Index Theorems and Noncommutative Topology

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Summary. These lecture notes are mainly devoted to a K-theory proof of the Atiyah-Singer index theorem. Some applications of the K-theory to noncommutative topology are also given.

Introduction

Topological K-theory for locally compact spaces was introduced by Atiyah and Singer in their proof of the index theorem for elliptic operators. During the last two decades, topological K-theory and elliptic operators have become important tools in topology. For instance, index theory for C^*-algebras was used to compute the K-theory of many "noncommutative" spaces, leading to the so called *Baum-Connes conjecture.* Also, G. Kasparov used K-theory and K-homology for C^*-algebras to investigate the *Novikov's conjecture on higher signatures* for large classes of groups. With the emergence of A. Connes non-commutative geometry, one can say that K-theory and elliptic theory for complex algebras have become usual tools in topology.
The purpose of this course is to introduce the main ideas of the Atiyah-Singer index theorem for elliptic operators. We also show how K-theory for C^*-algebras can be used to study the leaf space of a foliation. We would like to thank the referee for many suggestions making these notes more transparent.

1 Index of a Fredholm operator

1.1 Fredholm operators

Recall that an endomorphism $T \in B(H)$ of a Hilbert space H is called *compact* if it is a norm limit of finite rank endomorphisms of H or, equivalently, if the image $T(B)$ of the unit ball B of H is compact. Let H_1, H_2 be two Hilbert spaces and $T : H_1 \longrightarrow H_2$ be a linear continuous map. We say that T is *Fredholm* if there exists a continuous linear map $S : H_2 \longrightarrow H_1$ such that the operators $ST - I \in B(H_1)$ and $TS - I \in B(H_2)$ are compact. For instance, if the operator $K \in B(H)$ is compact, $T = I + K \in B(H)$ is Fredholm.

Exercise 1.1.1. *Let $K : [0,1] \times [0,1] \longrightarrow \mathbb{C}$ be a continuous map. Show that the bounded operator T defined on $H = L^2([0,1], dx)$ by $Tf(x) = \int_0^1 K(x,y)f(y)dy$ is compact. Is T a Fredholm operator?*

Exercise 1.1.2. *Let $(e_0, e_1, ...)$ be the natural orthonormal basis of $H = l^2(\mathbb{N})$. Show that the unilateral shift $S \in B(H)$ defined by $Se_n = e_{n+1}$ is Fredholm.*

Let us give a characterization of Fredholm operators involving only the kernel and the image of the operator.

Theorem 1.1.3. *Let $T : H_1 \longrightarrow H_2$ be a bounded operator. The following conditions are equivalent:*

(i) T is Fredholm;

(ii) $Ker(T)$ is a finite dimensional subspace of H_1 and $Im(T)$ is a closed finite codimensional subspace of H_2.

Proof $(i) \Longrightarrow (ii)$. If T is Fredholm, the restriction of the identity map of H_1 to $Ker(T)$ is compact since it is equal to the restriction of $I - ST$ to $Ker(T)$. It follows that the unit ball of $Ker(T)$ is compact and hence $Ker(T)$ is finite dimensional. On the other hand, since T^* is Fredholm, the subspace $Im(T)^\perp = Ker(T^*)$ is also finite dimensional so that we only have to prove that $Im(T)$ is closed. Let $y_n \in Im(T)$ be a sequence converging to $y \in H_2$, and write $y_n = Tx_n$ with $x_n \in Ker(T)^\perp$. The sequence (x_n) is bounded because if not we could choose a subsequence $(x_{n_k})_k$ with $\|x_{n_k}\| \xrightarrow[k \to +\infty]{} +\infty$. By compactnees of $ST - I$, we may assume in addition that

$$(ST - I)\left(\frac{x_{n_k}}{\|x_{n_k}\|}\right) \xrightarrow[k \to +\infty]{} z \in H_1.$$

Since $ST\left(\frac{x_{n_k}}{\|x_{n_k}\|}\right) = \frac{S(y_{n_k})}{\|x_{n_k}\|} \xrightarrow[k\to+\infty]{} 0$, we would get $\frac{x_{n_k}}{\|x_{n_k}\|} \longrightarrow -z$, a fact which implies $z \in Ker(T)^{\perp}$ and $\|z\| = 1$. On the other hand, we have

$$\frac{y_{n_k}}{\|x_{n_k}\|} = T\left(\frac{x_{n_k}}{\|x_{n_k}\|}\right) \longrightarrow -T(z)$$

so that $T(z) = 0$. We thus would have $z \in Ker(T) \cap Ker(T)^{\perp} = \{0\}$, a fact which contradicts $\|z\| = 1$. The sequence (x_n) is thus bounded. By compactness of $ST - I$, we may choose a subsequence $(x_{n_k})_k$ such that

$$(ST - I)(x_{n_k}) \xrightarrow[k\to+\infty]{} z \in H_1.$$

It follows that $x_{n_k} \xrightarrow[k\to+\infty]{} Sy - z \in H_1$ and hence $y = Tu$ with $u = Sy - z$. This shows that $Im(T)$ is closed.
$(ii) \Longrightarrow (i)$ By the Hahn-Banach theorem, $T_{|Ker(T)^{\perp}} : Ker(T)^{\perp} \longrightarrow Im(T)$ is an isomorphism. Let $S_1 : Im(T) \longrightarrow Ker(T)^{\perp}$ be the inverse of $T_{|Ker(T)^{\perp}}$, and consider the operator S which coincides with S_1 on $Im(T)$, and which is 0 on $Ker(T^*)$. We have $ST - I = -p_{Ker(T)}$, $TS - I = -p_{Ker(T^*)}$, where p_K denotes the orthogonal projection on the closed subspace K, so that $ST - I$ and $TS - I$ are finite rank operators. QED

In the sequel, we shall denote by $Fred(H_1, H_2)$ the set of Fredholm operators from H_1 to H_2. It is easy to see that $Fred(H_1, H_2)$ is an open subset of $B(H_1, H_2)$ equipped with the norm topology.

1.2 Toeplitz operators

Toeplitz operators are good examples of "pseudodifferential" operators on $\mathbb{S}^1$. Let $H = H^2(\mathbb{S}^1)$ be the *Hardy* space, i.e. the subspace of $L^2(\mathbb{S}^1)$ generated by the exponentials $e_n(t) = e^{i2\pi nt}(n = 0, 1, ...)$, and denote by P the orthogonal projection onto $H^2(\mathbb{S}^1)$.

Definition 1.2.1. *Let $\varphi : \mathbb{S}^1 \longrightarrow \mathbb{C}$ be a continuous map. We call Toeplitz operator of symbol φ the bounded operator $T_\varphi \in B(H)$ defined by:*

$$T_\varphi(f) = P(\varphi f),\ f \in H^2(\mathbb{S}^1).$$

For instance, T_{e_1} is the unilateral shift S and $T_{e_{-1}}$ its adjoint S^*.

Proposition 1.2.2. *Let $\varphi : \mathbb{S}^1 \longrightarrow \mathbb{C}$ be a non vanishing continuous map. Then T_φ is a Fredholm operator.*

Proof. It suffices to prove that $T_\varphi T_\psi - T_{\varphi\psi}$ is compact for any $\varphi, \psi \in C(\mathbb{S}^1)$. Indeed, for a non vanishing $\varphi : \mathbb{S}^1 \longrightarrow \mathbb{C}$, the inverse $\psi = \frac{1}{\varphi}$ is continuous and the above assertion will imply that $T_\varphi T_\psi - I$ and $T_\psi T_\varphi - I$ are compact. To show that $T_\varphi T_\psi - T_{\varphi\psi}$ is compact for any $\varphi, \psi \in C(\mathbb{S}^1)$ we may assume, by using the Stone-Weierstrass theorem and the continuity of the map

$$\varphi \in C(\mathbb{S}^1) \longrightarrow T_\varphi \in B(H),$$

that φ, ψ are trigonometric polynomials. By linearity, we are finally reduced to prove that $T_{e_n} T_{e_m} - T_{e_{n+m}}$ is compact for any $n, m \in \mathbb{Z}$. But we have:

$$(T_{e_n} T_{e_m} - T_{e_{n+m}})(e_k) = \begin{cases} -e_{n+m+k} & \text{if } -(n+m) \leq k < -m \\ 0 & \text{if not} \end{cases}$$

so that $T_{e_n} T_{e_m} - T_{e_{n+m}}$ is a finite rank operator, and hence is compact. QED

Exercise 1.2.3. *Let $\varphi : \mathbb{S}^1 \longrightarrow \mathbb{C}$ be a continuous map such that T_φ is compact. Show that $\varphi = 0$.*

1.3 The index of a Fredholm operator

Definition 1.3.1. *Let $T : H_1 \longrightarrow H_2$ be a Fredholm operator. The integer:*

$$Ind(T) = dimKer(T) - codimKer(T) = dimKer(T) - dimKer(T^*) \in \mathbb{Z}$$

is called the index of T.

