in \mathrm{H}^{2j}(E)$ can always be thought of as the j^{th} symmetric function in the formal variables x_i. In the same way for an oriented real vector bundle F, the total Pontryagin class $p(F)$ can be written

$$p(F) = \Pi_{j=1}^{k}(1 + x_j^2),$$

so that $p_j(F) := (-1)^j c_{2j}(F \otimes \mathbb{C})$ expresses as the j^{th} symmetric function in the variables x_i^2.

F. Hirzebruch proposed in the early sixties to construct characteristic classes by using the splitting principle together with the notion of multiplicative sequences [46]. Let us explain very briefly this approach.

Let $Q(z) = 1 + a_1 z + \cdots + a_k z^k + \cdots$ be a formal power series with constant term 1. We compute, depending on the allowed dimension, the product $Q(x_1)Q(x_2)\cdots Q(x_k)$ and one can prove that this product only depends on the symmetric functions in the variables x_j. Therefore, this product can be expressed in terms of the Chern classes and yields new characteristic classes, i.e.

$$Q(x_1)Q(x_2)\cdots Q(x_k) = 1 + Q_1(c_1) + Q_2(c_1, c_2) + \cdots + Q_k(c_1, \cdots, c_k) + \cdots .$$

Take for instance

$$Q(x) := \frac{x}{1 - e^{-x}} = 1 + x/2 + x^2/12 + \cdots .$$

The characteristic class that we obtains in this way is called the Todd class and denoted Td or $\mathrm{Td}(E)$ for a given complex vector bundle E. A straightforward computation gives

$$\begin{aligned} \mathrm{Td}(c_1, \cdots, c_k) = 1 + \mathrm{Td}_1(c_1) + \mathrm{Td}_2(c_1, c_2) + \cdots + \mathrm{Td}_k(c_1, \cdots, c_k) + \cdots , \\ \text{with } \mathrm{Td}_1(c_1) = c_1/2, \mathrm{Td}_2(c_1, c_2) = (c_2 + c_1^2)/12, \cdots . \end{aligned}$$

In four dimensions only the first and second classes are non trivial and the computation is complete. Hence for instance, the formula

$$\int_X \mathrm{Td}(E),$$

means $(1/12)\int_X (c_2(E) + c_1^2(E))$.

Remark 3. If we compute a product $Q(x_1)Q(x_2)\cdots Q(x_n)$ with $n > k$ then the homogeneous polynomials Q_j for $j \leq k$ are unchanged, which simplifies notaceably the calculations.

A similar connstruction works for real oriented bundles and one gets polynomials in the Pontryagin classes. As an example, take the formal power series $Q(x) = \frac{x/2}{Sh(x/2)}$, which yields the so called $\hat{A}$-polynomial and which is given in low dimensions by:

$$\hat{A}_1(p_1) = -p_1/24, \hat{A}_2(p_1, p_2) = \frac{-4p_2 + 7p_1^2}{2^7 \times 3^2 \times 5},$$

where the Pontryagin classes p_j are the symmetric functions in the variables x_i^2. In four dimensions for instance, only the first Pontryagin class is non trivial and we get $\int_X \hat{A}(F) = (-1/24)\int_X p_1(F)$.

To finish this review, we point out that if E is a complex vector bundle, then the Chern character $\mathrm{Ch}(E)$ [55] of E can also be redefined by using the splitting principle:

$$\mathrm{Ch}(E) = \sum_{j \geq 1} e^{x_j} = \dim(E) + c_1(E) + \cdots .$$

2.2 Statement of the index theorem

Let us give the explicit formula which computes the index of any elliptic pseudodifferential complex on a smooth closed manifold. Let (E,d) be an elliptic complex on X. Denote by $[\sigma(E,d)]$ the compactly supported K-theory class of the principal symbol of (E,d) in $K(T^*X)$. Then the Chern character $\mathrm{Ch}[\sigma(E,d)]$ of this symbol class belongs to $\mathrm{H}_c^{even}(T^*X,\mathbb{Q})$, the compactly supported even cohomology space of the total manifold T^*X. Let E be a real vector bundle over X, the $\hat{A}$ genus of E, $\hat{A}(E)$, is the characteristic class associated with the formal power series $\frac{x/2}{sh(x/2)}$, thus it is a polynomial in the Pontryagin classes of E. Recall also that the Todd characteristic class $\mathrm{Td}(V)$ of a complex vector bundle V over X is the characteristic class associated with the formal power series $\frac{x}{1-e^{-x}}$ and is a polynomial in the Chern classes of V.

Lemma 5. *[52] Let E be an oriented real vector bundle over X, then we have:*

$$\mathrm{Td}(E\otimes\mathbb{C}) = \hat{A}(E)^2.$$

Proof. If E is even dimensional then we can apply the splitting principle and use a formal decomposition [46]

$$E\otimes\mathbb{C} = (l_1\oplus\bar{l}_1)\oplus\cdots\oplus(l_n\oplus\bar{l}_n).$$

Therefore setting $x_j = c_1(l_j)$, the first Chern class of l_j, we deduce that:

$$\mathrm{Td}(E\otimes\mathbb{C}) = \Pi_{j=1}^n \frac{x_j}{1-e^{-x_j}}\frac{-x_j}{1-e^{x_j}}$$

which coincides with $\hat{A}(E)^2$. □

Theorem 3. *(The Atiyah-Singer index theorem)*

Let (E,d) be an elliptic complex over the closed n-dimensional manifold X as above. Then the analytic index of (E,d) is given by the AS-formula:

$$\mathrm{Ind}_a(E,d) = (-1)^n < \mathrm{Ch}[\sigma(E,d)]\hat{A}(X)^2, [T^*X] > .$$

where the total $\hat{A}$ class of X is pulled back to T^*X and where T^*X is endowed with the symplectic orientation.

The RHS of the index formula is a local topological data of the complex (E,d) and of the manifold X. In general an index formula is a formula which computes the index of an elliptic operator by the integral of a local data. Notice that the formula in Proposition 2 was of this type. Locality has a very precise meaning here, see [5] or [30]. In the present case of smooth closed manifolds, the Gilkey invariance theory [42] enables to unify the different local formulae one may obtain by different methods and to recover the above one.

As an example of a local formula, we indicate the Connes-Moscovici index theorem when we specialize to the manifold case [30]. The proof of Theorem 3 that we summarize below is the K-theory proof due to Atiyah and Singer.

2.3 Sketch of the original K-theory proof

Instead of giving all the technicalities behind the proof of the index theorem, we have preferred to try to explain the main steps and to be sketchy. The references are intended to clear up the details of the constructions and proofs. Let us first itemize the main ideas of the proof. We start with an elliptic pseudodifferential operator P on a smooth closed manifold X. If X were a sphere, then we would be able to carry out the computation using Bott periodicity. If X is any closed manifold then we can use the Whitney imbedding theorem to assume that X is a submanifold of a euclidian space $\mathbb{R}^N$ for large N ($N \geq 2n+1$). The problem is then to construct out of P an elliptic operator on $\mathbb{R}^N$ which would have an index and whose index would be equal to that of P. If $\mathcal{N}$ is a tubular neighborhood of X in $\mathbb{R}^N$, then the construction consists first in finding an elliptic operator on $\mathcal{N}$ and then in extending it to the whole euclidian space $\mathbb{R}^N$. The extension of an elliptic operator with index on $\mathcal{N}$ to $\mathbb{R}^N$ preserves the index by a weak version of an excision theorem, so that we are reduced to the case of $\mathcal{N} \to X$ which can be diffeomorphically identified with the normal bundle of X in $\mathbb{R}^N$.

Hence given now a vector bundle $\mathcal{N} \to X$ over X, we are led to the problem of constructing in a natural way out of P an elliptic operator on $\mathcal{N}$ whose index is equal to that of P. If X is reduced to the point, this problem is equivalent to finding a pseudodifferential operator (or a pseudodifferential complex) on $\mathbb{R}^q$ whose index is equal to 1. Such an operator exists and the properties of K-theory enable to produce the general topological construction for any X. Let us now give more details.

The idea on $\mathbb{R}^q$ is that the problem can be reduced to the case $q = 1$ thanks to multiplicative properties of the index.

Lemma 6. *Let b_q be the class in $K(T^*\mathbb{R}^q)$ of the complex of symbols over $T^*(\mathbb{R}^q)$ given by:*

$$0 \to \Lambda^0\mathbb{C}^q \xrightarrow{\sigma_q} \Lambda^1\mathbb{C}^q \xrightarrow{\sigma_q} \cdots \xrightarrow{\sigma_q} \Lambda^q\mathbb{C}^q \to 0,$$

where $\sigma_q(x,\xi)$ is exterior multiplication by $x+i\xi$. Then the analytic index of b_q exists and equals 1. The class $[b_q]$ in $K(T^*\mathbb{R}^q)$ is called the Bott generator of $K(T^*\mathbb{R}^q) \simeq \mathbb{Z}$.

Proof. There exists a sharp product $\sharp : K(T^*\mathbb{R}) \otimes \cdots \otimes K(T^*\mathbb{R}) \to K(T^*\mathbb{R}^q)$

[9] such that from the very definition of this product we have [58]

$$b_q = b_1 \sharp \cdots \sharp b_1.$$

Thus a multiplicativity property of the analytic index with respect to this sharp product, and that will not be expanded here, shows that we can assume $q = 1$. The operator whose index we thus need to compute is the operator $x + d/dx$ over $\mathbb{R}$. It is a classical result due to Hørmander that this operator is surjective. Since its kernel is one dimensional, the conclusion follows. $\square$

Before explaining the construction of the operator on $\mathcal{N}$ associated with P, let us clear up the excision property of the index. The idea is to construct an analytic index map on open manifolds. So if M is such a non compact manifold and if $i : M \hookrightarrow Y$ is an open imbedding in a closed manifold Y, then one naively defines the analytic index map as the composite of the index map $K(T^*Y) \to \mathbb{Z}$ with the K-theory extension map $i_* : K(T^*M) \to K(T^*Y)$. One then needs to check the independence of the imbedding (i, Y). It is an easy exercise to prove that this definition of the index coincides with taking the analytic index of the corresponding operator on M itself so that it does not depend on the imbedding [9]. Hence we have now reduced the index problem to the construction of an appropriate symbol class in $K(T^*\mathcal{N})$ out of the symbol class $[\sigma(P)] \in K(T^*X)$.

The rough idea is to take a sharp product of the class $[\sigma(P)]$ with a symbol on the vertical tangent bundle to the fibration $\mathcal{N} \to X$ corresponding to the generator b_q on each fibre. This is very similar to the Thom homomorphism in de Rham cohomology and this construction is called the Thom homomorphism (in K-theory).

More precisely now, let $\pi : E \to M$ be a complex vector bundle over the (not necessarily compact) manifold M. The dimension of the fibres of E is denoted q. The Thom complex of E is the complex $\lambda(E)$ over E given by:

$$0 \to \pi^*(\Lambda^0 E) \xrightarrow{\lambda} \pi^*(\Lambda^1 E) \xrightarrow{\lambda} \cdots \xrightarrow{\lambda} \pi^*(\Lambda^q E) \to 0,$$

where $\lambda(e) = ext(e)$ is the exterior multiplication by $e \in E$.

The above complex is exact off the zero section of E. However when M is not compact it does not define a compactly supported K-theory class of E. If (F, d) is a compactly supported complex over M defining a class in $K(M)$ then the sharp product of $\pi^*(F, d)$ by $\lambda(E)$ is a complex over E which is compactly supported and thus defines a class in $K(E)$. In summary we have a well defined additive map:

$$\begin{aligned} \lambda_E : K(M) &\longrightarrow \quad K(E), \\ [F, d] &\longmapsto \pi^*[F, d] \sharp \lambda(E). \end{aligned}$$

The homomorphism λ_E is called the Thom homomorphism of E.

Back to our situation $\mathcal{N} \to X$, we point out that the normal bundle to T^*X in $T^*\mathbb{R}^N$ is isomorphic to $\pi^*T^*\mathcal{N} \simeq \pi^*(\mathcal{N} \otimes \mathbb{C})$ if we denote by $\pi : T^*X \to X$ the projection. Therefore starting with our class $[\sigma(P)] \in K(T^*X)$ we construct using $\lambda_{\pi^*(\mathcal{N}\otimes\mathbb{C})}$ a class in $K(T^*\mathcal{N})$. Roughly speaking and formally, the index of this class can then be computed by first evaluating the index in the fibres of $T^*\mathcal{N} \to T^*X$ to obtain a class in $K(T^*X)$ and then computing the index of this latter. But because the construction of λ_E consists exactly in putting the class b_q on each fibre, we deduce that the vertical index image of $\lambda_{\pi^*(\mathcal{N}\otimes\mathbb{C})}([\sigma(P)])$ coincides with $[\sigma(P)]$, therefore the index is preserved by the Thom homomorphism. We denote by $i! : K(T^*X) \to K(T^*\mathbb{R}^N)$ the resulting homomorphism. So we have a commutative diagram:

$$\begin{array}{ccc} K(T^*X) & \xrightarrow{i!} & K(T^*\mathbb{R}^N) \\ \mathrm{Ind}_X \searrow & & \swarrow \mathrm{Ind}_{\mathbb{R}^N} \\ & \mathbb{Z} & \end{array}$$

The commutativity of this diagram is the heart of the index theorem. To give an explicit formula for the index in terms of characteristic classes we then simply apply the Chern character to this diagram and compute. To get more insight into the explicit constructions and proofs we again refer to [9].

We are then reduced to the problem of commutativity of the following diagram

$$\begin{tikzcd} K(M) \arrow[r] \arrow[d, "\mathrm{Ch}_M"] & K(E) \arrow[d, "\mathrm{Ch}_E"] \\ H_c^{even}(M) \arrow[r] & H_c^{even}(E) \end{tikzcd}$$

where the horizontal maps are the K-theory Thom homomorphism and the cohomological Thom homomorphism. It turns out that this diagram is not commutative and this is the reason why the Todd class appears in the index formula. The detailed computation using the Chern character is given in [10]. See also [52].

2.4 The heat expansion method

Thanks to the Gilkey invariance theory, there are many different proofs of the Atiyah-Singer index theorem. The aim as we have explained is to find a local formula by using only local tools. We have already met local tools of a given elliptic pseudodifferential operator in these notes. The Dixmier trace for instance is a local tool when the operator is in the right Dixmier ideal. More generally, the Wodzicki residue trace is a local tool and has the advantage over the famous homological formula of having a direct generalization to the Non Commutative setting. Good references are [30] and [20]. The Wodzicki residue is closely related with the heat expansion method, by using the Riemann Γ function. Let us therefore rather explain how the heat kernel method works and yields the index formula.