The main property of the index is its homotopy invariance:

Proposition 1.3.2. *For any norm continuous path $t \in [0,1] \longrightarrow T_t \in Fred(H_1, H_2)$ of Fredholm operators, we have $Ind(T_0) = Ind(T_1)$.*

The homotopy invariance of the index is a consequence of the continuity of the map $T \longrightarrow Ind(T)$ on $Fred(H_1, H_2)$. For a proof, see [15], theorem 2.3, page 224. Let us give a consequence of this homotopy invariance:

Corollary 1.3.3. *Let $T_1, T_2 \in B(H)$ be two Fredholm operators. Then, $T_1 T_2$ is a Fredholm operator such that $Ind(T_1 T_2) = Ind(T_1) + Ind(T_2)$.*

Proof. It is clear from the definition that T_1T_2 is Fredholm. For any $t \in [0, \pi/2]$, set $F_t = \begin{pmatrix} T_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} cost & sint \\ -sint & cost \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} cost & -sint \\ sint & cost \end{pmatrix}$. We thus define a homotopy of Fredholm operators between $F_0 = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ and $F_{\pi/2} = \begin{pmatrix} T_1T_2 & 0 \\ 0 & I \end{pmatrix}$. By proposition 1.3.2., we get:

$$Ind(T_1) + Ind(T_2) = Ind(F_0) = Ind(F_{\pi/2}) = Ind(T_1T_2). \text{ QED}$$

Exercise 1.3.4. *Let $F, K \in B(H_1, H_2)$. Assume that F is Fredholm and K compact. Show that $F + K$ is Fredholm and $Ind(F + K) = Ind(F)$.*

1.3.5 Meaning of the index. Let $T \in Fred(H_1, H_2)$. By the Hahn-Banach theorem, $T_{|Ker(T)^\perp} : Ker(T)^\perp \longrightarrow Im(T)$ is an isomorphism. If $Ind(T) = 0$, we can choose an isomorphism $R : Ker(T) \longrightarrow Im(T)^\perp$ and the operator $\widetilde{T}$ equal to T on $Ker(T)^\perp$ and to R on $Ker(T)$ is a finite rank perturbation of T which is an isomorphism. Conversely, if there exists a finite rank operator R such that $\widetilde{T} = T + R$ is an isomorphism, then $Ind(T) = Ind(\widetilde{T}) = 0$ (cf. exercise 1.3.4.). This shows that $Ind(T)$ is the obstruction to make T an isomorphism by a finite rank perturbation. Let us give another interpretation of the index of a Fredholm operator. To avoid unnecessary technicalities, we shall assume that $H_1 = H_2 = H$. Denote by $Calk(H) = B(H)/K(H)$ the quotient of the algebra $B(H)$ by the closed ideal $K(H)$ of compact operators on H. The Calkin algebra $Calk(H)$ is a Banach*-algebra[1] with unit for the quotient norm. Denote by $\pi : B(H) \longrightarrow B(H)/K(H) = Calk(H)$ the canonical projection. By definition, an operator $T \in B(H)$ is Fredholm if and only if $\pi(T)$ is invertible in $Calk(H)$.The index of T is, by the preceding discussion, the obstruction to lift $\pi(T)$ to some invertible element in $B(H)$. The following proposition shows that we can always choose an invertible lift $X \in M_2(B(H))$ for the 2×2 matrix $\begin{pmatrix} \pi(T) & 0 \\ 0 & \pi(T)^{-1} \end{pmatrix} \in M_2(Calk(H))$ with coefficients in $Calk(H)$, and that $Ind(T)$ can be interpreted as the formal difference of the projections:

[1] Recall that a Banach*-algebra B is a Banach algebra with an involution $x \longmapsto x^*$ such that $\|x^*\| = \|x\|$ for any $x \in B$. When B has a unit 1, we always ask that $\|1\| = 1$.

$$X\begin{pmatrix}1&0\\0&0\end{pmatrix}X^{-1}\in M_2(I+K(H)) \text{ and } \begin{pmatrix}1&0\\0&0\end{pmatrix}\in M_2(I+K(H)).$$

Proposition 1.3.6. *Let $T\in B(H)$ be a Fredholm operator. Denote by S the operator which is zero on $Ker(T^*)$ and which is equal on $Im(T)$ to the inverse of the isomorphism $T_{|Ker(T)^\perp}:Ker(T)^\perp\longrightarrow Im(T)$. Let e and f be the orthogonal projections on $Ker(T)$ and $Ker(T^*)$.*

(i) $X=\begin{pmatrix}T&f\\e&S\end{pmatrix}$ *is an invertible element in* $M_2(B(H))=B(H)\otimes M_2(\mathbb{C})$ *such that* $(\pi\otimes I_2)(X)=\begin{pmatrix}\pi(T)&0\\0&\pi(T)^{-1}\end{pmatrix}$;

(ii) $X\begin{pmatrix}1&0\\0&0\end{pmatrix}X^{-1}-\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}-f&0\\0&e\end{pmatrix}$, *so that*

$$Ind(T)=Trace\left(X\begin{pmatrix}1&0\\0&0\end{pmatrix}X^{-1}-\begin{pmatrix}1&0\\0&0\end{pmatrix}\right).$$

Proof. (i) By direct calculation, we see that $\begin{pmatrix}S&e\\-f&T\end{pmatrix}\in M_2(B(H))$ is an inverse for X. Since $ST=1-e$, we have $\pi(S)\pi(T)=1$ and hence $\pi(S)=\pi(T)^{-1}$. It follows that $(\pi\otimes I_2)(X)=\begin{pmatrix}\pi(T)&0\\0&\pi(T)^{-1}\end{pmatrix}$ and (i) is proved.
(ii) We get by direct computation:

$$X\begin{pmatrix}1&0\\0&0\end{pmatrix}X^{-1}-\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1-f&0\\0&e\end{pmatrix}-\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}-f&0\\0&e\end{pmatrix}. \text{ QED}$$

Exercise 1.3.7. *Let $T\in B(H_1,H_2)$ and assume that there exists $S\in B(H_2,H_1)$ and a positive integer n such that $(ST-I)^n$ and $(TS-I)^n$ are trace class operators.*
Show that T is Fredholm and prove that:

$$Ind(T)=Trace((ST-I)^n)-Trace((TS-I)^n).$$

Let us now compute the index of a Toeplitz operator. To this end, recall that the degree of a continuous map $\varphi:\mathbb{S}^1\longrightarrow\mathbb{C}$ which does not vanish is by definition the degree $\frac{1}{2\pi i}\int_0^1 \frac{\psi'(t)}{\psi(t)}dt$ of any smooth function $\psi : \mathbb{S}^1 \longrightarrow \mathbb{C}$ sufficiently close to φ.

Theorem 1.3.8. *For any non vanishing continuous map $\varphi : \mathbb{S}^1 \longrightarrow \mathbb{C}$, we have:*

$$Ind(T_\varphi) = -deg(\varphi).$$

Proof. Set $n = deg(\varphi)$. Since φ is homotopic to the map $t \in \mathbb{S}^1 \longrightarrow \frac{\varphi(t)}{|\varphi(t)|} \in \mathbb{S}^1$ whose degree is n, there exists by Hopf's theorem a continuous homotopy $(\varphi_t)_{0\leq t\leq 1}$ between $\varphi_0 = \varphi$ and $\varphi_1(s) = e^{i2\pi ns}$ such that $\varphi_t : \mathbb{S}^1 \longrightarrow \mathbb{C}$ is a continuous invertible map for each $t \in [0,1]$. Since the index of a Fredholm operator and the degree of a continuous invertible map are homotopy invariants, we get:

$$Ind(T_\varphi) = Ind(T_{\varphi_1}) = Ind(T_{e_n}) = n.Ind(T_{e_1}) = -n = -deg(\varphi),$$

and the proof is complete. QED

Exercise 1.3.9. *Let T be the operator on $l^2(\mathbb{Z})$ defined by:*

$$T(e_n) = \begin{cases} \frac{n}{\sqrt{1+n^2}} e_{n-1} & if\ n \geq 0 \\ \frac{n}{\sqrt{1+n^2}} e_n & if\ n \leq 0 \end{cases}$$

where $(e_n)_{n\in\mathbb{Z}}$ is the canonical orthonormal basis of $l^2(\mathbb{Z})$. Show that T is a Fredholm operator of index equal to 1.

2 Elliptic operators on manifolds

Elliptic operators on manifolds give rise to Fredholm operators, whose analytical index can be computed from the *principal symbol*, which is a purely topological data.

2.1 Pseudodifferential operators on $\mathbb{R}^n$

Pseudodifferential operators of order m on $\mathbb{R}^n$ generalize differential operators. They are constructed from *symbols* of order m. In what follows, we shall write as usually: $D_x^\alpha = \frac{\partial^{|\alpha|}}{i^{|\alpha|}\partial x^\alpha}$.

Definition 2.1.1. *Let $m \in \mathbb{R}$. A smooth matrix-valued function $p = p(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be a symbol of order m if there is, for any pair (α, β) of multiindices, a constant $C_{\alpha,\beta} \geq 0$ such that:*

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \text{ for all } x, \xi.$$

We shall denote by S^m the space of symbols of order m. Note that $S^m \subset S^{m'}$ if $m \leq m'$. We shall say that a symbol p of order m has a formal development $p \sim \sum_{j=1}^{\infty} p_j$ with $p_j \in S^{m_j}$ if there exists, for each positive integer m, an integer N such that $p - \sum_{j=1}^{k} p_j \in S^{-m}$ for any $k \geq N$. The following result (see for instance [16], proposition 3.4, page 179) is very close to Borel's result on the existence of a smooth function having a given Taylor expansion at some point:

Proposition 2.1.2. *For any formal series $\sum_{j=1}^{\infty} p_j$ with $p_j \in S^{m_j}$ and $m_j \longrightarrow -\infty$, there exists a symbol p of order m such that we have $p \sim \sum_{j=1}^{\infty} p_j$.*

To any $p \in S^m$ with values in $M_k(\mathbb{C})$, we associate a linear operator:

$$P = Op(p) : S(\mathbb{R}^n) \otimes \mathbb{C}^k \longrightarrow S(\mathbb{R}^n) \otimes \mathbb{C}^k$$

by the formula:

$$Pu(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i<x,\xi>} p(x,\xi)\widehat{u}(\xi)d\xi.$$

Here, $S(\mathbb{R}^n)$ is the Schwartz space of $\mathbb{R}^n$ and $\widehat{u}$ denotes the Fourier transform of $u \in S(\mathbb{R}^n) \otimes \mathbb{C}^k$. The fact that P defines a linear operator from the Schwartz space $S(\mathbb{R}^n) \otimes \mathbb{C}^k$ to itself is straightforward.

Definition 2.1.3. *The operators of the form $Op(p)$ with $p \in S^m$ are called pseudodifferential operators of order m on $\mathbb{R}^n$.*

The space of all pseudodifferential operators of order m will be denoted by Ψ^m. Differential operators on $\mathbb{R}^n$ are examples of pseudodifferential operators. Indeed, consider the symbol $p(x,\xi) = \sum_{|\alpha| \leq m} A^\alpha(x)\xi^\alpha$ of order m, where

m is a positive integer and the $A^\alpha(x)$ are smooth matrix valued functions on $\mathbb{R}^n$. Since $\widehat{D_x^\alpha u(\xi)} = \xi^\alpha \hat{u}(\xi)$, we have:

$$Op(p) = \sum_{|\alpha|\leq m} A^\alpha(x) D^\alpha.$$

For $s \in \mathbb{R}$, denote by $H^s(\mathbb{R}^n)$ the Sobolev space of exponent s in $\mathbb{R}^n$, i.e. the completion of the Schwartz space $S(\mathbb{R}^n)$ for the Sobolev s-norm:

$$\|u\|_s = \sqrt{\int (1+|\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi}.$$

Theorem 2.1.4. *For any $p \in S^m$ with values in $M_k(\mathbb{C})$ and compact x-support, the operator $P = Op(p)$ has a continuous extension:*

$$P : H^{s+m}(\mathbb{R}^n) \otimes \mathbb{C}^k \longrightarrow H^s(\mathbb{R}^n) \otimes \mathbb{C}^k.$$

For a proof of this theorem, see [16], proposition 3.2, page 178. Note that a pseudodifferential operator can have an order $-m < 0$. Such an operator is said to be *smoothing of order m*.

Definition 2.1.5. *A linear map $P : S(\mathbb{R}^n) \otimes \mathbb{C}^k \longrightarrow S(\mathbb{R}^n) \otimes \mathbb{C}^k$ that extends to a bounded linear map $P : H^{s+m}(\mathbb{R}^n) \otimes \mathbb{C}^k \longrightarrow H^s(\mathbb{R}^n) \otimes \mathbb{C}^k$ for all s and m is called infinitely smoothing.*

The space of all infinitely smoothing operators will be denoted by $\Psi^{-\infty}$. Since we have a continuous inclusion $H^s(\mathbb{R}^n) \subset C^q(\mathbb{R}^n)$ for any $s > (n/2) + q$ (Sobolev's embedding theorem), the image Pu of any $u \in H^s(\mathbb{R}^n) \otimes \mathbb{C}^k$ by an infinitely smoothing operator P is a smooth function. Two pseudodifferential operator P and P' will be called *equivalent* if $P - P' \in \Psi^{-\infty}$.

2.1.6 Kernel of a pseudodifferential operator. Any $P = Op(p) \in \Psi^m$ has a *Schwartz* (distribution) *kernel* $K_P(x, y)$ satisfying:

$$Pu(x) = < K_P(x, .), u(.) >$$

for any smooth compactly supported function u. Note that K_P is not a function on $\mathbb{R}^n \times \mathbb{R}^n$ in general. Its restriction to the complement of the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$ is given by a smooth function, but it may have singularities on the diagonal (see for instance the case of differential operators). Formally, we have from the formula defining $P = Op(p)$:

$$K_P(x,y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i<x-y,\xi>} p(x,\xi) d\xi,$$

but we have to point out that this integral does not converge in general. For $m < -n$, this integral makes sense and defines a continuous function K_P on $\mathbb{R}^n \times \mathbb{R}^n$ which is smooth outside the diagonal. In this case, P is an ordinary integral operator. For $m \geq -n$, we get from the above remark, by writing $P = P(1+\Delta)^{-l}(1+\Delta)^l$ where l is a positive integer such that $m - 2l < -n$:

$$Pu(x) = \sum_{|\alpha| \leq 2l} \int K_{\alpha,P}(x,y) D^\alpha u(y) dy$$

where the $K_{\alpha,P}$ are continuous functions that are smooth outside the diagonal. We thus have, in the distributional sense:

$$K_P(x,y) = \sum_{|\alpha| \leq 2l} (-1)^{|\alpha|} D_y^\alpha K_{\alpha,P}(x,y).$$

Definition 2.1.7. *Let $P = Op(p) \in \Psi^m$ be a pseudodifferential operator on $\mathbb{R}^n$.*

(i) We call P ε-local if we have:

$$supp(Pu) \subset \{x \in \mathbb{R}^n | dist(x, supp(u)) \leq \varepsilon\}$$

for any smooth compactly supported function u on $\mathbb{R}^n$:

(ii) We say that P has support in a compact set K if we have:

$$supp(Pu) \subset K \text{ and } (supp(u) \cap K = \varnothing) \Longrightarrow Pu = 0$$

for any smooth compactly supported function u on $\mathbb{R}^n$.