The first property that we recall is the MacKean-Singer cancellation formula. Let D be an elliptic differential operator of order one on a smooth closed n-manifold X with coefficients in a vector bundle E. Denote $\Delta = D^* \circ D = |D|^2$ and define for any $t > 0$ the bounded operator $e^{-t\Delta}$ by continuous functional calculus in $L^2(X, E)$. Because $(1 + \Delta)^{(n+1)/2} e^{-t\Delta}$ is a bounded operator for any $t > 0$, we know that the operator $e^{-t\Delta}$ is a trace class operator. Moreover the Schwartz kernel k_t of $e^{-t\Delta}$ is a smooth section over $X \times X$ of the bundle $\mathrm{Hom}(\pi_2^*(E), \pi_1^*(E))$ where $\pi_j : X \times X \to X$ is the projection on the j^{th} factor. More precisely, we have:

Lemma 7. *For any $t > 0$, the operator $e^{-t\Delta}$ is a regularizing operator and belongs to $\mathcal{P}^{-\infty}(X; E)$.*

Proof. We give a formal proof and refer to [42] for a rigorous justification. To show that $e^{-t\Delta}$ is regularizing, we must prove that it induces a bounded operator between each pair of Sobolev spaces. But recall that the Sobolev spaces can be defined using any positive elliptic operator instead of the Riemannian Laplace operator. So we can take Δ itself and therefore the proof is completed by observing that the operator $\Delta^s e^{-t\Delta}$ is bounded for any $s \in \mathbb{R}$. □

Proposition 8. *(MacKean-Singer)*

Let D be an elliptic differential operator of order 1. Then we have:

$$\mathrm{Ind}(D) = \mathrm{Tr}(e^{-tD^*D}) - \mathrm{Tr}(e^{-tDD^*}), \quad \forall t > 0.$$

Proof. The operator D furnishes an isomorphism between the eigenspace $E_0(\lambda)$ of D^*D and the eigenspace $E_1(\lambda)$ of DD^* corresponding to $|\lambda|^2 \neq 0$. Therefore their dimensions are equal and we have

$$\begin{aligned}
\mathrm{Tr}(e^{-tD^*D}) - \mathrm{Tr}(e^{-tDD^*}) &= \sum_{\lambda \in Sp(D)} e^{-t|\lambda|^2} (\dim(E_0(\lambda)) - \dim(E_1(\lambda))) \\
&= \dim(E_0(0)) - \dim(E_1(0)) \\
&= \mathrm{Ind}(D).
\end{aligned}$$

□

The MacKean-Singer formula enables to prove the index theorem. The method is an expansion of $\mathrm{Tr}(e^{-tD^*D}) - \mathrm{Tr}(e^{-tDD^*})$ when $t \sim 0$. It turns out that the restriction of the kernel k_t of e^{-tD^*D} to the diagonal of $X \times X$ admits an expansion near 0 of the form:

$$k_t(x,x) \sim \sum_{j \geq 0} t^{\frac{j-n}{2}} \alpha_j^{D^*D}(x),$$

where $\alpha_j^{D^*D}(x) \in \mathrm{End}(E_x)$ is a local data which defines a global section. The justification of this expansion is tedious but is a straightforward consequence of an application of the pseudodifferential calculus with parameter to produce a parametrix for the resolvent of D, see [42]. If D is a pseudodifferential operator, then one needs to add in this expansion terms involving $\mathrm{Log}(t)$.

Proposition 9. *The index of the operator D on the n-manifold X is given by the local formula*

$$\mathrm{Ind}(D) = \int_X \left[\mathrm{Tr}(\alpha_n^{D^*D}(x)) - \mathrm{Tr}(\alpha_n^{DD^*}(x))\right] dvol(x).$$

The equivalence between this formula and the Atiyah-Singer formula is a corollary of the invariance theory, see [5] and [42].

Let us recall now the link between the heat expansion method and the zeta functions. Recall that the complex powers Δ^z of Δ can be defined using holomorphic functional calculus. These powers are pseudodifferential operators, this is the Seeley theorem [61]. But then we have from the definition of the Riemann Γ-function and by using the Fubini theorem, for $Re(z) > 0$:

$$\Delta^{-z} = \frac{1}{\Gamma(z)} \int_0^{+\infty} t^z e^{-t\Delta} \frac{dt}{t}.$$

The asymptotic expansion of $\mathrm{Tr}(e^{-t\Delta})$ given above then shows that the zeta function $\zeta_z(\Delta) = \mathrm{Tr}(\Delta^{-z})$ of Δ, which at first converges only for large $Re(z)$, actually extends to a meromorphic function in the z-plane. Moreover it shows that the function $\Gamma(z)\zeta_z(\Delta)$ has at most simple poles at the points $z_j = (j-n)/2$, $j \geq 0$, and the residue is given for $j \neq n$ by:

$$Res_{z=z_j}(\Gamma(z)\zeta_z(\Delta)) = \int_X \alpha_j^{\Delta}(x) dvol(x),$$

while $\zeta_0(\Delta)$ exists and coincides with $\int_X \alpha_n^{\Delta}(x) dvol(x)$. This proves that the index of D can be related with the regularized values $\zeta_0(D^*D)$ and $\zeta_0(DD^*)$ of the zeta functions. This remark is due to [5]. Other links with residues of zeta functions appeared later but we will not expand them here since the local Connes-Moscovici index theorem is explained in parallel lectures.

3 Corollaries and examples

The first observation is that the Atiyah-Singer theorem shows that the index of any elliptic pseudodifferential operator from the Kontsevich-Vishik class [51] on an odd dimensional manifold is trivial, this is true for instance for differential operators. The principal symbol of such operators satisfies $\sigma(x, -\xi) = (-1)^m \sigma(x, \xi)$ and therefore, the homotopy class is unchanged under the involution $(x, \xi) \mapsto (x, -\xi)$. This shows that the analytic index map, which only depends on the homotopy class of the principal symbol is unchanged under this $\mathbb{Z}_2$-action. However, the homology class $[T^*X]$ which appears in the RHS of the AS-formula is multiplied by $(-1)^{\dim(X)}$.

This remark shows the importance of the index theorem for Toeplitz operators briefly explained in the first section. Notice that these operators are order zero pseudodifferential operators in the sense of [9], and thus furnish invariants for the odd case.

Let us restrict ourselves now to even spin manifolds and denote again by $D \otimes E$ the Dirac operator on X that we have twisted by the hermitian vector bundle E. We are led to the following index problem:

Give a local computation of the Chern-character of the Dirac operator.

In other words:

Find a local representative for the Atiyah map:

$$K(X) \longrightarrow \mathbb{Z}, \text{ defined by } [E] \longmapsto \operatorname{Ind}[(D \otimes E)_+].$$

Theorem 4. *Denote by $\hat{A}(X) \sqcap [X]$ the Poincaré dual of the $\hat{A}$ genus of the manifold X. Then the rational homology class $\hat{A}(X) \sqcap [X]$ satisfies:*

$$\operatorname{Ind}[(D \otimes E)_+] = < \operatorname{Ch}(E), \hat{A}(X) \sqcap [X] > .$$

In other words, the index theorem can be rewritten as:

$$\operatorname{Ind}[(D \otimes E)_+] = \int_X \operatorname{Ch}(E) \hat{A}(X).$$

Proof. We give the details of the now classical proof for the convenience of the reader. Recall that the $\hat{A}$ characteristic class is associated with the generating function $\frac{x/2}{sh(x/2)}$. We must compute the integrand in the Atiyah-Singer index formula when the elliptic pseudodifferential operator P is $(D \otimes E)_+$. Since X is oriented the index formula in the Atiyah-Singer theorem becomes:

$$\operatorname{Ind}[(D \otimes E)_+] = (-1)^m \int_X \pi! \operatorname{Ch}(\sigma((D \otimes E)_+)) \hat{A}(X)^2,$$

where $\pi! : \mathrm{H}^*(T^*X, \mathbb{Q}) \to \mathrm{H}^*(X, \mathbb{Q})$ is integration along the fibres and m is the dimension of X. The K-theory class of $\sigma((D\otimes E)_+)$ in $K(T^*X)$ is represented by $(\pi^*(S^+\otimes E), \pi^*(S^-\otimes E), c)$ where $c(x,\xi) : S_x^+\otimes E_x \to S_x^-\otimes E_x$ is $c(\xi)\otimes Id_{E_x}$ and $c(\xi)$ is Clifford multiplication by ξ acting from S_x^+ to S_x^-. Therefore we have:

$$\pi! \operatorname{Ch}(\sigma((D \otimes E)_+)) = \operatorname{Ch}(E)\pi! \operatorname{Ch}(\sigma(D_+))$$

So we need to prove that

$$\pi! \operatorname{Ch}(\pi^*(S^+), \pi^*(S^-), c) = (-1)^m \hat{A}(X)^{-1}.$$

The restriction $\mu(TX) = i^*(\pi^*(S^+), \pi^*(S^-), c)$ to X is exactly the class $\mu(TX) = [S^+] - [S^-]$ in K(X). If $TX \simeq F \oplus F'$ where F and F' are two vector bundles over X then we have in $K(X)$:

$$\mu(TX) = \mu(F)\mu(F').$$

Therefore we can use the splitting principle to assume that TX decomposes into $E_1\oplus\cdots\oplus E_m$ where each E_j is a 2-plane bundle and $E_j\otimes\mathbb{C} \simeq l_j\oplus\bar{l}_j$ where $E_j \simeq l_j$ as oriented bundles. It follows that the spin bundle S is the tensor product of the corresponding spin bundles. But each E_j is isomorphic as an oriented real 2-plane to the complex line bundle l_j. We have assumed that X is a spin manifold, therefore each complex line bundle l_j has a spin structure. This is equivalent to the fact that the first Chern class $c_1(l_j)$ of l_j is even. Hence l_j admits a square root, i.e. a bundle $l_j^{1/2}$ such that $l_j^{1/2} \otimes l_j^{1/2} \simeq l_j$. In this case it is easy to see that the spin bundle of l is exactly

$$S(l) = l^{1/2} \oplus \bar{l}^{1/2},$$

where $S^+(E_j) = \bar{l}_j^{\,1/2}$ and $S^-(E_j) = l_j^{1/2}$. So

$$\mu(E_j) = [\bar{l}_j^{\,1/2}] - [l_j^{1/2}].$$

Hence

$$\operatorname{Ch}(\mu(E_j)) = e^{-x_j/2} - e^{x_j/2},$$

where x_j is the first Chern class of E_j. Finally we get:

$$\operatorname{Ch}(\mu(TX)) = \Pi_{1\leq j\leq m}(e^{-x_j/2} - e^{x_j/2}) = (-1)^m e(X)\hat{A}(X)^{-1},$$

where $e(X)$ is the Euler class defined by $e(X) = \Pi_{1\leq j\leq m} x_j$. On the other hand,

$$\operatorname{Ch}(\mu(TX)) = i^* \operatorname{Ch}(\pi^*(S^+), \pi^*(S^-), c) = i^* \circ i! \circ \pi! \operatorname{Ch}(\pi^*(S^+), \pi^*(S^-), c),$$

where $i!$ is the Thom isomorphism $i! : \mathrm{H}^*(X, \mathbb{Q}) \to \mathrm{H}^*(TX, \mathbb{Q})$ which is the inverse of the integration along the fibres $\pi!$. From the definition of $i!$ we know that

$$(i^* \circ i!)[\omega] = e(X)[\omega].$$

Therefore we get

$$\mathrm{Ch}(\mu(TX)) = e(X)\pi! \,\mathrm{Ch}(\pi^*(S^+), \pi^*(S^-), c),$$

and so:

$$\pi! \,\mathrm{Ch}(\pi^*(S^+), \pi^*(S^-), c) = (-1)^m \hat{A}(X)^{-1},$$

as announced. The proof is then complete. $\square$

If we take for E the spin vector bundle S itself, then the Dirac operator $D \otimes S$ with coefficients in S coincides with the operator $d + d^*$ acting on differential forms. Indeed, if ∇ is the Levi-Civita connection on the exterior algebra $\Lambda^*_{\mathbb{C}}(X)$ then we can recover the de Rham differential by the formula [52][page 123]:

$$d\omega = \sum_{j \geq 1} e_j \wedge \nabla_j \omega \text{ while } d^* \text{ is given by } d^*\omega = -\sum_{j \geq 1} i(e_j)\nabla_j \omega,$$

with $i(e_j)$ the interior product by e_j. But recall that Clifford multiplication by e_j can be read through the linear isomorphism $(S \otimes S \simeq) \,\mathrm{Cliff}_{\mathbb{C}}(X) \cong \Lambda^*_{\mathbb{C}}(X)$ as:

$$e_j * \omega = e_j \wedge \omega - i(e_j)\omega.$$

Let us decompose now the bundle $S \otimes S$ into $S_+ \otimes S$ and $S_- \otimes S$ exactly as we did for any coefficient bundle E. Then we have:

Lemma 8. *The operator $[D \otimes S]_+$ is exactly the positive part of the Atiyah-Hirzebruch Signature operator, and thus:*

$$\mathrm{Ind}([D \otimes S]_+) = \mathrm{Sgn}(X) = 2^m \int_X \hat{L}(X) = \int_X L(X),$$

where $\hat{L}(X)$ and $L(X)$ are the $\hat{L}$-characteristic polynomial and the Hirzebruch L-polynomial associated respectively with the power series $\frac{x/2}{th(x/2)}$ and $\frac{x}{th(x)}$.

Proof. The decomposition of S into S^+ and S^- is obtained by using left multiplication by the involution $\omega_{\mathbb{C}} = i^m e_1 * \cdots * e_{2m}$ which is a globally defined section of $\mathrm{Cliff}_{\mathbb{C}}(X)$, [52][page 136]. But under the linear isomorphism $\mathrm{Cliff}_{\mathbb{C}}(X) \simeq \Lambda^*_{\mathbb{C}}(X)$, left multiplication by $\omega_{\mathbb{C}}$ corresponds up to sign to the

Riemannian Hodge operator $*$. This is proved in [52][page 129]. We deduce that the Clifford grading gives in this case exactly the signature grading.

Applying the above index theorem for twisted Dirac operators, we obtain:

$$\mathrm{Ind}([D \otimes S]_+) = \mathrm{Sgn}(X) = \int_X \mathrm{Ch}(S)\hat{A}(X).$$

But the Chern character of S can be computed using again the splitting principle. In fact this principle tells us that we can assume that the tangent bundle TX is the direct sum of 2-plane bundles of the form

$$TX \simeq \oplus_j E_j \text{ where } E_j \otimes \mathbb{C} \simeq l_j \oplus \bar{l}_j.$$

Thus writing the Chern character of each $S(l_j)$ as $e^{x_j/2} + e^{-x_j/2}$ where x_j is the first Chern class $c_1(l_j)$ of l_j, we obtain:

$$\mathrm{Ch}(S) = \Pi_{1\leq j\leq m}(e^{x_j/2} + e^{-x_j/2}).$$

Now with respect to the same generators, $\hat{A}(X)$ is by definition the characteristic class associated with the generating function $\frac{x/2}{sh(x/2)}$ and thus we have:

$$\hat{A}(X) = \Pi_{1\leq j\leq m}\frac{x_j/2}{sh(x_j/2)},$$

The product of the two expressions then gives:

$$\mathrm{Ch}(S)\hat{A}(X) = 2^m \Pi_{1\leq j\leq m}\frac{x_j/2}{th(x_j/2)}.$$

□

Remark 4. The index formula for the signature operator given above holds even if the manifold is (only oriented and) not spin.