The following proposition summarizes classical results used to construct pseudodifferential operators.

Theorem 2.1.8. *(i) For any formal series $\sum_{j=1}^{\infty} p_j$ with $p_j \in S^{m_j}$ and $m_j \longrightarrow -\infty$, there exists $P = Op(p) \in \Psi^{m_1}$, unique up to equivalence, such that*

$p \sim \sum_{j=1}^{\infty} p_j$;

(ii) Let $a(x,y,\xi)$ *be a smooth matrix-valued function on* $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ *with compact* x *and* y*-support. Assume that there exists* $m \in \mathbb{R}$ *and, for each* α, β, γ*, a constant* $C_{\alpha\beta\gamma}$ *such that:*

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi)| \leq C_{\alpha\beta\gamma}(1+|\xi|)^{m-|\gamma|}.$$

Then, the formula:

$$(Pu)(x) = \frac{1}{(2\pi)^n} \int\int e^{i<x-y,\xi>} a(x,y,\xi) u(y) dy d\xi$$

defines a pseudodifferential operator $P = Op(p)$ *of order* m *whose symbol* p *has the asymptotic development* $p \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha D_y^\alpha a)(x,x,\xi)$;
(iii) For any $P = Op(p) \in \Psi^m$ *whose symbol* p *has compact* x*-support and any* $\varepsilon > 0$*, there exists an* ε*-local pseudodifferential operator* $P_\varepsilon = Op(p) \in \Psi^m$ *such that* $P - P_\varepsilon \in \Psi^{-\infty}$.

For a proof, we refer to [16], chap. III, § 3.

Exercise 2.1.9. *Let* a *be as in theorem 2.1.8 (ii), and assume in addition that* $a(x,y,\xi)$ *vanishes for all* (x,y) *in a neighbourhood of the diagonal. Show that* $P = Op(p)$ *is infinitely smoothing.*

Exercise 2.1.10. *Let* $P = Op(p) \in \Psi^m$*. Show that for any pair* (φ, ψ) *of smooth real valued functions with compact support, the operator* $Q(u) = \psi P(\varphi u)$ *is also pseudodifferential of order* m*. Deduce that if* U *is an open subset of* $\mathbb{R}^n$*, we have for any* $u \in H^s(\mathbb{R}^n)$*:*

$$u_{|U} \in C^\infty \Longrightarrow Pu_{|U} \in C^\infty.$$

The following theorem summarizes the main rules of symbolic calculus on pseudodifferential operators.

Theorem 2.1.11. *(i) Let* $P = Op(p) \in \Psi^l$ *and* $Q = Op(q) \in \Psi^m$ *be pseudodifferential operators. Then, the product* $R = PQ$ *is a pseudodifferential operator* $R = Op(r) \in \Psi^{l+m}$ *whose symbol* r *has the formal development:*

$$r \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha p)(D_x^\alpha q);$$

(ii) Let $P = Op(p) \in \Psi^m$ be a pseudodifferential operator. Then, the formal adjoint P^* is a pseudodifferential operator $P^* = Op(p^*) \in \Psi^m$ whose symbol p^* has the formal development:

$$p^* \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \overline{p}^t;$$

(iii) Let $P = Op(p) \in \Psi^m$ be a pseudodifferential operator. Let $\varphi : U \longrightarrow V$ be a C^∞-diffeomorphism from an open subset U of $\mathbb{R}^n$ onto an open subset $V = \varphi(U) \subset \mathbb{R}^n$. For any pair (α, β) of smooth functions with compact support in U such that $\beta = 1$ in a neighbourhood of $supp(\alpha)$, the operator:

$$Q(u)(x) = (\alpha P \beta)(u \circ \varphi)(\varphi^{-1}(x))$$

is a pseudodifferential operator $Q = Op(q) \in \Psi^m$ whose symbol q has the formal development:

$$q(\varphi(x), \xi) \sim \alpha(x) \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} D_y^\alpha e^{i(\varphi(y) - \varphi(x) - \frac{\partial \varphi}{\partial x}(y-x)).\xi}|_{y=x} D_\xi^\alpha p(x, (\frac{\partial \varphi}{\partial x}(x))^t \xi),$$

where $(\frac{\partial \varphi}{\partial x}(x))$ denotes the Jacobi matrix.

For a proof, see [8], chap. I, § 1.3. Assume in (iii) that P has compact support $K \subset U$, and denote by $\varphi_* P$ the compactly supported pseudodifferential operator defined by $(\varphi_* P)(u) = P(u \circ \varphi) \circ \varphi^{-1}$. By (iii), we get:

$$q(\varphi(x), \xi) \sim p(\varphi^{-1}(x), (\frac{\partial \varphi}{\partial x}(x))^t \xi) \ \ (mod\ S^{m-1}),$$

where q is the symbol of $\varphi_* P$. This shows that, modulo symbols of lower order, the symbol of a pseudodifferential operator transforms by change of variable like a function on the cotangent bundle. This observation leads to the following definition:

Definition 2.1.12. *Let $P = Op(p) \in \Psi^m$. The principal symbol $\sigma(P)$ of P is by definition the ressidue class of p in S^m / S^{m-1}.*

If $Op(p) = \sum_{|\alpha| \leq m} A^\alpha(x) D^\alpha$, a representative of $\sigma(P)$ is given by the symbol:

$$\sigma_P(x,\xi) = \sum_{|\alpha|=m} A^\alpha(x)\xi^\alpha.$$

For instance, the principal symbol of the Cauchy-Riemann operator $\frac{\partial}{\partial \overline{Z}} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}$ in $\mathbb{R}^2$ is: $\partial_{\frac{\partial}{\partial \overline{z}}}(x,\xi) = i\xi_1 - \xi_2$.

2.2 Pseudodifferential operators on manifolds

Let M be a n-dimensional smooth compact Riemannian manifold without boundary. Denote by $\pi : T^*M \longrightarrow M$ the canonical projection. Let E, F be smooth complex vector bundles over M.

Definition 2.2.1. *A linear operator $P : C^\infty(M,E) \longrightarrow C^\infty(M,F)$ is called pseudodifferential of order $m \in \mathbb{R}$ if, for every open chart U on M trivializing E and F and any $\varphi, \psi \in C_c^\infty(U)$, the localized operator $\varphi P \psi$ is pseudodifferential of order m (with compact x-support) on the chart U, viewed as an open subset of $\mathbb{R}^n$.*

As above, we identify here φ with the multiplication operator by φ. We shall denote by $\Psi^m(M;E,F)$ the space of all pseudodifferential operators of order m acting from the sections of E to the sections of F. Let $S^m(T^*M)$ be the set of all $p \in C^\infty(T^*M)$ such that the pullback to any local chart of M is in S^m, and define analogously $S^m(T^*M, Hom(\pi^*E, \pi^*F))$. By theorem 2.1.11 (iii), any $P \in \Psi^m(M;E,F)$ has a *principal symbol:*

$$\sigma(P) \in S^m(T^*M, Hom(\pi^*E,\pi^*F))/S^{m-1}(T^*M, Hom(\pi^*E,\pi^*F)).$$

Let us denote by *dvol* the Riemannian measure on M and by $H^s(M,E)$ the Sobolev space of exponent $s \in \mathbb{R}$ for the sections of the vector bundle E over M. This space is equipped with the norm $\|u\|_s = \sum_{i=1}^{q} \|\varphi_i u\|_s$, where $(\varphi_1, ..., \varphi_q)$ is a smooth partition of unity subordinate to a covering of M by charts trivializing E. From theorem 2.1.4, we get:

Theorem 2.2.2. *Any $P \in \Psi^m(M;E,F)$ extends, for any $s \in \mathbb{R}$, to a continuous map $P : H^{s+m}(M,E) \longrightarrow H^s(M,F)$.*

Exercise 2.2.3. *Show that any $P \in \Psi^m(M;E,E)$ with $m \leq 0$ defines a bounded operator in $L^2(M,E)$.*

Definition 2.2.4. *A linear operator* $P : C^\infty(M,E) \longrightarrow C^\infty(M,F)$ *is called infinitely smoothing if it extends to a bounded map* $P : H^{s+m}(M,E) \longrightarrow H^s(M,F)$ *for any* $s \in \mathbb{R}$.