The above example of the signature operator is important to understand the Connes-Moscovici Diff-equivariant spectral triple [30]. One of the corollaries of the index formula concerns *integrality* [10]. The first immediate but relevent observation is that the characteristic numbers obtained for various operators in the RHS of the AS-formula are integers. Let us give now two examples, probably the most known, to explain such integrality results. Many other corollaries can be consulted in the literature, for instance in [8, 52] and the references therein, many of them rather use the equivariant version of the Atiyah-Singer index theorem. This last version is omitted in these introductive notes.

Proposition 10.

(1) We have for four dimensional oriented manifolds:

$$\mathrm{Sgn}(X) = -8 \int_X \hat{A}(X).$$

(2) (Rochlin) If X is a four dimensional spin manifold then $\int_X \hat{A}(X)$ is an even integer and the signature of X is a multiple of 16.

Proof. (1) In this case we have only two generators x_1 and x_2, so that:

$$\hat{A}(X) = \frac{x_1/2}{sh(x_1/2)} \frac{x_2/2}{sh(x_2/2)} = (1-\frac{(x_1/2)^2}{6})(1-\frac{(x_2/2)^2}{6}) = 1-(1/24)(x_1^2+x_2^2).$$

But $p_1(X) = x_1^2 + x_2^2$ is the first Pontryagin class of X, therefore we get:

$$\int_X \hat{A}(X) = -(1/24) \int_X p_1(X).$$

On the other hand, let us compute the signature of X using also the index formula. We have:

$$4\frac{x_1/2}{th(x_1/2)} \frac{x_2/2}{th(x_2/2)} = 2(1 + (1/12)x_1^2)(1 + (1/12)x_2^2) = 4(1 + (1/12)p_1(X)).$$

Therefore we get:

$$\mathrm{Sgn}(X) = (1/3) \int_X p_1(X) = -8 \int_X \hat{A}(X).$$

(2) When X is spin, $\int_X \hat{A}(X)$ is the index of the Dirac operator and is integer. In fact it is an even integer and this can be justified as follows:

In the four dimenisonal case, Clifford multiplication by the operator $J = e_1 * e_3$ yields a well defined endomorphism of S which preserves the grading. In addition it satisfies $J^2 = 1$ and preserves the kernel of D. Since $J\psi$ is orthogonal to ψ, we deduce that $\mathrm{Ker}(D_+)$ and $\mathrm{Ker}(D_-)$ are even dimensional. The conclusion follows by using (1). □

A simple consequence is that an oriented four manifold whose signature is not a multiple of 16 cannot carry a spin structure. A simple example is given by $P_2(\mathbb{C})$, in this case we deduce that:

$$\int_X \hat{A}(X) = -1/8,$$

cannot even be an index.

We can also apply the index theorem to twisted signature operators. Let E be a smooth complex vector bundle over the smooth closed manifold X.

We can extend the operator D_+ to the sections of $\Lambda^*_{\mathbb{C}}(X) \otimes E$ to get a new elliptic first order differential operator D^+_E, called the twisted (by E) signature operator. The index theorem for this operator can then be written as:

$$\mathrm{Ind}(D^+_E) = \int_X \mathrm{Ch}(E)L(X).$$

The integrality corollary of this formula yields a simple proof of a classical theorem due to R. Bott:

Corollary 1. *Let E be a complex vector bundle over the even sphere $\mathbb{S}^{2n}$. Then the top Chern number of E is a multiple of $(n-1)!$. In other words:*

$$\int_{\mathbb{S}^{2n}} c_n(E) \in (n-1)!\ \mathbb{Z}.$$

Proof. The cohomology spaces of the sphere $\mathbb{S}^{2n}$ are trivial except in dimensions 0 and $2n$. Therefore we have:

$$[\mathrm{Ch}(E) \sqcap L(X)]_{2n} = \mathrm{Ch}_n(E) + L_{n/2}(X).$$

But the signature of the sphere is trivial since $\mathrm{H}^n(\mathbb{S}^{2n}) = 0$, thus $\int_X L_{n/2}(X) = 0$ and we get:

$$\mathrm{Ind}(D^+_E) = \int_X \mathrm{Ch}_n(E) = n \times \int_X c_n(E)/n! = \frac{1}{(n-1)!}\int_X c_n(E) \in \mathbb{Z}.$$

□

Other corollaries of the index formula concern some rigidity theorems. Again we refer to [52], for instance from page 280 to page 290. Besides the importance of the Atiyah-Singer index theorem on its own, the methods of proof are ready to be generalized to more complicated situations. The index theorem admits for instance equivariant versions for actions of groups. The case of the action of a compact Lie group led Atiyah and Segal [8] to prove a Lefschetz fixed point formula. Some references, among others, concerning the equivariant approach are [3, 4, 8, 10]. See also the nice paper [66] where E. Witten shows how such Lefschetz formulae may be applied to the study of loop spaces.

Chapter II. The Murray-Von Neumann index theory

The introduction of von Neumann algebras in the framework of Non Commutative Geometry is natural. It unifies many known von Neumann index problems [1, 23, 62], and opens up the way to the investigation of new non commutative phenomena. Von Neumann spectral triples are also expected to play an important part in a reformulation of local Quantum Field Theory as pointed out to me by C. Brouder. We take the opportunity of the present notes to give some further developments which involve in a fundamental way von Neumann algebras. The experimental example chosen to this end is the case of flat bundles, say Galois suspensions. This choice is motivated by the fact that, first and in many respects, this example already embodies all the typical constructions that give rise to von Neumann algebras, and second because the von Neumann index theorem for measured foliations [23] turned out to be a fundamental tool in solving a physical problem concerning the computation of the *range of the trace* in the Bellissard approach to quasi-crystals. To end this chapter and in the last section, we show how Connes' Non Commutative Geometry can be rearranged to deal with von Neumann algebras.

The results of Section 4 and Section 5 were obtained in collaboration with T. Fack [16]. The author wishes to thank him for allowing these new results to be expounded in this school.

4 Dixmier traces in von Neumann algebras

In this section, we shall denote by $\mathcal{M}$ a von Neumann algebra acting on a Hilbert space H. For more information about von Neumann algebras, we refer to [32] but also to [45] where a nice survey, motivated by local quantum field theory, is given. The von Neumann algebras considered here will be type II_∞ von Neumann algebras.

4.1 Dixmier trace and residue of zeta functions

To show the existence of a non normal trace on the algebra $B(H)$ of all bounded operators on an infinite dimensional separable Hilbert space H, J. Dixmier [33] constructed in 1966 a unitarily invariant state Tr_ω on the ideal $L^{1,\infty}(H)$ of all compact operators T on H whose singular numbers $\mu_0(T) \geq \mu_1(T) \geq ...$ satisfy

$$\sum_{k=0}^{n} \mu_k(T) = O(\mathrm{Log}(n)) \text{ when } n \to \infty.$$

The Dixmier trace of such operator was recalled in 1.2.

Let Δ be the Riemannian Laplacian operator with principal symbol $\sigma(\Delta)(x,\xi) = |\xi|$ for a given Riemannian metric on M, then Formula 1.1 gives:

$$\int_M f\ dvol = \frac{n \times (2\pi)^n}{vol(\mathbb{S}^n)} \times \mathrm{Tr}_\omega(f\Delta^{-n}), \quad \forall f \in C^\infty(M).$$

This led A. Connes to reinterprete the Dixmier trace Tr_ω as the correct operator theoretical substitute for integration of infinitesimals of order one in Non Commutative Geometry. We shall now extend the Dixmier trace to the case of semi-finite von Neumann algebras.

Let us denote by $\mathcal{M}$ an infinite semi-finite von Neumann algebra acting on a Hilbert space H and equipped with a faithful normal semi-finite trace τ. For any τ-measurable operator T in H, denote by

$$\mu_t(T) = \mathrm{Inf}_{t>0}\{\|TE\|, E = E^* = E^2 \in \mathcal{M}, \tau(1-E) \leq t\}$$

the t^{th} generalized s-number of T.

An element $T \in \mathcal{M}$ is called τ-compact if $\lim_{t\to\infty} \mu_t(T) = 0$. The set of all τ-compact elements in $\mathcal{M}$ is a norm closed ideal of $\mathcal{M}$ that we shall denote by $\mathcal{K}(\mathcal{M},\tau)$. By [37][page 304], the ideal $\mathcal{K}(\mathcal{M},\tau)$ is the norm closure of the ideal $\mathcal{R}(\mathcal{M},\tau)$ of all elements X in $\mathcal{M}$ whose final support $r(X) = \mathrm{Supp}(X^*)$ satisfies $\tau(r(X)) < \infty$.

Definition 7. An element $T \in \mathcal{M}$ is called a τ-Dixmier operator if:

$$\sigma_t(T) = \int_0^t \mu_s(T)ds = O(\mathrm{Log}(1+t)), \text{ when } t \to +\infty.$$

The set of τ-Dixmier operators is a vector space that we shall denote by $L^{1,\infty}(\mathcal{M},\tau)$. It is clearly a Banach space for the norm:

$$\|T\|_{1,\infty} = \mathrm{Sup}_{t>0} \frac{\sigma_t(T)}{\mathrm{Log}(t+1)}$$

and an ideal in $\mathcal{M}$ which contains $L^1(\mathcal{M},\tau) \cap \mathcal{M}$. We have for any $T \in L^{1,\infty}(\mathcal{M},\tau)$ and any $t > 0$:

$$\mu_t(T) \leq \frac{\sigma_t(T)}{t} \leq \|T\|_{1,\infty} \frac{\mathrm{Log}(1+t)}{t},$$

which shows that for any $\epsilon > 0$:

$$L^1(\mathcal{M},\tau) \cap \mathcal{M} \subset L^{1,\infty}(\mathcal{M},\tau) \subset L^{1+\epsilon}(\mathcal{M},\tau) \cap \mathcal{M} \quad (\subset \mathcal{K}(\mathcal{M},\tau)).$$

Let ω be a state on the C^*-algebra $C_b(\mathbb{R}_+)$ vanishing on $C_0(\mathbb{R}_+)$. Such states do exist thanks to the Hahn-Banach theorem. For any bounded function f in $\mathbb{R}_+$, let $M(f)$ be the mean defined by

$$M(f)(x) = \frac{1}{\mathrm{Log}(x)} \int_a^x f(t) \frac{dt}{t} \quad \text{ where } a > e \text{ is fixed.}$$

Since $M(f) \in C_b(\mathbb{R}_+)$ we can set as in [30]:

$$l_\omega(f) = \omega(M(f)).$$

Definition 8. For any state ω as above, the Dixmier trace $\tau_\omega(T)$ of a positive operator $T \in L^{1,\infty}(M, \tau)$ is defined by:

$$\tau_\omega(T) := l_\omega(t \to \frac{\sigma_t(T)}{\mathrm{Log}(t+1)}).$$

We point out that this definition depends on the choice of the state ω. We shall fix such ω for the rest of this subsection. As in the classical case (cf [30]) one may prove that we have for any positive operators $T, S \in L^{1,\infty}(\mathcal{M}, \tau)$:

$$\tau_\omega(T + S) = \tau_\omega(T) + \tau_\omega(S)$$

This additivity property then enables to extend τ_ω to a positive linear form on the Dixmier ideal $L^{1,\infty}(\mathcal{M}, \tau)$. We also easily check that $\tau_\omega(ST) = \tau_\omega(TS)$ for any $T \in L^{1,\infty}(\mathcal{M}, \tau)$ and any $S \in \mathcal{M}$. The proof used in the classical case (cf [29]) clearly extends to our setting.

Note that $\tau_\omega(T) = 0$ whenever $T \in \mathcal{M} \cap L^1(\mathcal{M}, \tau)$.

Definition 9. Let $\mathcal{M}$ be as before an infinite semi-finite von Neumann algebra acting on a Hilbert space H and equipped with a normal faithful positive trace τ. A positive self-adjoint operator T on H is called τ-discrete if $(T + \lambda)^{-1} \in \mathcal{K}(\mathcal{M}, \tau)$ for any $\lambda > 0$.

It may be proved (cf [60][page 48]) that $T = \int_0^{+\infty} \lambda dE_\lambda$ is τ-discrete if and only if one of the two following properties holds:

(i) $\forall \lambda \in \mathbb{R}$, $\tau(E_\lambda) < +\infty$;

(ii) $\exists \lambda_0 > 0$ such that $(T + \lambda_0)^{-1} \in \mathcal{K}(\mathcal{M}, \tau)$.

For such positive τ-discrete operator T, the function

$$N_T(\lambda) := \tau(E_\lambda)$$

is well defined on $\mathbb{R}$; moreover, it is non decreasing, positive and right continuous. This is the state density of T.

Definition 10. Let $T = \int_\epsilon^{+\infty} \lambda dE_\lambda$ be a positive self-adjoint τ-discrete operator with spectrum in $[\epsilon, +\infty)$, where $\epsilon > 0$. The zeta function ζ_T of T is defined by:

$$\zeta_T(z) = \int_\epsilon^{+\infty} \lambda^z dN_T(\lambda),$$

for any complex argument z such that the above integral converges.

Since we have for any $R > 1$:

$$\int_1^R \lambda^z dN_T(\lambda) = \int_0^{\mathrm{Log}(R)} e^{zt} d\alpha(t),$$

where $\alpha(t) = N_T(e^t)$, we know from the classical Laplace-Stieltjes transform theory that the integral

$$\int_\epsilon^{+\infty} \lambda^z dN_T(\lambda)$$

converges for $Re(z) < -d_T$ and diverges for $Re(z) > -d_T$, where:

$$d_T := \overline{\lim}_{t\to+\infty} \frac{\mathrm{Log}(\alpha(t))}{t} = \overline{\lim}_{\lambda\to+\infty} \frac{\mathrm{Log}(N_T(\lambda))}{\mathrm{Log}(\lambda)}.$$

Moreover, ζ_T is analytic in the half-plane $\{Re(z) < -d_T\}$ and $z = -d_T$ is a singularity of ζ_T if $d_T < +\infty$. For more informations about the Laplace-Stieltjes transform, we refer to [65].

We also have:

$$d_T = \mathrm{Inf}\{\mu \geq 0, T^{-z} \in L^1(\mathcal{M}, \tau) \text{ for } Re(z) > \mu\}.$$

Indeed, it follows from the normality of the trace τ that:

$$\mathrm{Sup}_{R>0} \int_0^R \lambda^x dN_T(\lambda) = \mathrm{Sup}_{R>0}\, \tau(E_R(T)T^x) = \tau(T^x), \text{ for any } x \in \mathbb{R}.$$

In particular:

$$d_T < +\infty \Longleftrightarrow \exists x \leq 0 \text{ such that } T^x \in L^1(\mathcal{M}, \tau).$$

Sometimes, d_T is called the quantum τ-dimension of the operator T.