We shall denote by $\Psi^{-\infty}(M;E,F)$ the space of all infinitely smoothing operators $P : C^\infty(M,E) \longrightarrow C^\infty(M,F)$. It is straightforward to check that $\Psi^{-\infty}(M;E,F) = \bigcap_m \Psi^m(M;E,F)$.

Exercise 2.2.5. *Let* $P : C^\infty(M,E) \longrightarrow C^\infty(M,F)$ *be a linear operator. Show that* $P \in \Psi^{-\infty}(M;E,F)$ *if and only if* P *can be written as an integral operator:*

$$Pu(x) = \int_M K(x,y)u(y)dvol(y),$$

where $K(x,y) \in Hom_{\mathbb{C}}(E_y, F_x)$ *varies smoothly with* x *and* y *on* M.

Let us now collect some classical results about pseudodifferential operators on manifolds.

Theorem 2.2.6. *Let* M, E, F *be as above.*

(i) Any $P \in \Psi^m(M;E,F)$ *can be written as finite sum:*

$$P = \sum_{i=1}^{q} P_i + R,$$

where R *is an infinitely smoothing operator on* M *and each* P_i *is an order* m *pseudodifferential operator compactly supported by a local chart* U_i *trivializing* E *and* F *(more precisely,* $P_i = \varphi_i P_i \psi_i$ *where* φ_i, ψ_i *are smooth functions on* M *with compact support on* U_i*);*

(ii) For any $P \in \Psi^m(M;E,F)$*, any open subset* U *of* M *and any* $u \in H^s(M,E)$ *such that* $u_{|U}$ *is* C^∞*, the section* Pu *of* F *is* C^∞ *over* U*;*

(iii) Let $P \in \Psi^m(M;E,E)$ *be a pseudodifferential operator of order* $m \leq 0$*, viewed as a bounded operator in* $L^2(M,E)$ *(cf. exercise 2.2.5).*
If $m < 0$*,* P *is a compact operator;*
If $m < -n/2$*,* P *is Hilbert-Schmidt operator of the form:*

$$Pu(x) = \int_M K_P(x,y)u(y)dvol(y),$$

where $K_P(x,y) \in Hom(E_y, F_x)$ *varies continuously with* x *and* y *on* M*;*
If $m < -n/p$*,* P *belongs to the Schatten class* C_p [2] *;*
In particular, if $m < -n$*,* P *is a trace-class operator and* $Tr(P) = \int_M tr(K_P(x,x))dvol(x)$*.*

(iv) If $P \in \Psi^m(M;E,F)$ *and* $Q \in \Psi^r(M;F,G)$*, then:*

$$QP = Q \circ P \in \Psi^{m+r}(M;E,G);$$

(v) Let $P \in \Psi^m(M;E,F)$ *and assume that* P *has a formal adjoint i.e. there exists an operator* $P^* : C^\infty(M,F) \longrightarrow C^\infty(M,E)$ *such that:*

$$< Pu, v >_{L^2(M,F)} = < u, P^*v >_{L^2(M,E)}$$

for any smooth sections u *and* v*. Then,* $P^* \in \Psi^m(M;F^*,E^*)$*.*

Let us end up this section by giving a specific example of a differential operator on a manifold.

2.2.7 The signature operator. Let M be a compact oriented Riemannian manifold without boundary, of dimension $n = 4k$. Denote by d the exterior derivative $d : C^\infty(M, \Lambda^*(T^*M \otimes \mathbb{C})) \longrightarrow C^\infty(M, \Lambda^*(T^*M \otimes \mathbb{C}))$. The metric g on M induces a scalar product on $\Lambda^p(T_x^*M)$ by the formula:

$$< a_I dx^I | b_J dx^J > = p! g^{i_1 j_1} ... g^{i_p j_p} a_I b_J,$$

where $I = (i_1, ..., i_p)$, $J = (j_1, ..., j_p)$ and $g^{ij} = < dx^i | dx^j >$. Denote by δ the formal adjoint of d with respect to this scalar product. To describe δ, let us introduce the *Hodge-star* operation. Let $vol = \sqrt{g} dx^1 \wedge ... \wedge dx^n$ be the volume form. The *Hodge star* operator $* : \Lambda^p(T_x^*M) \longrightarrow \Lambda^{n-p}(T_x^*M)$ is by definition the only linear map satisfying:

$$< \alpha | \beta > vol = \alpha \wedge *(\beta) \text{ for any } \alpha, \beta \in \Lambda^p(T_x^*M).$$

It is easy to check that $\delta = - * d *$, so that δ is an order one differential operator on M. Set:

$$D = d + \delta;$$

[2] Recall that C_p is the space of compact operators T acting on a separable Hilbert space H such that $Tr(| T |^p) < +\infty$. For more information on the Schatten class C_p, we refer to [19].

we thus define a self-adjoint order one diffferential operator. The signature operator is obtained by restriction of D to the positive part of some grading on $C^\infty(M, \Lambda^*(T^*M \otimes \mathbb{C}))$ that we shall now describe. To this end, note that we have $*(*\alpha) = (-1)^p id$, so that the operator:

$$\varepsilon = i^{2k+p(p-1)} * : \Lambda^p(T^*M \otimes \mathbb{C}) \longrightarrow \Lambda^{4k-p}(T^*M \otimes \mathbb{C})$$

defines a grading on $\Lambda^*(T^*M \otimes \mathbb{C})$, i.e. $\varepsilon^2 = 1$. The ± 1-eigenspaces $E_\pm = \Lambda^\pm(T^*M \otimes \mathbb{C})$ of ε give rise to a direct sum decomposition:

$$\Lambda^*(T^*M \otimes \mathbb{C}) = E_+ \oplus E_-,$$

and since $D = d + \delta$ anticommutes with the grading ε, it decomposes to give rise to operators $D_\pm : C^\infty(M, \bigwedge^+(T^*M \otimes \mathbb{C})) \longrightarrow C^\infty(M, \Lambda^-(T^*M \otimes \mathbb{C}))$. The *signature operator* is by definition the operator

$$D_+ : C^\infty(M, E_+) \longrightarrow C^\infty(M, E_-).$$

It is an order one differential operator on M with principal symbol:

$$\sigma(D_+)(x, \xi) = i(ext(\xi) - int(\xi)),$$

where $ext(\xi)$ is the exterior multiplication by ξ and $int(\xi)$ its adjoint. Note that $int(\xi)$ is just interior multiplication by ξ since we have for any local orthonormal basis $(e_1, ..., e_n)$ for T^*M:

$$int(e_1)(e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_p}) = \begin{cases} e_{i_2} \wedge ... \wedge e_{i_p} & if\ i_1 = 1 \\ 0 & if\ i_1 > 1. \end{cases}$$

2.3 Analytical index of an elliptic operator

Let M be a n-dimensional smooth compact manifold without boundary and denote by π the projection $T^*M \longrightarrow M$. Let $P : C^\infty(M, E) \longrightarrow C^\infty(M, F)$ be a pseudodifferential operator of order m, where E, F are smooth complex vector bundles over M. Recall that P has a principal symbol $\sigma_P \in S^m/S^{m-1}$, where $S^k = S^k(T^*M, Hom(\pi^*E, \pi^*F))$.

Definition 2.3.1. *We say that $P \in \Psi^m(M; E, F)$ is elliptic if its principal symbol σ_P has a representative $p \in S^m(T^*M, Hom(\pi^*E, \pi^*F))$ which is pointwise invertible outside a compact set in T^*M and satisfies the estimate*

$$|p(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$$

for some constant C and some riemannian metric on M.

Example 2.3.2. The signature operator D_+ on a $4k$-dimensional compact oriented manifold M without boundary is elliptic, since we have:

$$\sigma(D_+)(x,\xi)^2 = -(ext(\xi) - int(\xi))^2 = \|\xi\|^2 id.$$

The following result shows that an elliptic pseudodifferential operator on a compact manifold M is invertible modulo infinitely smoothing operators:

Theorem 2.3.3. *Let $P \in \Psi^m(M; E, F)$ be an order m elliptic pseudodifferential operator on M. Then, there exists $Q \in \Psi^{-m}(M; F, E)$ such that:*

$$QP - I \in \Psi^{-\infty}(M, E, E) \text{ and } PQ - I \in \Psi^{-\infty}(M, F, F).$$

The operator Q is called a *parametrix* for P.

Sketch of proof. We shall essentially prove that P admits a parametrix locally.

Case of an elliptic operator on $\mathbb{R}^n$. Let P be a pseudodifferential operator of order m on $\mathbb{R}^n$ whose principal symbol p is pointwise invertible outside a compact set in $T^*\mathbb{R}^n$ and satisfies the estimate $|p(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$ for some constant C. We are going to construct a parametrix Q for P from a formal development $q \sim \sum_{k=0}^{\infty} q_k$ of its symbol, where $q_k \in S^{-m-k}$. By adding some infinitely smoothing operator to P, we may assume that P is 1-local and ask that Q is 1-local too. Since the formal development of the symbol of QP is given by:

$$\sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q)(D_x^\alpha p) = \sum_k \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_k)(D_x^\alpha p),$$

where $(D_\xi^\alpha q_k)(D_x^\alpha p)$ is a symbol of order $-k-|\alpha|$, it is natural to determine $q_k \in S^{-m-k}$ in such a way that:

$$\begin{cases} q_0 p - I \in S^{-\infty} \\ q_k p + \sum_{j=0}^{k-1} \left[\sum_{|\alpha|+j=k} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_j)(D_x^\alpha p) \right] \in S^{-\infty} \; for \; k = 1, 2, ... \end{cases}$$

Let us now solve these equations. Note that we only have to choose q_0, because we can determine inductively $q_1, q_2, ...$ from q_0 by setting:

$$(*) \qquad q_k = -\sum_{j=0}^{k-1} \left[\sum_{|\alpha|+j=k} \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha q_j)(D_x^\alpha p) \right] q_0.$$

To get q_0, we set $q_0(x,\xi) = \theta(|\xi|)p(x,\xi)^{-1}$, where $\theta : \mathbb{R}^+ \longrightarrow [0,1]$ is a smooth function such that $\theta(t) = 0$ for $t \leq C$ and $\theta(t) = 1$ for $t \geq 2C$. It is easy to check that $q_0 \in S^{-m}$. Consider now the asymptotic series $\sum_{k=0}^{\infty} q_k$ where the q_k's are given by (*). There exists a 1-local pseudodifferential Q of order $-m$ with symbol q satisfying $q \sim \sum_{k=0}^{\infty} q_k$. From the formula for the symbol of a product, we get that $QP - I$ is infinitely smoothing. In the same way, we get a pseudodifferential operator Q' such that $PQ' - I$ is infinitely smoothing. But we have, modulo infinitely smoothing operators:

$$Q \equiv Q(PQ') = (QP)Q' \equiv Q',$$

so that Q is a parametrix for P.

General case. To avoid technical difficulties, we shall only consider the case of a *differential* operator $P \in \Psi^m(M; E, F)$. Choose a covering of M by open charts U_α trivializing E and F, with local coordinates

$$x_\alpha : U_\alpha \longrightarrow x_\alpha(U_\alpha) = \mathbb{R}^n$$

such that the open subsets $\Omega_\alpha = \{m \in U_\alpha | \; |x_\alpha(m)| < 1\}$ cover M. Let $(\varphi_\alpha)_\alpha$ be a partition of unity subordinate to the Ω_α. Then the restriction P_α of P to U_α, viewed as differential operator on $\mathbb{R}^n$, has a parametrix Q_α that we may assume to be 1-local. Since Q_α is 1-local, the operators $\varphi_\alpha Q_\alpha$ and $Q_\alpha \varphi_\alpha$ have compact support in $\Omega'_\alpha = \{m \in U_\alpha | \; |x_\alpha(m)| < 2\} \subset U_\alpha$, and hence make sense as pseudodifferential operators in $\Psi^{-m}(M; E, F)$. Set:

$$Q = \sum_\alpha \varphi_\alpha Q_\alpha \in \Psi^{-m}(M; E, F) \text{ and } Q' = \sum_\alpha Q_\alpha \varphi_\alpha \in \Psi^{-m}(M; E, F).$$

We have:

$$PQ' - I = \sum_\alpha (PQ_\alpha \varphi_\alpha - \varphi_\alpha) = \sum_\alpha (P_\alpha Q_\alpha - I)\varphi_\alpha = \sum_\alpha R_\alpha \varphi_\alpha,$$

where $R_\alpha = P_\alpha Q_\alpha - I$ is a 1-local infinitely smoothing operators in U_α, so that $R_\alpha \varphi_\alpha \in \Psi^{-\infty}(M; E, F)$. It follows that $PQ' - I$ is infinitely smoothing. In the same way, $QP - I$ is infinitely smoothing, and since we have

$$Q - Q' \in \Psi^{-\infty}(M; E, F)$$

as in the first step, the proof is complete. QED

Corollary 2.3.4. *Let $P \in \Psi^m(M; E, F)$ be an order m elliptic pseudodifferential operator on a compact manifold M. For any $s \in \mathbb{R}$, the operator P extends to a Fredholm operator $P_s : H^{s+m}(M, E) \longrightarrow H^s(M, F)$ whose index $Ind(P_s)$ is independent of s.*

Proof. By theorem 2.3.3, there exists $Q \in \Psi^{-m}(M; F, E)$ such that $PQ - I$ and $QP - I$ are infinitely smoothing. Denote by $Q_s : H^s(M, F) \longrightarrow H^{s+m}(M, E)$ the unique extension of Q to $H^s(M, F)$. Since an infinitely smoothing operator from $H^r(M, E)$ into itself is compact, we get that $Q_s P_s - I$ and $P_s Q_s - I$ are compact, and hence P_s is Fredholm. Since $Q_s P_s - I$ is infinitely smoothing, we have $u = -(Q_s P_s - I)u \in C^\infty(M, E)$ for any $u \in Ker(P_s)$, and hence:

$$Ker(P_s) = Ker(P) \subset C^\infty(M, E).$$

In the same way, we get $Ker(P_s^*) = Ker(P^*) \subset C^\infty(M, F)$, where P^* is the formal adjoint of P, and hence $Ind(P_s)$ is independent of s. QED

Definition 2.3.5. *Let $P \in \Psi^m(M; E, F)$ be an elliptic operator on a compact manifold M. The index $Ind(P_s)$ of any extension $P_s : H^{s+m}(M, E) \longrightarrow H^s(M, F)$ is called the analytical index of P and is denoted by $Ind(P)$.* Let us now compute the analytical index of the signature operator on a compact riemannian manifold M without boundary. We assume here that M is $4k$-dimensional and oriented. Recall that the *signature* $\sigma(M)$ of M is by definition the signature of the symmetric bilinear form

$$\begin{array}{ccc} H^{2k}(M, \mathbb{C}) \times H^{2k}(M, \mathbb{C}) & \longrightarrow & \mathbb{C} \\ ([\omega_1], [\omega_2]) & \longrightarrow & \displaystyle\int_M \omega_1 \wedge \omega_2 \end{array}$$

induced by the cup-product in cohomology.