Theorem 5. *[16] Let T be a positive τ-discrete operator with spectrum in $[\epsilon, +\infty)$, $\epsilon > 0$. If $0 < d_T < +\infty$, the following conditions are equivalent:*

(i) $(x + d_T)\zeta_T(x) \to A$ when $x \to -d_T, x \in (-\infty, -d_T[$;

(ii) $T^{-d_T} \in L^{1,\infty}(\mathcal{M}, \tau)$ and

$$\tau_\omega(T^{-d_T}) = \lim_{t\to+\infty} \frac{1}{\mathrm{Log}(1+t)} \int_0^t \mu_s(T^{-d_T}) ds = -A/d_T. \tag{4.1}$$

The proof of this theorem is given in [16]. The Dixmier trace is strongly related to the asymptotics of the spectrum. For instance we have the following

Proposition 11. *[60] Let T be as in Theorem 5. Assume that there exists $\delta > 0$ such that ζ_T admits a meromorphic extension to $\{Re(z) < -d_T + \delta\}$ with a simple pole at $z = -d_T$. Then we have:*

$$\lim_{\lambda \to +\infty} \frac{N_T(\lambda)}{\lambda^{d_T}} = -\frac{Res_{-d_T}(\zeta_T)}{d_T} = \tau_\omega(T^{-d_T}).$$

This proposition generalizes the Guillemin spectral estimate to a more general class of pseudodifferential operators.

Some of the above properties have been explicited. In the case of measured foliated manifolds in [16]. In particular, there exists a generalization of the Connes formula expressing the Dixmier trace by a local residue.

5 Index theory in von Neumann algebras

We define in this section τ-Fredholm operators and their τ-index in type II von Neumann algebras. We also show how the famous Calderon formula can be generalized to the setting of von Neumann algebras, the main technical difficulty being that the spectrum of a τ-compact operator is not discrete in general. This formula will be used in the next section to give a polynomial formula for the Chern-Connes character of a von Neumann spectral triple. In the present section, we shall also explain how von Neumann algebras are involved in the index theory of measured suspensions.

5.1 *Definitions and properties*

As before, let $\mathcal{M}$ be a von Neumann algebra in a Hilbert space H, equipped with a semi-finite normal faithful trace τ.

Lemma 9. *For any τ-compact projection $e \in \mathcal{M}$, we have $\tau(e) < +\infty$.*

Proof. Since $e = e^* = e^2$, we have $\mu_t(e) \in \{0, 1\}$. But $\mu_t(e) \to 0$ as $t \to +\infty$ by hypothesis, so that there exists t_0 such that

$$\mu_t(e) = 0, \quad \text{for } t \geq t_0,$$

and hence $\tau(e) = \int_0^{t_0} \mu_t(e) dt < +\infty$. □

Definition 11. An operator $T \in \mathcal{M}$ is called τ-Fredholm if there exists $S \in \mathcal{M}$ such that $1 - ST$ and $1 - TS$ are τ-compact.

Proposition 12. *If $T \in \mathcal{M}$ is τ-Fredholm, the projections p_T and p_{T^*} onto $\mathrm{Ker}(T)$ and $\mathrm{Ker}(T^*)$ are τ-finite.*

Proof. Let S be as in Definition 11. The projections $p_T = (1 - ST)p_T$ and $p_{T^*} = (1 - TS)^* p_{T^*}$ are τ-compact, and Lemma 9 gives the result. $\square$

Definition 12. The index $\mathrm{Ind}_\tau(T)$ of a τ-Fredholm operator T is defined by:

$$\mathrm{Ind}_\tau(T) := \tau(p_T) - \tau(p_{T^*}), \tag{5.1}$$

where p_T and p_{T^*} are the projections onto the kernel of T and T^* respectively.

The following proposition generalizes the Calderon formula and computes $\mathrm{Ind}_\tau(T)$ from the powers of $1-ST$ and $1-TS$ in the setting of semi-finite von Neumann algebras. This formula will be used to get a polynomial formula for the Chern-Connes character (see 6.2).

Proposition 13. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a normal faithful semi-finite (positive) trace τ and $T \in \mathcal{M}$. Assume that there exists $p \geq 1$ and an operator $S \in \mathcal{M}$ such that:*

$$1 - ST \in L^p(\mathcal{M}, \tau) \text{ and } 1 - TS \in L^p(\mathcal{M}, \tau).$$

Then T is τ-Fredholm and we have for any integer $n \geq p$:

$$\mathrm{Ind}_\tau(T) = \tau((1 - ST)^n) - \tau((1 - TS)^n). \tag{5.2}$$

Proof. [16] The operator T is τ-Fredholm because $L^p(\mathcal{M}, \tau) \cap \mathcal{M} \subset \mathcal{K}(\mathcal{M}, \tau)$. To prove the proposition, we may assume that $n = 1$. Indeed, let $S \in \mathcal{M}$ be such that:

$$A = 1 - ST \text{ and } B = 1 - TS$$

are in $L^p(\mathcal{M}, \tau)$ (and hence in $L^n(\mathcal{M}, \tau)$ for any $n \geq p$) and set:

$$S' = S(1 + B + B^2 + ... + B^{n-1}).$$

We have:

$$1 - TS' = B^n \text{ and } 1 - S'T = A^n.$$

Replacing S by S', A by $A^n \in L^1(\mathcal{M}, \tau)$ and B by $B^n \in L^1(\mathcal{M}, \tau)$, we are thus reduced to the case where $n = 1$. When $A = 1 - ST$ and $B = 1 - TS$ are in $L^1(\mathcal{M}, \tau)$, we get from the relations $Ap_T = p_T$ and $p_{T^*}B = p_{T^*}$ the equality:

$$\mathrm{Ind}_\tau(T) = \tau(Ap_T) - \tau(p_{T^*}B).$$

To prove that $\mathrm{Ind}_\tau(T) = \tau(A) - \tau(B)$, it thus suffices to show that:

$$\tau(eAe) = \tau(fBf),$$

where $e = 1 - p_T$ and $f = 1 - p_{T^*}$. To this end, set $V := fTe$. We clearly have:

$$TeA = TA = BT = BfT,$$

and hence:

$$V(eAe) = (fBf)V.$$

If the intertwining operator V from $e(H)$ to $f(H)$ were invertible (the inverse would then be automatically in $e\mathcal{M}f$ by the bicommutant von Neumann theorem), we would get by cyclicity of the trace:

$$\tau(eAe) = \tau(fBf), \tag{5.3}$$

and Calderon's formula would be proved.

Although V is not necessarily invertible here, it is injective with dense range from $e(H)$ to $f(H)$. It turns out that this is enough to prove (5.3) (see [16][Lemma 2]), and Calderon's formula follows. $\square$

5.2 An experimental example: Suspended actions

The index theorem in the case of non compact manifolds has been actively studied under different assumptions. The first significant result was obtained by Atiyah in [1]. His method proved to be very fruitful in the applications of von Neumann algebras to index theory on the one hand, and in the non commutative harmonic analysis of discrete groups on the other hand.

The novelty of Atiyah's approach was the meaning that he gave to the index in the sense of the Murray-von Neumann dimension theory described in the previous subsections.

In many geometric situations, Atiyah's approach helped to perform new strategies to attack singular index problems. One of the famous examples with immediate applications to physics is the Shubin computation of a von Neumann index for almost periodic operators and the reader is encouraged to read Shubin's paper [62] which will not be expounded in these notes. We point out on the other hand that Connes' measured index formula holds for any closed foliated manifold which admits a holonomy invariant transverse measure [23]. One of the applications of such theorem is to the study of the so called solenoid where the transverse structure is modelled on a Cantor space [56][page 42]. Before embarking in the flat bundles world, we point out that the index theory for foliations was the first important application of non commutative C^*-algebras to geometry and it led A. Connes to introduce his famous non commutative foliation C^*-algebra as a topological counterpart for the singular (in general) space of leaves.

Let T be a smooth (for simplicity) closed manifold of dimension q and let $\pi : \tilde{M} \to M$ be a Galois covering over the smooth closed connected p-manifold M with group Γ. Let $\varrho : \Gamma \to \mathrm{Diff}(T)$ be a representation of Γ and denote by V the suspended smooth closed manifold $V = \tilde{M} \times_\Gamma T$, where $\tilde{M} \times_\Gamma T$ is a usual notation for the quotient manifold of $\tilde{M} \times T$ under the free and proper action of Γ on the right given by:

$$(\tilde{x}, t)g = (\tilde{x}g, \varrho(g^{-1})t), \quad \forall (\tilde{x}, t) \in \tilde{M} \times T \text{ and } \forall g \in \Gamma.$$

Then we have a commutative diagram of natural projections:

$$\begin{tikzcd}
\tilde{M} \times T \arrow[r] \arrow[d] & V \arrow[d] \\
\tilde{M} \arrow[r] & M
\end{tikzcd}$$

Hence V fibres over M with fibres diffeomorphic to the manifold T. The foliation F of V that we are interested in, is the one whose leaves are the quotients in V of the manifolds $\tilde{M} \times \{t\}$ when t runs over T. We assume that there exists a (positive for simplicity) measure μ on T which is invariant under the action of the group Γ through ϱ.

So T is a measure space which we have assumed smooth for simplicity and on which the group Γ acts by measure preserving diffeomorphisms. We fix a Lebesgue measure on the manifold M and lift it to a Lebesgue measure ν to $\tilde{M}$. This measure is defined more precisely by choosing a fundamental domain U in $\tilde{M}$, i.e. a subset U of $\tilde{M}$ such that

$$U \cap gU = \emptyset, \forall g \in \Gamma \setminus \{1\} \text{ and } \tilde{M} = \cup_{g \in \Gamma} Ug,$$

and setting for any Borel subspace A of $\tilde{M}$:

$$\nu(A) = \sum_{g \in \Gamma} \nu[\pi(Ag \cap U)], \text{ where } \pi : \tilde{M} \to M \text{ is the projection.}$$

Note that U has non empty interior $int(U)$ and because we have taken a Lebesgue measure ν on $\tilde{M}$, we know for instance that $\nu(int(U)) = \nu(U)$. Furthermore we also deduce that:

$$\nu(\tilde{M} \setminus \cup_{g \in \Gamma} int(U)g) = 0.$$

The leaves of V are quotients of $\tilde{M}$ by isotropy (in T) subgroups of Γ and since the measure ν on $\tilde{M}$ is Γ-invariant it also induces a well defined measure still denoted ν on each leaf of (V, F), and hence a Lebesgue measure on the leaf manifold $\mathcal{F}$. The leaf manifold $\mathcal{F}$ of (V, F) is the p-manifold which is the discrete union of the leaves. Again we need to be more precise about the construction of the measure ν on each leaf: This is easy since we just choose again a fundamental domain U_{Γ_1} for the corresponding subgroup Γ_1 of Γ and set for any Borel subset A in a leaf L (which is diffeomorphic to $\tilde{M}/\Gamma_1$) of (V, F):

$$\nu(A) = \nu(\pi_1^{-1}(A) \cap U_{\Gamma_1}), \text{ where } \pi_1 : \tilde{M} \to L \text{ is the projection.}$$

For any $t \in T$ we define U_t to be the projection in V of $U \times \{t\}$ and we set

$$U_V := \cup_{t \in T} U_t \subset V.$$

Using the measure ν on the leaf manifold together with the Γ-invariant positive measure μ, we can define a positive measure ν_μ on the whole manifold V. This is true on each foliated manifold but in our case this is trivial because we can define this measure in the interior of U_V as the tensor product measure and set for any Borel subset A of V:

$$\nu_\mu(A) := \nu_\mu(A \cap int(U_V)).$$

So in the measure sense the situation is quasi-trivial. We denote by

$$\mathcal{M} := [L^\infty(T, \mu) \otimes B(L^2(\tilde{M}, \nu))]^\Gamma$$

the algebra of $B(L^2(\tilde{M}, \nu))$-valued μ-a.e. bounded functions on T which are Γ-equivariant. The algebra $\mathcal{M}$ can also be described as the commutant in the von Neumann algebra $L^\infty(T, \mu) \otimes B(L^2(\tilde{M}))$ of the diagonal left representation of Γ in $L^2(\tilde{M}, \nu) \otimes L^2(T, \mu)$. Therefore it is a von Neumann algebra which acts on the Hilbert space

$$\mathcal{H} := L^2(T, \mu) \otimes L^2(\tilde{M}, \nu) \cong L^2(T, \mu) \otimes L^2(U, \nu) \otimes l^2(\Gamma).$$

Let ψ be the characteristic function of U and denote also by ψ the operator in $L^2(\tilde{M})$ corresponding to multiplication by ψ. Then we can define a positive functional on $\mathcal{M}$ by setting:

$$\mathrm{Tr}_{\Gamma,\mu}(T) := \int_T \mathrm{Tr}(\psi \circ T_t \circ \psi) d\mu(t).$$

Let $T \in L^\infty(T, \mu) \otimes B(L^2(\tilde{M}, \nu))$. Then for μ-a.e. $t \in T$, T_t admits a Schwartz kernel k_t. It is clear from the definition of k that $T \in \mathcal{M}$ iff for any $g \in \Gamma$ and any $(\tilde{x}, \tilde{y}) \in \tilde{M}$, we have:

$$k_{g^{-1}(t)}(\tilde{x}g, \tilde{y}g) = k_t(\tilde{x}, \tilde{y}).$$

Hence if $T \in \mathcal{M}$, then k induces a kernel k_0 which is a distribution on the quotient groupoid

$$G := \frac{\tilde{M} \times \tilde{M} \times T}{\Gamma}$$

defined by

$$k_0([\tilde{x}, \tilde{x}', t]) = k_t([\tilde{x}], [\tilde{x}']).$$

One can show that the functional $\mathrm{Tr}_{\Gamma,\mu}$ is a trace [1]. Let us recall some useful terminology. A $\mathrm{Tr}_{\Gamma,\mu}$-measurable operator T is a Hilbert-Schmidt operator in the Hilbert space $\mathcal{H}$ if T^*T is a $\mathrm{Tr}_{\Gamma,\mu}$-trace class operator. So if k_t is the Schwartz distributional kernel of T_t defined for μ a.e $t \in T$, then T is a (Γ, μ)-Hilbert-Schmidt operator iff

$$\int_{T \times U \times U} |k_t(\tilde{x}, \tilde{y})|^2 d\nu(\tilde{x}) d\nu(\tilde{y}) d\mu(t) < +\infty.$$

If T is a $\mathrm{Tr}_{\Gamma,\mu}$-measurable operator in $\mathcal{H}$, then the polar decomposition $T = U|T|$ of T defines a partial isometry of $\mathcal{H}$ which lives in $\mathcal{M}$ and the positive operator $|T|$ is a $\mathrm{Tr}_{\Gamma,\mu}$-measurable operator in $\mathcal{H}$ [22]. Therefore a (Γ, μ)-trace class operator is exactly the product of two (Γ, μ)-Hilbert-Schmidt operators. Assume that for μ-a.e. t, the kernel k_t of T_t is a smooth function on $\tilde{M} \times \tilde{M}$ and that the induced kernel k^0 is compactly supported in $\frac{\tilde{M} \times \tilde{M} \times T}{\Gamma}$, then T is of (μ, Γ)-trace class and we have:

$$\mathrm{Tr}_{\Gamma,\mu}(T) = < \nu_\mu, k^0 > = \int_V k^0([\tilde{x}, \tilde{x}, t]) d\nu_\mu([\tilde{x}, t]),$$

where ν_μ is the measure on V induced by the Lebesgue measure on the leaves and the invariant measure μ on T. The justification of this assertion is a rephrasing of the proof given in [1] with the extra term T here.

The trace $\mathrm{Tr}_{\Gamma,\mu}$ is obviously faithful and semi-finite.

Remark 5. An operator A in the Hilbert space $L^2(\tilde{M})$ whose kernel k is integrable on the diagonal is not necessarily trace-class.

Let E be a continuous longitudinally smooth vector bundle over V of dimension N. We shall denote by $\tilde{E}$ the pull back of E to $T \times \tilde{M}$. The restriction of $\tilde{E}$ to each $\{t\} \times \tilde{M}$ enables to define $\psi^\infty(T \times \tilde{M}, \tilde{E})$ as the space of continuous families $(T_t)_{t \in T}$ of classical pseudodifferential operators on the manifold $\tilde{M}$. Thus $\psi^\infty(T \times \tilde{M}, \tilde{E})$ fibres continuously over T with typical fibre the space $\psi^\infty(\tilde{M}, \mathbb{C}^N)$ (where $\psi^\infty(T \times \tilde{M}, \tilde{E})$ is endowed with the topology of kernels, see [57]). Note that the group Γ acts on $\psi^\infty(T \times \tilde{M}, E)$ so that this fibration is equivariant.

Definition 13. (1) A longitudinal pseudodifferential operator P of order m on the foliated manifold V with coefficients in the smooth vector bundle E is a smooth section of the Γ-equivariant bundle $\psi^m(T \times \tilde{M}, \tilde{E})$ over T,

$$P : T \to \psi^m(T \times \tilde{M}, \tilde{E}),$$

which is Γ-equivariant.

(2) The principal symbol of the operator P is defined as the family $\sigma(P) = (\sigma_t(P))_{t \in T}$ such that $\sigma_t(P)$ is the principal symbol of the operator P_t on $\tilde{M}$. Hence $\sigma(P)$ is a continuous Γ-equivariant section of the bundle

$$C_m^\infty((T^*(\tilde{M}) \setminus \tilde{M}) \times T, \mathrm{Hom}(\pi^*(\tilde{E}))),$$

where $\pi : (T^*(\tilde{M}) \setminus \tilde{M}) \times T \to \tilde{M} \times T$ is the projection and C_m^∞ means smooth in sections which are positively m-homogeneous in the sense that

$$\sigma(t, \tilde{x}, \lambda\xi) = \lambda^m \sigma(t, \tilde{x}, \xi), \quad \forall \lambda > 0.$$

A pseudodifferential operator of order m is elliptic if for any $t \in T$, $\sigma(P)(t, \tilde{x}, \xi)$ is an automorphism of the vector space $E_{t,x}$. We point out that again by using the Schwartz kernel of P on each $\tilde{M}$ inside $T \times \tilde{M}$, we can construct an operator P^0 on the quotient manifold V. It is (longitudinally) elliptic if P is elliptic. Let us briefly define now the von Neumann index in the foliated bundle V. The groupoid of interest in V is the groupoid $G := \frac{\tilde{M} \times \tilde{M} \times T}{\Gamma}$ which is the quotient of the family indexed by T of groupoids $\tilde{M} \times \tilde{M} \times \{t\}$ by the diagonal action of Γ.

The ν_μ-measure of any smooth compactly supported kernel k in G is then finite. The space of such kernels will be denoted by $C_c^\infty(G)$.

We shall only consider longitudinal pseudodifferential operators on V with compactly supported distributional kernel and we denote by $\psi^m(V|T; E)$ the space of those operators. As usual we set:

$$\psi^\infty(V|T; E) = \cup_{m \in \mathbb{Z}} \psi^m(V|T; E) \text{ and } \psi^{-\infty}(V|T; E) = \cap_{m \in \mathbb{Z}} \psi^m(V|T; E).$$

The Schwartz theorem enables to identify $\psi^{-\infty}(V|T; E)$ with the ideal $C_c^\infty(G, \mathrm{End}(E))$ of longitudinally smooth continuous sections. Thus we have a short exact sequence of algebras:

$$0 \to C_c^{\infty,0}(G, \mathrm{End}(E)) \hookrightarrow \psi^\infty(V|T; E) \to \mathcal{A}(V|T; E) \to 0,$$

and $\mathcal{A}(V|T; E)$ is called the algebra of longitudinal complete symbols on (V, F). This is a filtered algebra by the order of the pseudodifferential operators. The symbol map induces an isomorphism between the graded algebra of $\mathcal{A}(V|T; E)$ and the graded algebra of Γ-equivariant homogeneous sections of $\mathrm{End}(\tilde{E})$ over $T \times T^*(\tilde{M}) \setminus \tilde{M}$. It turns out that the parametrix theorem recalled in 1.3 is true in the foliation case and we have:

Proposition 14. *If $P \in \psi^\infty(V|T;E)$ is elliptic then its class in $\mathcal{A}(V|T;E)$ is invertible.*

Proof. The proof is classical. For any $t \in T$, T_t is an elliptic operator on the non compact (in general) manifold $\tilde{M}$. Therefore we can find a parametrix S_t which is a pseudodifferential operator on $\tilde{M}$ such that $id - S_t \circ T_t$ and $id - T_t \circ S_t$ are regularizing operators in the manifold $\tilde{M}$. From the construction of the parametrix, we easily deduce that we can take the family $(S_t)_{t\in T}$ Γ-invariant. It remains then to check that S can be chosen so that it induces a compactly supported kernel in the quotient groupoid $(\tilde{M} \times \tilde{M} \times T)/\Gamma$. But the parametrix S_t being constructed locally we know that we can take it almost local in the sense that its support in $\tilde{M} \times \tilde{M}$ is contained in a small neighborhood of the diagonal. Therefore the Schwartz kernel of the quotient operator of S can be taken compactly supported in the groupoid $(\tilde{M} \times \tilde{M} \times T)/\Gamma$ as allowed, see also [1]. □

In other words, compactly supported elliptic pseudodifferential operators admit parametrices modulo the algebra $C_c^\infty(G, \mathrm{End}(E))$. We point out, but this will not be used here, that this last algebra is (algebraically) Morita equivalent to the algebra $C_c^\infty(G)$ and the latter is also Morita equivalent to the algebra $C^\infty(T) \rtimes_{alg} \Gamma$, the algebraic crossed product algebra of the algebra $C^\infty(T)$ of smooth functions on T by the action of the discrete group Γ. Therefore, the above exact sequence enables to define a Γ-index as in [40], i.e. an index which belongs to the K-theory group of the algebra $C^\infty(T) \rtimes_{alg} \Gamma$.

Proposition 15. *Let P be a pseudodifferential operator in $\psi^{-k}(V|T)$ with $k \geq 0$. Then P extends to a bounded operator which belongs to $\mathcal{M}$ and we have:*

(i) If $k < -p/2$ then P is a $(\Gamma, \mu)-$Hilbert Schmidt operator;

(ii) If $k < -p$ then P is a (Γ, μ)-trace class operator in $\mathcal{M}$;

(iii) If $k = -p$, then the operator P belongs to the Dixmier ideal associated with $\mathcal{M}$ and the trace on $\mathcal{M}$ induced by μ. The Dixmier trace $\mathrm{Tr}^\omega_{\Gamma,\mu}$ associated with the trace $\mathrm{Tr}_{\Gamma,\mu}$ on $\mathcal{M}$ and any state ω on the C^*-algebra $C_b(\mathbb{R}^*_+)/C_0(\mathbb{R}^*_+)$, is then given by:

$$\mathrm{Tr}^\omega_{\Gamma,\mu}(P) = \frac{1}{p(2\pi)^p} \int_{T\times\mathbb{S}^*(\tilde{M})} \psi(\tilde{x})\sigma(P_t)(\tilde{x},\tilde{\xi})d\nu_\mu(\tilde{x},t)d\xi,$$

where $\sigma(P_t)$ is the principal symbol of P_t, so it is positively $(-p)$-homogeneous in $\tilde{\xi}$ and ψ is again the characteristic function of the fundamental domain U.

Proof. (i) and (ii) are proved in [23]. (iii) is proved in [16]. □

Let us recall now the construction of the Ruelle-Sullivan morphism associated with the Γ-invariant measure μ on the foliated bundle V. Smooth differential forms in the leaf direction of V can be identified with Γ-invariant differential forms in the $\tilde{M}$-direction of $\tilde{M}\times T$. So a longitudinal differential k-form ω on V is a smooth map

$$\omega : T \to \Omega^k(\tilde{M}),$$

which is Γ-equivariant. We shall denote by $\Omega^{k,0}(V|T)$ the space of longitudinal k-forms on V. The longitudinal differential d_ℓ can then be defined as the composition with the de Rham differential on $\tilde{M}$. We get in this way a longitudinal de Rham complex [19]:

$$d_\ell : \Omega^{k,0}(V|T) \to \Omega^{k+1,0}(V|T).$$

The cohomology of this longitudinal complex will be denoted by $\mathrm{H}^{*,0}(V|T)$. Using again the characteristic function ψ of a fundamental domain U in $\tilde{M}$, we define:

$$C_{\mu,\Gamma} : \Omega^{p,0} \to \mathbb{C} \quad \text{by} \quad C_{\mu,\Gamma}(\omega) := \left\langle \int_{\tilde{M}} \otimes\mu, (\psi\otimes id_T)\omega(\psi\otimes id_T) \right\rangle.$$

Because we are working with invariant differential forms, the map $C_{\mu,\Gamma}$ induces a well defined map on $\mathrm{H}^{p,0}(V|T)$ [19].

There is another way to define the Ruelle-Sullivan current in the foliated manifold V itself. This is classical and uses a partition of unity construction. It turns out that this Ruelle-Sullivan map C_μ on V when evaluated at a longitudinal differential form ω is exactly equal to the evaluation of the above map $C_{\mu,\Gamma}$ at the Γ-invariant longitudinal form $\tilde{\omega}$ which is the lift of ω to $\tilde{M}\times T$. This is easy to check and the justification is then omitted.

Let now $\tilde{D}$ be any (scalar for simplicity) order one elliptic longitudinal differential operator on $\tilde{M}\times T$ which is Γ-invariant. Denote by D the induced operator on the quotient closed foliated manifold V. Then D is a longitudinally elliptic scalar differential operator. The projections onto the kernel and cokernel, denoted respectively $\tilde{p}$ and $\tilde{p}'$, of $\tilde{D}$ belong to the von Neumann algebra $\mathcal{M}$. We have already observed that the regularizing operators are $\mathrm{Tr}_{\Gamma,\mu}$ class operators in the von Neumann algebra $\mathcal{M}$, therefore the operator $\tilde{D}$ is a $\mathrm{Tr}_{\Gamma,\mu}$ Fredholm operator in the sense of Section 4. Hence $\tilde{p}$ and $\tilde{p}'$ are $\mathrm{Tr}_{\Gamma,\mu}$ finite projections and we define the (Γ,μ)-index of $\tilde{D}$ by:

$$\mathrm{Ind}_{\Gamma,\mu}(\tilde{D}) := \mathrm{Tr}_{\Gamma,\mu}(\tilde{p}) - \mathrm{Tr}_{\Gamma,\mu}(\tilde{p}') \in \mathbb{R}.$$

Indeed the operators $\tilde{p}$ and $\tilde{p}'$ have μ a.e bounded kernels which are smooth in restriction to each leaf $\tilde{M}\times\{t\}$ of $\tilde{M}\times T$. In the smooth closed foliated manifold V, we can also define the μ-index $\mathrm{Ind}_\mu(D)$ of D using partitions of unity in V and the holonomy invariant transverse measure μ [23].

Theorem 6. *Let $\tilde{D}$ be any Γ-invariant differential operator on $\tilde{M}\times T$ which is elliptic in the $\tilde{M}$ direction. Then the operator $\tilde{D}$ admits a von Neumann (Γ,μ)-index $\mathrm{Ind}_{\Gamma,\mu}(\tilde{D})$ with respect to the von Neumann algebra $\mathcal{M}$ and this real number coincides with the μ-index of the underlying longitudinal operator D on the foliation of V:*

$$\mathrm{Ind}_{\Gamma,\mu}(\tilde{D}) = \mathrm{Ind}_\mu(D)$$

This theorem generalizes Atiyah's result on coverings and the proof is an easy generalization of Atiyah's method, see [1]. The triviality of the foliated manifold $\tilde{M}\times T$ enables to deduce the measured index theorem for many geometric operators from the Atiyah covering theorem without foliations. The main property of these operators is that they are independent of the T-component and are already Γ-invariant on $\tilde{M}$. Assume for simplicity that M is a spin even dimensional manifold. Let $\tilde{D}$ be the operator obtained as the tensor product of the Dirac operator on $\tilde{M}$ by the identity operator on T. So for any $t\in T$, $\tilde{D}_t$ is independent of t and is just the Dirac operator of $\tilde{M}$ constructed out of the Dirac operator on M. The operator $\tilde{D}$ is obviously a Γ-invariant differential operator which is elliptic in the $\tilde{M}$ direction. For any $t\in T$, the Γ-index of the operator D_t is then equal to the spin genus of M thanks to the covering index theorem [1]. One then immediately checks that the (Γ,μ)-index of $\tilde{D}$ is simply the mean, with respect to the measure μ of the Γ-indices. Rewriting the spin genus of M as an integral over the fundamental domain U of the $\hat{A}$ genus of $\tilde{M}$, we deduce the measured index theorem in this case.

Note that most of the geometric operators are constructed in this way, so the index theorem for them is easy to prove in this case. However, when we consider for instance the Dirac operator along the leaves of V twisted by a vector bundle E over V, the lift depends on the T-variable and the index theorem is not obvious. Nevertheless, the same result holds in the general case and is a consequence of the analysis carried out by A. Connes in [23].

So Connes' measured index theorem [23] provides information on the solutions of longitudinal elliptic equations on the non compact manifold $\tilde{M}\times T$, whenever the operator which is involved is Γ-invariant as a family operator over T.