Theorem 2.3.6. *The index of the signature operator on M is equal to the signature $\sigma(M)$ of M.*

Proof. Since $D = d + \delta$ anticommutes with the grading ε, we have:

$$D = \begin{pmatrix} 0 & D_+^* \\ D_+ & 0 \end{pmatrix}$$

where D_+ is the signature operator on M. But $D^2 = (d+\delta)^2 = d\delta + \delta d$ is nothing but the Hodge-Laplace operator Δ, so that we get:

$$\Delta = D^2 = \begin{pmatrix} D_+^* D_+ & 0 \\ 0 & D_+ D_+^* \end{pmatrix},$$

and hence $D_+^* D_+ = \Delta_+$ (resp.$D_+ D_+^* = \Delta_-$) is the restriction of Δ to the $+1$ (resp. -1) eigenspace of ε. It follows that:

$$\begin{aligned} Ind(D_+) &= dim(Ker D_+) - dim(Ker D_+^*) \\ &= dim(Ker D_+^* D_+) - dim(Ker D_+ D_+^*) \\ &= dim(Ker \Delta_+) - dim(Ker \Delta_-) \end{aligned}$$

and hence:

$$\begin{aligned} Ind(D_+) = (&dim(Ker \Delta_+^{2k}) - dim(Ker \Delta_-^{2k})) \\ &+ \sum_{p=0}^{2k-1} (dim(Ker \Delta_+^p) - dim(Ker \Delta_-^p)), \end{aligned}$$

where we denote by $\Delta_\pm^p$ the restriction of $\Delta_\pm$ to the ε-invariant subspace

$$C^\infty(M, \Lambda_{\mathbb{C}}^p(T^*M) \oplus \Lambda_{\mathbb{C}}^{4k-p}(T^*M)) \text{ for } p = 0, 1, ..., 2k.$$

But we have:
$\omega \in Ker \Delta_\pm^p \Longleftrightarrow \omega = \alpha \pm \varepsilon(\alpha)$ for a harmonic p-form α,
and since the map $\alpha + \varepsilon(\alpha) \longrightarrow \alpha - \varepsilon(\alpha)$ induces an isomorphism between $Ker(\Delta_+^p)$ and $Ker(\Delta_-^p)$ for $p = 0, 1, ..., 2k-1$, we get:

$$Ind(D_+) = dim(Ker \Delta_+^{2k}) - dim(Ker \Delta_-^{2k}).$$

By Hodge theory, $Ker(\Delta^{2k})$ identifies with $H^{2k}(M, \mathbb{C})$. Denote by $H_\pm$ the ± 1 eigenspace of $\varepsilon = *$ on $2k$-harmonic forms. Since we have:

$$\int_M \omega \wedge \omega = \pm \int_M \omega \wedge (*\omega) = \pm < \omega|\omega > \text{ for any } \omega \in H_\pm,$$

the signature form is positive definite on H_+ and negative definite on H_-, so that finally $Ind(D_+) = \sigma(M)$. QED

Exercise 2.3.7. *Show that the analytical index of the De Rham operator D on a compact oriented Riemannian manifold M without boundary is given by:*

$$Ind(D) = \sum_{i=0}^{dim(M)} (-1)^i dim(H^i(M, \mathbb{C})).$$

3 Topological K-theory

The analytical index of an elliptic operator P of order m on a compact manifold M is computable from its principal symbol $\sigma(P)$ of order m. When $m = 0$, this principal symbol yields an element in the K-thery group (with compact support) of T^*M, and the analytical index can be viewed as a map $[\sigma(P)] \in K^0(T^*M) \longrightarrow Ind(P) \in \mathbb{Z}$. The aim of this section is to introduce the topological K-theory of locally compact spaces, in order to give a topological description of the above index map. Since it does not require more effort, we shall simultaneously introduce the topological K-functor $A \longrightarrow K_*(A) = K_0(A) \oplus K_1(A)$ for C^*-algebras.

3.1 The group $\mathbf{K^0(X)}$

Definition 3.1.1. *Let X be a compact space. The K-theory group $K^0(X)$ is the abelian group generated by the isomorphism classes of complex vector bundles over X with the relations: $[E \oplus F] = [E] + [F]$ for any pair (E, F) of vector bundles over X.*

Every element of $K^0(X)$ is thus a difference $[E] - [F]$, where E and F are complex vector bundles over X. In this representation, we have (with obvious notations): $[E]-[F]=[E']-[F'] \Longleftrightarrow (\exists G)$ *such that* $E\oplus F'\oplus G\cong E'\oplus F\oplus G$. Denote by $[\tau]$ the K-theory class of the trivial bundle of rank 1 over X. Since there exists, for any complex vector bundle E over a compact space X, a vector bundle F over X such that $E\oplus F\cong X\times\mathbb{C}^n$ (trivial bundle of rank n), any element in $K^0(X)$ can be written in the form $[E]-n[\tau]$ for some bundle E over X and $n\in\mathbb{N}$.

Example 3.1.2. A vector bundle over a point is a finite dimensional complex vector space, and two such vector bundles are isomorphic if and only if they have same dimension. Henceforth, $K^0(p^t)$ is isomorphic to $\mathbb{Z}$.

Exercise 3.1.3. *Show that* $K^0(S^1)\cong\mathbb{Z}$.

Note that any continuous map $f:X\longrightarrow Y$ between compact spaces induces a group homomorphism $f^*:K^0(Y)\longrightarrow K^0(X)$ by $f^*([E])=[f^*(E)]$, where $f^*(E)=\{(x,\zeta)\in X\times E|f(x)=\pi(\zeta)\}$ is the pull-back to X of the complex vector bundle $E\stackrel{\pi}{\longrightarrow}Y$ over Y.

Definition 3.1.4. *If X is a locally compact space X, the K-theory group $K^0(X)$ (with compact support) is by definition the kernel of the map $K^0(\widehat{X})\longrightarrow K^0(\{\infty\})=\mathbb{Z}$ induced by the inclusion of $\{\infty\}$ into the one-point compactification $\widehat{X}=X\cup\{\infty\}$ of X.*

Racall that a continuous map $f:X\longrightarrow Y$ between locally compact spaces is called *proper* if $f^{-1}(K)$ is compact for any compact subset K of Y. From the definition of $K^0(X)$, it is clear that $X\longrightarrow K^0(X)$ is a contravariant functor from the category of locally compact spaces with proper continuous maps to abelian groups.

Exercise 3.1.5. *Show that we have $K^0(\widehat{X})=K^0(X)\oplus\mathbb{Z}$ for any non-compact locally compact space X. Prove that $K^0(\mathbb{R})=\{0\}$.*

Exercise 3.1.6. *Let X be a compact space that can be written as a disjoint union of two open subspaces X_1 and X_2. Prove that $K^0(X)\cong K^0(X_1)\oplus K^0(X_2)$.*

The main property of the functor $X\longrightarrow K^0(X)$ is its homotopy invariance:

Theorem 3.1.7. *Let X, Y be two locally compact spaces and $f_t : X \longrightarrow Y$ $(0 \leq t \leq 1)$ be a continuous path of proper maps from X to Y. Then, we have:*

$$f_0^* = f_1^* : K^0(Y) \longrightarrow K^0(X).$$

For a proof of this result, see [10], theorem 1.25, p. 56.