Corollary 2. *Under the above assumptions, we have the following index formula:*

$$\mathrm{Ind}_{\Gamma,\mu}(\tilde{D}) = < \mathrm{Ch}(\tilde{E})\hat{A}(\tilde{M}), [C_{\Gamma,\mu}] > .$$

Proof. We apply Connes' measured index theorem together with Theorem 6 and obtain:

$$\mathrm{Ind}_{\Gamma,\mu}(\tilde{D}) =< \mathrm{Ch}(E)\hat{A}(F), [C_\mu] >,$$

where F is the longitudinal bundle of V and $[C_\mu]$ is the Ruelle-Sullivan current associated with the invariant measure μ on (V, F). But in our case, it remains to compute the integral of the top-dimensional component of the differential form $\mathrm{Ch}(E)\hat{A}(F)$ on U and then integrate the result against μ on T. This is easily justified by the fact that $V \setminus (Int(U) \times T)$ is Lebesgue negligible. Thus the proof is complete. □

6 Type II Non Commutative Geometry

The data proposed by A. Connes to define a "geometry" is a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a $*$-algebra represented in a Hilbert space $\mathcal{H}$ and D is an unbounded densely defined self-adjoint operator with a summability condition. To work with such a spectral triple, A. Connes imposes some constraints on the interaction of D with $\mathcal{A}$. This formalism has been very useful especially in exploring index theory for singular spaces. We extend in this section some known results in non commutative geometry to the setting of von-Neumann algebras. The main idea is simple, we replace the algebra $B(\mathcal{H})$ of all bounded operators in $\mathcal{H}$ by a von-Neumann algebra $\mathcal{M}$, i.e. by the commutant of some unitary group in $B(\mathcal{H})$. In the II_∞ factor case, the comparability of projections involves a semi-finite trace which is not the usual trace on $B(\mathcal{H})$, and which can be used to define the index of von Neumann Fredholm operators as in the previous sections.

Any p-dimensional von Neumann spectral triple $(\mathcal{A}, \mathcal{M}, D)$ defines an analytic index map:

$$\mathrm{Ind}_{D,\tau} : K_*(\mathcal{A}) \to \mathbb{R},$$

which is essentially given in the even case by the formula:

$$\mathrm{Ind}_{D,\tau}([e]) := \tau(P_{\mathrm{Ker}(D_e)}) - \tau(P_{\mathrm{Ker}(D_e)^*}) = \dim_\tau(\mathrm{Ker}(D_e)) - \dim_\tau(\mathrm{Ker}(D_e)^*).$$

This index map may be computed by using the natural pairing $< .,. >$ between cyclic cohomology and cyclic homology of the algebra $\mathcal{A}$.

More precisely, we shall see that there exists a natural cyclic $p-$cocycle $\mathrm{Ch}_\tau(\mathcal{M}, D)$ such that:

$$\mathrm{Ind}_{D,\tau}(x) =< \mathrm{Ch}_\tau(\mathcal{M}, D), \mathrm{Ch}(x) >,$$

where $< .,. >$ denotes the natural pairing and $\mathrm{Ch}(x)$ is the Chern-Connes character of x as defined in [25] or in [41].

6.1 Von Neumann spectral triples

In view of polynomial formulae, we shall restrict ourselves to finite dimensional spectral triples. The general case can be treated similarly by extending the definition of θ-summability.

Definition 14. By a von Neumann $p-$summable spectral triple we mean a triple $(\mathcal{A}, \mathcal{M}, D)$ where $\mathcal{M} \subset B(\mathcal{H})$ is a von Neumann algebra faithfully represented in a Hilbert space $\mathcal{H}$ and endowed with a (positive) normal semi-finite faithful trace τ, $\mathcal{A}$ is a $*$-subalgebra of the von Neumann algebra $\mathcal{M}$, and D is a τ-measurable self-adjoint operator such that:

(i) $\forall a \in \mathcal{A}$, the operator $a(D+i)^{-1}$ belongs to the Dixmier ideal $L^{p,\infty}(\mathcal{M}, \tau)$;

(ii) Every element $a \in \mathcal{A}$ preserves the domain of D and the commutator $[D, a]$ belongs to $\mathcal{M}$;

(iii) For any $a \in \mathcal{A}$, the operators a and $[D, a]$ belong to $\cap_{n\in\mathbb{N}} Dom(\delta^n)$, where δ is the unbounded derivation of $\mathcal{M}$ given by $\delta(b) = [|D|, b]$.

When H is $\mathbb{Z}_2$-graded with $\mathcal{A}$ even and D odd, we say that the von Neumann-spectral triple is even and denote by γ the grading involution assumed to belong to $\mathcal{M}$.

In (i) the ideal $L^{p,\infty}(\mathcal{M}, \tau)$ is the ideal composed of operators $T \in \mathcal{M}$ such that $(T^* \circ T)^{p/2}$ belongs to the Dixmier ideal $L^{1,\infty}(\mathcal{M}, \tau)$.

Examples. (1) Let M be a smooth closed (Riemannian) spin manifold of dimension n. Let D be the Dirac operator associated with the spin structure. Then $(\mathcal{A} = C^\infty(M), \mathcal{M} = B(\mathcal{H}), D)$ where $\mathcal{H}$ is the L^2-space of spinors, is a n-summable von Neumann spectral triple. As pointed out by A. Connes, it is possible to recover the Gauss-Riemann calculus on M from $(\mathcal{A}, \mathcal{M}, D)$ [29].

(2) Let (V, F) be a (closed) foliated manifold with a holonomy invariant transverse measure Λ and $p \geq 1$ the dimension of the leaves of (V, F). Let D be a generalized Dirac operator along the leaves of (V, F) acting on sections of a hermitian vector bundle E. Denote by $W^*_\Lambda(V, F; E)$ the von Neumann algebra associated with Λ and E. The holonomy invariant transverse measure Λ gives rise to a semi-finite trace τ_Λ on $W^*_\Lambda(V, F; E)$ given by the formula:

$$\tau_\Lambda(T) := \int_{V/F} \mathrm{Tr}(T_L) d\Lambda(L).$$

Then $(\mathcal{A} = C^\infty(V), W^*_\Lambda(V, F; E), D)$ is a $p-$summable von Neumann spectral triple which is not a type I spectral triple [16].

(3) Let $\Gamma \hookrightarrow \tilde{M} \to M$ be a Galois covering over a closed manifold M of dimension n. Let $\tilde{D}$ be the Γ covering of a generalized Dirac operator D on M, and consider the von Neumann algebra $\mathcal{M}$ of bounded Γ-invariant operators

as defined in [1] with its von Neumann trace Tr_Γ. Then $(\mathcal{A} = C^\infty(M), \mathcal{M}, D)$ is a n-summable von Neumann spectral triple as can be deduced easily from the results of [1].

(4) Let $D = \sum_i a_i(x) D_x^i + b(x)$ be a first order uniformly elliptic differential operator with almost periodic coefficients on $\mathbb{R}^n$. As proved by Shubin in [62], even if not written in these terms, the index of such an operator can be defined by looking at a spectral triple $(\mathcal{A}, \mathcal{M}, \tilde{D})$ that we shall briefly describe. The operator $\tilde{D}$ is a direct integral over the Bohr compactification $\mathbb{R}^n_B$ of operators $\tilde{D}_x$ defined by:

$$\tilde{D}_x = \sum a_i(x+y) D_y^i + b(x+y).$$

So $\tilde{D}$ acts on $L^2(\mathbb{R}^n_B \times \mathbb{R}^n)$. The algebra $\mathcal{A}$ is the algebra $\mathrm{CAP}^\infty(\mathbb{R}^n)$ of smooth almost periodic functions on $\mathbb{R}^n$, and $\mathcal{M}$ is the von Neumann crossed product algebra $L^\infty(\mathbb{R}^n) \rtimes \mathbb{R}^n_{discrete}$ which is a II_∞ factor.

6.2 The analytic Chern-Connes character

As usually, we shall replace D by $sgn(D)$ the sign of D. Let

$$D = F_1 |D|$$

be the polar decomposition of the self-adjoint operator D. To get rid of the possible non invertibility of F_1, we replace the Hilbert space $H = \mathrm{Ker}(D)^\perp \oplus \mathrm{Ker}(D)$ by

$$\mathcal{H} = H \oplus \mathrm{Ker}(D) \simeq H_1 \oplus H_2 \oplus H_3, \tag{6.1}$$

where $H_1 = \mathrm{Ker}(D)^\perp$, $H_2 = \mathrm{Ker}(D)$ and H_3 is an extra copy of $\mathrm{Ker}(D)$. Denote by e_1, e_2 and e_3 the projections onto H_1, H_2 and H_3 respectively. According to the splitting (6.1) of $\mathcal{H}$, set:

$$F = \begin{pmatrix} F_1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = F_1 + V \text{ where } V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

We thus define F, V in the semi-finite von Neumann algebra

$$\tilde{\mathcal{M}} = \mathcal{M} \oplus p_D \mathcal{M} p_D \subset B(\mathcal{H}),$$

which is equipped with the trace $\tilde{\tau} = \tau \oplus \tau$. Finally, embed $\mathcal{A}$ in $\tilde{\mathcal{M}}$ by $a \to a \oplus 0$. We have:

Lemma 10.

(1) $F = F^*$ *and* $F^2 = 1$*;*

(2) $\forall a \in \mathcal{A}, [F, a] \in L^{p,\infty}(\tilde{\mathcal{M}}, \tilde{\tau})$*;*

(3) $\forall a \in \mathcal{A}, aFa = aF_1a$*;*

(4) $\forall a \in \mathcal{A}$, $a(F - F_1) \in L^{p,\infty}(\tilde{\mathcal{M}}, \tilde{\tau})$ *and hence* $a(F - F_1) \in \mathcal{K}(\tilde{\mathcal{M}}, \tilde{\tau})$.

Proof. (1) Trivial.

(2) Note first that ap_D (and hence and p_Da) belongs to $L^{p,\infty}(\mathcal{M}, \tau)$, for any $a \in \mathcal{A}$. Indeed, we have:

$$ap_D = a(D + i)^{-1}(D + i)p_D = ia(D + i)^{-1}p_D \in L^{p,\infty}(\mathcal{M}, \tau).$$

On the other hand we have $[F_1, a] \in L^{p,\infty}(\mathcal{M}, \tau)$ for any $a \in \mathcal{A}$. Indeed we get by easy computations:

$$[F_1, a] = [F_1, a](F_1+p_D)(F_1+p_D) = [F_1, a]F_1(F_1+p_D)+[F_1, a]p_D(F_1+p_D) = [F_1, a]F_1(F_1 + p_D) + F_1ap_D(F_1 + p_D),$$

where $F_1ap_D(F_1+p_D) \in L^{p,\infty}(\mathcal{M}, \tau)$ by the preceding observation. Moreover, we have by straightforward computation:

$$[F_1, a]F_1 = [D, a](D + i)^{-1} + i[F_1, a]F_1(D + i)^{-1} - F_1[|D|, a](D + i)^{-1},$$

and hence $[F_1, a]F_1 \in L^{p,\infty}(\mathcal{M}, \tau)$. It follows that $[F_1, a]F_1(F_1 + p_D) \in L^{p,\infty}(\mathcal{M}, \tau)$ and finally,

$$[F_1, a] \in L^{p,\infty}(\mathcal{M}, \tau) \text{ for any } a \in \mathcal{A},$$

as announced. But we have:

$$[F, a] = [F_1, a] + [V, a].$$

where:

$$[V, a] = Vp_Da(1 - e_3) - (1 - e_3)ap_DVe_3,$$

by straightforward computation. Since $[F_1, a], ap_D$ and p_Da are in the Dixmier ideal $L^{p,\infty}(\mathcal{M}, \tau)$, we finally get:

$$[F, a] \in L^{p,\infty}(\tilde{\mathcal{M}}, \tilde{\tau})$$

and (2) is proved.

(3) is obvious.

(4) We have $a(F - F_1) = aV$ and hence

$$aV(aV)^* = aVV^*a^* = ap_Da.$$

It follows that:

$$\mu_s(aV) = \mu_s((aV)(aV)^*)^{1/2} = \mu_s((ap_D)(ap_D)^*)^{1/2} = \mu_s(ap_D), \text{ for any } s > 0,$$

and the result follows since we know that $ap_D \in L^{p,\infty}(\mathcal{M}, \tau)$. $\square$

We are now in position to define the index map

$$\mathrm{Ind}_{D,\tau} : K_*(\mathcal{A}) \to \mathbb{R}.$$

Assume for simplicity that the spectral triple $(\mathcal{A}, \mathcal{M}, D)$ is even and denote by γ the grading involution. For any self-adjoint idempotent $e \in M_n(\mathcal{A})$, the operator:

$$T = e \circ (F \otimes 1_n) \circ e = e \circ (F_1 \otimes 1_n) \circ e$$

anticommutes with γ and satisfies:

$$T^2 - e = e \circ [F \otimes 1_n, e] \circ (F \otimes 1_n)e = e \circ [F \otimes 1_n, e] \circ [F \otimes 1_n, e], \qquad (6.2)$$

since $F^2 = 1$. It follows that $T^2 - e \in L^{p/2,\infty}(e(\tilde{\mathcal{M}} \otimes \mathrm{End}(\mathbb{C}^n))e, \tilde{\tau} \otimes \mathrm{Tr})$ and hence T is a $(\tilde{\tau} \otimes \mathrm{Tr})$-Fredholm operator in the von Neumann algebra $e(\tilde{\mathcal{M}} \otimes \mathrm{End}(\mathbb{C}^n))e$ acting on $e(\mathcal{H}^n)$.

Denote by $\mathrm{Ind}_{D,\tau}(e)$ the $(\tilde{\tau}\otimes\mathrm{Tr})$-index of the positive part of T acting from $e(H^n_+)$ to $e(H^n_-)$. Now if e, e' are two self-adjoint idempotents representing a class $[e]-[e']$ in $K_0(\mathcal{A})$, then the number $\mathrm{Ind}_{D,\tau}(e) - \mathrm{Ind}_{D,\tau}(e')$ only depends on the class of $[e] - [e'] \in K_0(\mathcal{A})$. The τ-index map thus induces a group morphism:

$$\mathrm{Ind}_{D,\tau} : K_0(\mathcal{A}) \to \mathbb{R}.$$

So, any even von Neumann spectral triple $(\mathcal{A}, \mathcal{M}, D)$ gives rise to an index map:

$$\mathrm{Ind}_{D,\tau} : K_0(\mathcal{A}) \to \mathbb{R},$$

that we shall describe in Theorem 7 as a pairing with some (polynomial) cyclic cocycle over $\mathcal{A}$. Note that the odd case can be treated similarly, see [16]. Let ϕ be a cyclic k-cocycle on the algebra $\mathcal{A}$. As in [25], we shall denote for any $N \geq 1$, by $\phi\sharp\,\mathrm{Tr}$ the cyclic k-cocycle on $M_N(\mathcal{A})$ given by:

$$(\phi\sharp\,\mathrm{Tr})(a^0 \otimes A^0, \cdots, a^k \otimes A^k) := \phi(a^0, \cdots, a^k)\,\mathrm{Tr}(A^0 \cdots A^k),$$

for any $(a^0, \cdots, a^k) \in \mathcal{A}^{k+1}$ and any $(A^0, \cdots, A^k) \in M_N(\mathbb{C})$.

Theorem 7. *Let $(\mathcal{A}, \mathcal{M}, D)$ be an even von Neumann-spectral triple of dimension p with grading involution γ. Let F be the symmetry associated with D as above so that $F^2 = 1$. Then the formula:*

$$\phi_{2k}(a^0, ..., a^{2k}) = (-1)^k \tau(\gamma a^0 [F, a^1] \cdots [F, a^{2k}]);$$

defines, for $k > p/2$, a $2k$-cyclic cocycle on the algebra $\mathcal{A}$ and we have for any projection e in $M_N(\mathcal{A})$:

$$\mathrm{Ind}_{D,\tau}(e) := \mathrm{Ind}_\tau((eFe)_+) = (\phi_{2k}\sharp\,\mathrm{Tr})(e, \cdots, e),$$

independently of the choice of k.

Proof. The proof follows the lines of [25].

That ϕ_{2k} is cyclic is evident. Let $(a^0, \cdots, a^{2k+1}) \in \mathcal{A}^{2k+2}$. Then we have:

$$b(\phi_{2k})(a^0, \cdots, a^{2k+1}) = (-1)^k \tau(\gamma[a^0[F, a^1] \cdots [F, a^{2k}], a^{2k+1}]) \\ = (-1)^k \tau([a^0[F, a^1] \cdots [F, a^{2k}], \gamma a^{2k+1}]) = 0.$$

Therefore the cochain ϕ_{2k} is a cyclic cocycle on $\mathcal{A}$. To compute the τ-index of $(e \circ (F \otimes id_N) \circ e)_+$, we apply the Calderon formula. From the relation (6.2), we deduce that the operator $T = (e \circ (F \otimes id_N) \circ e)_+$ is τ-Fredholm in $e(\mathcal{M} \otimes \mathrm{End}(\mathbb{C}^N))e$ with parametrix given by $S = (e \circ (F \otimes id_N) \circ e)_-$. Moreover, $e - ST$ as well as $e - TS$ are in $L^k(e(\mathcal{M} \otimes \mathrm{End}(\mathbb{C}^N))e, \tau \otimes \mathrm{Tr})$. Therefore Proposition 13 gives:

$$\mathrm{Ind}_\tau((eFe)_+) = (\tau \sharp \mathrm{Tr})(\gamma \circ (e - e \circ (F \otimes id_N) \circ e)^2)^k).$$

Computing $(e - e \circ (F \otimes id_N) \circ e)^2)^k$ and using the relation $e \circ [F \otimes 1_N, e] \circ e = 0$, we obtain [13]:

$$(e - e \circ (F \otimes id_N) \circ e)^2)^k = (-1)^k e \circ [F \otimes 1_N, e]^{2k},$$

and hence the conclusion. □

So associated with any von Neumann spectral triple, there is an index problem which can be stated as follows:

"Give a local formula for the traced index map $K_0(\mathcal{A}) \longrightarrow \mathbb{R}$."

Using Theorem 7, we see that the index problem can be completely stated in the the cyclic cohomology world. It can be stated as follows:

"Find a local even cyclic cocycle ψ on $\mathcal{A}$ such that:

$$\forall e \in K_0(\mathcal{A}), < \psi, e > = < \phi, e >,$$

where ϕ is the cyclic cocycle defined in Theorem 7."

This index problem reduces to the index problem solved by A. Connes and H. Moscovici in [30] if one takes the usual von Neumann algebra of all bounded operators in a Hilbert space with the usual trace.

Examples. In the examples listed after Definition 14, the index problem becomes:

(1) In the first example of Riemannian geometry, we recover the classical index problem which was solved by Atiyah and Singer in [9,29] and which was recalled in the first chapter of these notes. More precisely, we have already observed that the Dirac operator D on a spin closed Riemannian manifold, yields a spectral triple $(\mathcal{A}, B(\mathcal{H}), D)$. The index map in this case thus coincides

with the Atiyah map of Section 1 and the local computation of this map is precisely given by the Atiyah-Singer index theorem.

(2) In the case of measured foliations we recover the measured index problem which was solved by A. Connes in [23].

(3) In the case of Galois coverings, we recover the index problem which was solved by M.F Atiyah in [1].

(4) For almost periodic operators, we obtain the Shubin index problem that was solved in [62]. The index map yields here a morphism:

$$\mathrm{Ind}_{D,\tau} : K^0(\mathbb{R}^n_B) \to \mathbb{R},$$

where $\mathbb{R}^n_B$ is the Bohr compactification of $\mathbb{R}^n$. For further details we refer to [17] where a new proof of Shubin's index theorem is derived.

Up to normalization constants, the sequence ϕ_n of Theorem 7 can be arranged to represent a periodic cyclic cocycle on $\mathcal{A}$ [25], i.e. up to appropriate constants, we have:

$$S(\phi_n) = \phi_{n+2}.$$

The periodic cyclic class obtained is called the Chern-Connes character of the von Neumann spectral triple. In [17] we give a local formula for this Chern-Connes character using Wodzicki residues and following the method of [30]. This local formula unifies all the examples listed above and gives a complete solution to the von Neumann index problem.

To end these notes we give the local formula for the Hochschild class of the Chern-Connes character. This formula is proved in [16].

Theorem 8. *Let $I : \mathrm{HC}^*(\mathcal{A}) \to \mathrm{HH}^*(\mathcal{A})$ be the natural map which corresponds to forgetting the cyclicity of the cocycles. Then the pairing of the Hochschild class $I(\mathrm{Ch}(\mathcal{M}, D))$ of the Chern-Connes character $\mathrm{Ch}(\mathcal{M}, D)$ of the even $(2r)$-von Neumann spectral triple is given by the following local formula:*

$$< I(\mathrm{Ch}(\mathcal{M}, D)), \sum_i a_i^0 \otimes a_i^1 \otimes \cdots \otimes a_i^{2r} >= C_r \sum_i \tau_\omega(\gamma \circ a_i^0 \circ [D, a_i^1] \circ \cdots \circ [D, a_i^{2r}],$$

where γ is the grading involution, C_r is a constant depending only on r and $\sum_i a_i^0 \otimes a_i^1 \otimes \cdots \otimes a_i^{2r}$ is any Hochschild cycle over $\mathcal{A}$.

6.3 The equivariant case

In this last subsection, we briefly explain how the constructions of the previous subsections can be generalized to deal with group actions. For simplicity, we shall only give the details for bounded Fredholm modules, the case of equivariant spectral triples can be easily carried out in a similar way. We point out the close relation of the equivariant von Neumann non commutative geometry with the Lefschetz fixed point theory, see for instance [13–15].

Let $\mathcal{A}$ be as before a (unital) $*$-algebra over $\mathbb{C}$. Let Γ be a given group which acts on $\mathcal{A}$ by automorphisms.

Definition 15. A Γ-invariant p-summable von Neumann Fredholm module over $(\mathcal{A}, \Gamma)$ is a couple $(\mathcal{M}, F)$ such that:

- $\mathcal{M}$ is a von Neumann algebra of operators in a Hilbert space H, with a positive faithful normal trace τ;
- The couple $(\mathcal{A}, \Gamma)$ admits a faithful covariant *-representation (π, U) in $\mathcal{M}$;
- The operator F belongs to $\mathcal{M}$ and satisfies:
  (i) $F^* = F$ and $F^2 = 1$;
  (ii) For any $a \in \mathcal{A}$, the commutator $[F, \pi(a)]$ belongs to the Shatten space $L^p(\mathcal{M}, \tau)$;
  (iii) F is Γ-invariant, i.e.
$$[F, U(\gamma)] = 0, \quad \forall \gamma \in \Gamma.$$

The von Neumann Fredholm module $(\mathcal{M}, F)$ is even if in addition the Hilbert space H admits a $\mathbb{Z}_2$-grading automorphism α which lives in $\mathcal{M}$ and if $\mathcal{A}$ and $U(\Gamma)$ are even for the grading while F is odd for the grading.

We have assumed in the above definition that $\forall \gamma \in \Gamma$, the unitary operator $U(\gamma)$ belongs to the von Neumann algebra $\mathcal{M}$. This assumption simplifies our discussion and can be lightened. Note then that any Γ-invariant von Neumann Fredholm module on $\mathcal{A}$ gives rise to a von Neumann Fredholm module over the discrete crossed product algebra $\mathcal{A} \rtimes \Gamma$, given by the same couple $(\mathcal{M}, F)$.

We shall concentrate on the even case, say the case of even Γ-invariant von Neumann Fredholm modules. The notation $(\mathcal{M}, F)$ hides the Hilbert space H and the trace τ on $\mathcal{M}$ and these data will then be implicit in the sequel. Since the operator $[F, \pi(a)]$ in the above definition is also bounded, Condition (ii) implies in fact that the commutator is a τ-compact operator.

The representation (π, U) is assumed covariant. This means that the action of Γ on $\mathcal{M} \subset B(H)$ preserves the algebra $\mathcal{A}$ and that the induced action coincides with the original one on $\mathcal{A}$, i.e.
$$U(\gamma) \circ \pi(a) \circ U(\gamma^{-1}) = \pi(\gamma(a)) \quad \forall \gamma \in \Gamma \text{ and } \forall a \in \mathcal{A},$$

Let now $(\mathcal{M}, F)$ be a fixed even Γ-invariant von Neumann Fredholm module which is p-summable and denote by $\alpha \in \mathcal{M}$ the $\mathbb{Z}_2$ grading involution.

Then since $(\mathcal{M}, F)$ is a von Neumann Fredholm module over the discrete crossed product algebra $\mathcal{A} \rtimes \Gamma$, we can define its Chern-Connes character as a cyclic cocycle on $\mathcal{A} \rtimes \Gamma$ by [25]:

$$\mathrm{Ch}(\mathcal{M}, F)(b^0, \cdots, b^p) := (-1)^{p/2} \tau(\alpha \circ b^0 \circ [F, b^1] \circ \cdots \circ [F, b^p]),$$

for any $(b^0, \cdots, b^p) \in (\mathcal{A} \rtimes \Gamma)^{p+1}$. Note that $\mathrm{Ch}(\mathcal{M}, F)(b^0, \cdots, b^p)$ is trivial unless p is even. Thus we assume that p is even. Note also that in restriction to $\mathcal{A}$, $\mathrm{Ch}(\mathcal{M}, F)$ induces a cyclic cocycle on $\mathcal{A}$.

Assume for simplicity and for the rest of this subsection that Γ is a finite group and denote by $R(\Gamma)$ the representation ring of Γ. We can state:

Theorem 9. *Let $V : \Gamma \to \mathrm{End}(X)$ be a finite dimensional unitary representation of Γ and let e be a Γ-invariant projection in $\mathcal{A} \otimes \mathrm{End}(X)$. Then:*

(i) For any $\gamma \in \Gamma$, the formula

$$< \mathrm{Ch}^\gamma(\mathcal{M}, F), [e] >:= (\mathrm{Ch}_\Gamma(\mathcal{M}, F) \sharp \,\mathrm{Tr})(e \circ [U(\gamma) \otimes V(\gamma)], e, \cdots, e),$$

defines a pairing with the equivariant K-theory of $\mathcal{A}$;

(ii) The pairing of (i) respects the prime ideal $I_\gamma = \{\chi \in R(\Gamma), \chi(\gamma) = 0\}$ of $R(\Gamma)$ associated with the conjugacy class of γ in Γ and induces an additive map:

$$\mathrm{Ch}^\gamma(\mathcal{M}, F) : K_0^\Gamma(\mathcal{A})_\gamma \to \mathbb{C};$$

(iii) The mapping $\gamma \mapsto < \mathrm{Ch}^\gamma(\mathcal{M}, F), [e] >$ is a central function on Γ and hence the Chern-Connes character furnishes an additive map:

$$< \mathrm{Ch}_\Gamma(\mathcal{M}, F), . >: K_0^\Gamma(\mathcal{A}) \to \hat{R}(\Gamma),$$

where $\hat{R}(\Gamma)$ is the space of central functions on the finite group Γ.

The proof of this theorem is given in [13]. See also 6.4.

Let us define now the equivariant index map associated with any equivariant von Neumann Fredholm module. Let $V : \Gamma \to \mathrm{End}(X)$ be a finite dimensional unitary representation of the finite group Γ. To any Γ-invariant projection $e \in \mathcal{A} \otimes \mathrm{End}(X)$, we associate the operator $eFe := e \circ (F \otimes 1_X) \circ e$ acting on the Γ-Hilbert space $e(H \otimes X)$. Recall that a τ-Fredholm operator in a sub-von Neumann algebra $\mathcal{N}$ of $\mathcal{M}$ is an operator in $\mathcal{N}$ which is invertible in $\mathcal{N}$ modulo τ-compact operators in $\mathcal{N}$ [16].

Lemma 11. *The Γ-invariant self-adjoint operator $e \circ (F \otimes 1_X) \circ e$ is a τ-Fredholm operator in the von Neumann algebra $e(\mathcal{M} \otimes \mathrm{End}(X))e$.*

Proof. We have with obvious notations:

$$(eFe)^2 - e = e[F, e]Fe = e[F, e][F, e].$$

Therefore the operator eFe is τ-Fredholm with inverse (modulo τ-compact operators) given by eFe itself. Finally we point out that eFe is Γ-invariant, since e and F are Γ-invariant. $\square$

The positive part $(eFe)_+$ of the operator eFe admits a von Neumann Γ-index which lives in the ring $\hat{R}(\Gamma)$ of complex central functions on Γ. More precisely, if $p_{e,F}$ and $p'_{e,F}$ are respectively the projections onto the kernel and cokernel of $(eFe)_+$, then the von Neumann Γ-index $\mathrm{Ind}_\Gamma((eFe)_+) : \Gamma \to \mathbb{C}$ of $(eFe)_+$ is defined by:

$$\mathrm{Ind}_\Gamma((eFe)_+)(\gamma) := (\tau\sharp\,\mathrm{Tr})([U(\gamma)\otimes V(\gamma)]\circ p_{e,F}) \\ - (\tau\sharp\,\mathrm{Tr})([U(\gamma)\otimes V(\gamma)]\circ p'_{e,F}).$$

That this is a central function is obvious from the trace property of τ and the Γ-invariance of $p_{e,F}$ and $p'_{e,F}$.

Theorem 10. *With the above notations we have the following polynomial equivariant index formula in $\hat{R}(\Gamma)$:*

$$\mathrm{Ind}_\Gamma((eFe)_+) =< \mathrm{Ch}_\Gamma(\mathcal{M}, F), [e] > .$$

Proof. Let $e \in \mathcal{A}\otimes \mathrm{End}(X)$ be any Γ-invariant projection with $V : \Gamma \to \mathrm{End}(X)$ a finite dimensional unitary representation of Γ as above. Denote by P the Γ-invariant τ-Fredholm operator:

$$P := [e(F\otimes id_X)e]_+ : e(\mathcal{H}^+\otimes X) \to e(\mathcal{H}^-\otimes X).$$

Denote by Q the parametrix of P which is the negative part of the operator $e(F\otimes id_X)e$. Then we have:

$$e - QP = -e\circ [F\otimes 1_X, e]\circ [F\otimes 1_X, e],$$

and the corresponding result for $e - PQ$. Therefore:

$$e - QP \in L^{p/2}(e(\mathcal{M}\otimes \mathrm{End}(X))e, \tau\sharp\,\mathrm{Tr}) \text{ and} \\ e - PQ \in L^{p/2}(e(\mathcal{M}\otimes \mathrm{End}(X))e, \tau\sharp\,\mathrm{Tr}).$$

Hence we can apply the equivariant Calderon formula (See Lemma 5.2 below) to compute the equivariant von Neumann index of P. This gives:

$$\mathrm{Ind}_\Gamma((eFe)_+)(\gamma) = \mathrm{Ind}^\Gamma(P)(\gamma) = \\ (\tau\sharp\,\mathrm{Tr})([U(\gamma)\otimes V(\gamma)]\circ (1-QP)^{p/2}) - (\tau\sharp\,\mathrm{Tr})(U(\gamma)\circ (1-PQ)^{p/2}).$$

Using the supertrace defined by $S\tau(T) := \tau(\alpha\circ T)$ where α is the involution associated with the $\mathbb{Z}_2$-grading of H, we can rewrite the Γ-index as:

$$\mathrm{Ind}_\Gamma((eFe)_+)(\gamma) = (S\tau\sharp\,\mathrm{Tr})(U(\gamma)\circ (e - [e(F\otimes id_X)e]^2)^{p/2}).$$

The above computation of $e-[e(F\otimes id_X)e]^2$ then gives:

$$\mathrm{Ind}_\Gamma((eFe)_+)(\gamma)=(-1)^{p/2}(S\tau\sharp\,\mathrm{Tr})(U(\gamma)\circ e\circ[F,e]^p),$$

and hence the conclusion. □

Lemma 12. *(Equivariant Calderon formula)*

Let $U:\Gamma\to\mathcal{U}(\mathcal{H})$ be a unitary representation of the compact Lie group Γ in a Hilbert space $\mathcal{H}$. Assume that conjugation by the elements of Γ in the algebra $B(\mathcal{H})$ of bounded operators in $\mathcal{H}$ preserves a von Neumann algebra $\mathcal{N}$ admiting a faithful normal trace τ. Let P,Q be two Γ-invariant operators in $\mathcal{N}$ such that $1-QP$ and $1-PQ$ belong the Schatten ideal $L^n(\mathcal{N},\tau)$ for some $n\geq 1$. Then

$$\forall\gamma\in\Gamma,\mathrm{Ind}^\Gamma(P)(\gamma)=\tau(U(\gamma)\circ[1-QP]^n)-\tau(U(\gamma)\circ[1-PQ]^n).$$

Proof. The proof of this lemma is classical. It is given in [16] in the non equivariant case but still works in the equivariant setting with minor modifications. □

Corollary 3. *Let $(B(H),F)$ be a Γ-invariant even Fredholm module over the Γ-algebra $\mathcal{A}$, with the usual trace. Then for any $\gamma\in\Gamma$, we have:*

$$\left\langle \mathrm{Ch}^\gamma(B(H),F),K^\Gamma(\mathcal{A})\right\rangle\quad\subset R(\Gamma).$$

This is an integrality theorem. When Γ is for instance the cyclic group $\mathbb{Z}_n$, the above corollary becomes:

$$\left\langle \mathrm{Ch}^\gamma(B(H),F),K^\Gamma(\mathcal{A})\right\rangle\quad\subset \mathbb{Z}[e^{2i\pi/n}].$$

6.4 Back to the equivariant periodic homology

Let again $\mathcal{B}$ be a $\mathbb{C}$-algebra with unit. The complex of Hochschild homology of $\mathcal{B}$ is defined as the complex $(C_*(\mathcal{B}),b)$, where $C_k(\mathcal{B}):=\mathcal{B}\otimes(\mathcal{B}/\mathbb{C})^{\otimes_k}$ and:

$$b(b^0\otimes\cdots\otimes b^k)=\sum_{j=0}^{k-1}(-1)^j b^0\otimes\cdots\otimes b^{j-1}\otimes b^jb^{j+1}\otimes\cdots\otimes b^k+$$
$$(-1)^k b^kb^0\otimes b^1\otimes\cdots\otimes b^{k-1}.$$

In the same way we can define the operator $B:C_k(\mathcal{B})\to C_{k+1}(\mathcal{B})$ as the transpose of the cohomological operator B. Then again we have $b^2=0$, $B^2=0$ and $bB+Bb=0$. This enables to define Hochschild, cyclic and periodic cyclic homologies of $\mathcal{B}$ [53].

Assume in addition that $\mathcal{B}$ is endowed with an action of the finite group Γ. The subspace of $C_*(\mathcal{B})$ generated by differences $\gamma(x)-x$ where γ runs over Γ and x runs over $C_*(\mathcal{B})$ and the action of Γ on $C_*(\mathcal{B})$ is the diagonal one, is preserved by the operators b and B. We can therefore consider the quotient bicomplex that we denote by $(C_*^\Gamma(\mathcal{B}), b, B)$. The homologies thus obtained will be denoted by $\mathrm{HH}_*^\Gamma(\mathcal{B})$, $\mathrm{HC}_*^\Gamma(\mathcal{B})$ and $\mathrm{HP}_*^\Gamma(\mathcal{B})$ and called respectively, the co-invariant Hochschild, cyclic and periodic cyclic homology of $\mathcal{B}$.

Assume now that $\mathcal{B} = \mathcal{A} \rtimes \Gamma$ is the discrete crossed product algebra considered in the previous sections. Denote by $C_*^{equiv}(\mathcal{B})$ the quotient of the Hochschild chains by the subspace generated by differences of the type

$$b^0 \otimes \cdots \otimes b^j U(\gamma) \otimes b^{j+1} \otimes \cdots \otimes b^k - b^0 \otimes \cdots \otimes b^j \otimes \gamma(b^{j+1}) U(\gamma) \otimes \cdots \otimes b^k.$$

Again the operators b and B induce well defined operators on this quotient space and the homologies will be denoted by $\mathrm{HH}_*^{equiv}(\mathcal{B})$, $\mathrm{HC}_*^{equiv}(\mathcal{B})$ and $\mathrm{HP}_*^{equiv}(\mathcal{B})$ respectively and called the co-equivariant homologies. We have a natural chain map

$$C_*^\Gamma(\mathcal{B}) \to C_*^{equiv}(\mathcal{B}),$$

which induces morphisms between the corresponding homologies. If $V : \Gamma \to \mathrm{End}(X)$ is a finite dimensional representation of Γ, then we can consider the discrete crossed product algebra $[\mathcal{A} \otimes \mathrm{End}(X)] \rtimes \Gamma$. If $b^0 \otimes \cdots \otimes b^k$ is a Hochschild chain on $[\mathcal{A} \otimes \mathrm{End}(X)] \rtimes \Gamma$, then we define a Hochschild chain on $\mathcal{A} \rtimes \Gamma$ as the centre projection $< b^0 \otimes \cdots \otimes b^k >$ of $b^0 \otimes \cdots \otimes b^k$ defined using an orthonormal frame of X by:

$$< b^0 \otimes \cdots \otimes b^k > := \sum_{i_0, \cdots, i_k} b^0_{i_0 i_1} \otimes \cdots \otimes b^k_{i_k i_0}$$

Lemma 13. *The projection*

$$< . >: C_*([\mathcal{A} \otimes \mathrm{End}(X)] \rtimes \Gamma) \to C_*(\mathcal{A} \rtimes \Gamma),$$

is a chain map between the periodic bicomplexes and it induces well defined maps between the co-invariant bicomplexes and the co-equivariant bicomplexes.

Proof. That $< . >$ is a chain map is classical [53]. Let us compute the expression:

$$\begin{aligned} &< b^0[U(\gamma) \otimes V(\gamma)] \otimes b^1 \otimes \cdots \otimes b^k > - \\ &\qquad < b^0 \otimes \gamma(b^1)[U(\gamma) \otimes V(\gamma)] \otimes b^2 \cdots \otimes b^k > . \end{aligned} \tag{6.3}$$

We notice that:

$$\gamma(b^1)_{ij} = \sum_{k,h} V(\gamma)_{ik}\gamma(b^1_{kh})V(\gamma^{-1})_{hj},$$

Thus we obtain that the difference (6.3) is equal to:

$$\sum_{i_0,i_0',i_1,\cdots,i_k} [V(\gamma)_{i_0'i_1} b^0_{i_0i_0'} U(\gamma) \otimes b^1_{i_1i_2} \cdots b^k_{i_ki_0} -$$
$$b^0_{i_0i_0'} \otimes V(\gamma)_{i_0'i_1}\gamma(b^1)_{i_1i_2}U(\gamma) \otimes b^2_{i_2i_3} \otimes \cdots \otimes b^k_{i_ki_0}].$$

Therefore the operator $< . >$ induces a well defined operator between the co-equivariant quotient bicomplexes.

A similar computation gives the result for the co-invariant homologies. □

Proposition 16. *Assume that $V : \Gamma \to \mathrm{End}(X)$ is a finite dimensional representation of Γ. Let e be a Γ-invariant projection in $\mathcal{A} \otimes \mathrm{End}(X)$. We define the sequence $(\mathrm{Ch}_n(e,\gamma))_{n\geq 0}$ by:*

$$\mathrm{Ch}_n(e,\gamma) := (-1)^n \frac{(2n)!}{n!} < (e-1/2)[U(\gamma)\otimes V(\gamma)]\otimes e\otimes\cdots\otimes e > \in C_{2n}(\mathcal{A}\rtimes\Gamma),$$

Then we have:

(i) For any $\gamma \in \Gamma$, the co-equivariant class $[\mathrm{Ch}(e,\gamma)]$ of $\mathrm{Ch}(e,\gamma) := (\mathrm{Ch}_n(e,\gamma))_{n\geq 0}$ is $(b+B)$ closed;

(ii) For any $[e] \in K_0^\Gamma(\mathcal{A})$ and for any $n \geq 0$, the map:

$$\Gamma \to C_*^{equiv}(\mathcal{A}\rtimes\Gamma) \text{ defined by } \gamma \mapsto [\mathrm{Ch}_n(e,\gamma)] \in C_{2n}^{equiv}(\mathcal{A}\rtimes\Gamma),$$

is a central function on Γ.

(iii) For any $\gamma \in \Gamma$, the co-equivariant periodic cyclic homology class of $\mathrm{Ch}(e,\gamma)$ only depends on the equivariant K-theory class of e in $\mathcal{A}$;

Proof. (i) The computation of $b(\mathrm{Ch}_n(e,\gamma))$, gives:

$$b(\mathrm{Ch}_{n+1}(e)) = \frac{(-1)^{n+1}}{2}\frac{(2n+2)!}{(n+1)!} < U(\gamma)\otimes e\otimes\cdots\otimes e > .$$

The computation in $C_*^{equiv}(\mathcal{A}\rtimes\Gamma)$ of $B([\mathrm{Ch}_n(e,\gamma)])$ gives:

$$B([\mathrm{Ch}_n(e,\gamma)]) = (2n+1)(-1)^n\frac{(2n)!}{n!}[< U(\gamma)\otimes e\otimes\cdots\otimes e >].$$

Hence we obtain:

$$b([\mathrm{Ch}_{n+1}(e,\gamma)]) + B([\mathrm{Ch}_n(e,\gamma)]) = \frac{(2n+1)!(-1)^{n+1}}{n!}\left(\frac{2n+2}{n+1} - 2\right) = 0.$$

(ii) Let $(\gamma_1, \gamma_2) \in \Gamma^2$ and let us compare $\mathrm{Ch}_n(e, \gamma_1\gamma_2)$ with $\mathrm{Ch}_n(e, \gamma_2\gamma_1)$. We have:

$$[\mathrm{Ch}_n(e, \gamma_1\gamma_2)] = (-1)^n \frac{(2n)!}{n!}$$
$$[< (e - 1/2)[(U(\gamma_1) \otimes V(\gamma_1)) \circ (U(\gamma_2) \otimes V(\gamma_2))] \otimes e \otimes \cdots \otimes e >] .$$

Since we are using the co-equivariant classes, we can move $U(\gamma_2) \otimes V(\gamma_2)$ to the right in this expression and use the Γ-invariance of e to move $U(\gamma_2) \otimes V(\gamma_2)$ until we obtain:

$$\mathrm{Ch}_n(e, \gamma_1\gamma_2) = (-1)^n \frac{(2n)!}{n!}$$
$$< (U(\gamma_2) \otimes V(\gamma_2))(e - 1/2)(U(\gamma_1) \otimes V(\gamma_1)) \otimes e \otimes \cdots \otimes e > .$$

Again using the Γ-invariance of e we finally deduce that:

$$\mathrm{Ch}_n(e, \gamma_1\gamma_2) = \mathrm{Ch}_n(e, \gamma_2\gamma_1)$$

(iii) The proof given in [53] still works in the co-equivariant setting. $\square$

The above Chern-Connes character is actually defined on the Γ-equivariant K-theory of the crossed product algebra $\mathcal{A} \rtimes \Gamma$. This is obvious from the proof. Note that if the algebra $\mathcal{A}$ can be endowed with a Banach norm, then the above Chern-Connes character belongs to the equivariant entire cyclic homology of $\mathcal{A}$ as defined in [29] or as defined in [41].

The central function which assigns to each $\gamma \in \Gamma$ the class of $\mathrm{Ch}(e, \gamma)$ in $\mathrm{HP}_*^{equiv}(\mathcal{A} \rtimes \Gamma)$ is called the equivariant Chern-Connes character of the projection e and denoted by $\mathrm{Ch}^{equiv}(e)$. It belongs to $\hat{R}(\Gamma) \otimes \mathrm{HP}_*^{equiv}(\mathcal{A} \rtimes \Gamma)$.

Note that $\mathrm{Ch}(e, \gamma)$ is not a cycle in the co-invariant periodic cyclic homology. In summary, we have:

$$\mathrm{Ch}^{equiv} : K^\Gamma(\mathcal{A}) \to \hat{R}(\Gamma) \otimes \mathrm{HP}_*^{equiv}(\mathcal{A} \rtimes \Gamma),$$

and for any equivariant von Neumann Fredholm module $(\mathcal{M}, F)$, we have seen that

$$\mathrm{Ch}_\Gamma(\mathcal{M}, F) \in \mathrm{HP}_\Gamma^*(\mathcal{A} \rtimes \Gamma),$$

is the image of an element $\mathrm{Ch}_{equiv}(\mathcal{M}, F)$ in $\mathrm{HP}^*_{equiv}(\mathcal{A} \rtimes \Gamma)$.

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