Exercise 3.1.8. *By using theorem 3.1.7, show that $K^0(]0,1]) = \{0\}$.*

Exercise 3.1.9. *By using the correspondence between complex vector bundles over a compact space X and idempotents in matrix algebras over $C(X)$, try to give a proof of theorem 3.1.7.*

3.2 Fredholm operators and Atiyah's picture of $\mathbf{K^0(X)}$

Let X be a compact space and denote by H a separable infinite dimensional Hilbert space. Since the product of two Fredholm operators is Fredholm, the space $[X, Fred(H)]$ of homotopy classes of continuous functions from X to $Fred(H)$ has a natural semigroup structure. The following description of $K^0(X)$ by continuous fields of Fredholm operators can be found in [1]:

Theorem 3.2.1. *There is a group isomorphism $Ind : [X, Fred(H)] \longrightarrow K^0(X)$ such that: $Ind \circ f_* = f^* \circ Ind$ for any continuous map $f : X \longrightarrow Y$ of compact spaces.*

Sketch of proof. Naively, we would like to define the index map Ind by setting:

$$Ind([T]) = [(KerT_x)_{x \in X}] - [(KerT_x^*)_{x \in X}] \in K^0(X), \tag{1}$$

where $[T]$ denotes the the homotopy class of the continuous map

$$x \in X \longrightarrow T_x \in Fred(H).$$

But since the dimension of $Ker(T_x)$ is not locally constant in general, $(KerT_x)_{x \in X}$ and $(KerT_x^*)_{x \in X}$ are not vector bundles over X so that the heuristic formula (1) does not make sense. To overcome this difficulty, fix a point $x_0 \in X$ and consider the map:

$$\widetilde{T}_x : (\zeta, \eta) \in Ker(T^*_{x_0}) \oplus H \longrightarrow \zeta + T_x\eta \in H,$$

which is defined for x in a neighborhood of x_0. Since $\widetilde{T}_{x_0}$ is surjective, $\widetilde{T}_x$ is surjective for x in some neighborhood of x_0 and we get by the homotopy invariance of the index:

$$dim(Ker\widetilde{T}_x) = Ind(\widetilde{T}_x) = Ind(\widetilde{T}_{x_0}) = dim\ Ker(T_{x_0}) = Constant.$$

Now, by using a partition of unit, it is easy to patch together such local constructions (in the neighborhood of any point $x \in X$) to construct a finite number of continuous maps $\zeta_i : X \longrightarrow H (i = 1, 2, ..., N)$ satisfying the following two conditions:

(i) For any $x \in X$, the map

$$\widetilde{T}_x : (\lambda, \eta) \in \mathbb{C}^N \oplus H \longrightarrow \widetilde{T}_x(\lambda, \eta) = \Sigma_{i=1}^N \lambda_i \zeta_i(x) + T_x\eta \in H$$

is surjective;

(ii) The function $x \longrightarrow dim(Ker\widetilde{T}_x)$ is locally constant.

Now, $(Ker\widetilde{T}_x)_{x\in X}$ is a vector bundle over X by (ii) and we can define correctly the index map $Ind : [X, Fred(H)] \longrightarrow K^0(X)$ by setting (in view of (i)):

$$Ind([T]) = [(Ker\widetilde{T}_x)_{x\in X}] - [\mathbb{C}^N] \in K^0(X).$$

It remains to check that Ind is a well defined map which is a group isomorphism. It is straghtforward to check that Ind is a well defined group homomorphism. To prove that Ind is an isomorphism, we can check that

$$[X, GL(H)] \longrightarrow [X, Fred(H)] \xrightarrow{Ind} K^0(X) \longrightarrow 0$$

is an exact sequence (this is not hard) and use the contractibility of $GL(H)$ (Kuiper's theorem[3]) to get $[X, GL(H)] = 0$. QED

3.3 Excision in K-theory

Let Y be a closed subspace of a locally compact space X. The relative K-theory group $K^0(X, Y)$ is defined as a quotient of the set $Q(X, Y)$ of triples (E_0, E_1, σ) where E_0, E_1 are complex vector bundles over X that are direct factors of trivial bundles, and $\sigma \in Hom(E_0, E_1)$ is a morphism of vector bundles such that:

[3] For a proof of Kuiper's theorem, see [14].

(i) There is a compact subset K of X such that $\sigma_{|X-K} : E_{0|X-K} \longrightarrow E_{1|X-K}$ is an isomorphism;

(ii) $\sigma_{|Y} : E_{0/Y} \longrightarrow E_{1|Y}$ is an isomorphism.

If σ is an isomorphism, the triple (E_0, E_1, σ) is called *degenerate*. There are obvious notions of sum, isomorphism and homotopy of pairs of triples in $Q(X,Y)$. Let us say that $(E_0, E_1, \sigma) \in Q(X,Y)$ and $(E'_0, E'_1, \sigma') \in Q(X,Y)$ are *equivalent* if there exist degenerate triples $(F_0, F_1, \rho), (F'_0, F'_1, \rho') \in Q(X,Y)$ and isomorphisms of bundles $\theta_0 : E_0 \oplus F_0 \longrightarrow E'_0 \oplus F'_0, \theta_1 : E_1 \oplus F_1 \longrightarrow E'_1 \oplus F'_1$ such that $(E_0 \oplus F_0, E_1 \oplus F_1, \sigma \oplus \rho)$ is homotopic to $(E_0 \oplus F_0, E_1 \oplus F_1, \theta_1^{-1}(\sigma' \oplus \rho')\theta_0)$ in $Q(X,Y)$.

Definition 3.3.1. *Let Y be a closed subspace of a locally compact space X. The quotient of $Q(X,Y)$ by the above equivalence relation is denoted by $K^0(X,Y)$.*

$K^0(X,Y)$ is clearly an abelian group for the direct sum. The excision property can be expressed as follows:

Theorem 3.3.2. (Excision). *For any closed subspace Y of a locally compact space X, we have natural isomorphisms:*

$$K^0(X,Y) \cong K^0(X-Y) \approx K^0(X/Y, \{\infty\}),$$

where X/Y denotes the one-point compactification of $X-Y$ obtained by identifying all points in Y to a single point $\{\infty\}$.

Sketch of proof. The second isomorphism is a tautology. To prove the first isomorphism, we may restrict our attention to the case where X is compact, since we have a natural isomorphism $K^0(X,Y) \cong K^0(\widehat{X}, \widehat{Y})$ where $\widehat{X}$ is the one point compactification of X. Let Z be the compact space obtained by gluing two copies $X_0 = X_1 = X$ of X along the common part $Y_0 = Y_1 = Y$ and denote by $i : X_1 \longrightarrow Z$ the natural inclusion. Since there is an obvious retraction $\rho : Z \longrightarrow X_1$, one can show that the natural exact sequence:

$$0 \longrightarrow K^0(Z - X_1) \cong K^0(X-Y) \xrightarrow{j^*} K^0(Z) \xrightarrow{i^*} K^0(X_1) \longrightarrow 0$$

is split exact, so that $K^0(Z) \cong K^0(X-Y) \oplus K^0(X_1)$. We can now construct the isomorphism $K^0(X,Y) \xrightarrow{\cong} K^0(X-Y)$. Let $[E_0, E_1, \sigma] \in K^0(X,Y)$ and consider the complex vector bundle F over Z obtained by identifying E_0 and E_1 over Y via the isomorphism $\sigma_{|Y}$. Since the element $[F]-[\rho^*(E_1)] \in K^0(Z)$ belongs to $Ker(i^*)$, there is a unique element $\chi(E_0, E_1, \sigma) \in K^0(X-Y)$ such that $j^*(\chi(E_0, E_1, \sigma)) = [F]-[\rho^*(E_1)]$. It is now straightforward to check that the map $(E_0, E_1, \sigma) \longrightarrow \chi(E_0, E_1, \sigma)$ defines an isomorphism from $K^0(X, Y)$ to $K^0(X-Y)$. QED

Definition 3.3.3. *Let X be a locally compact space. We call quasi isomorphism over X any triple (E_0, E_1, σ) where E_0, E_1 are complex vector bundles over X and $\sigma \in Hom(E_0, E_1)$ a morphism of vector bundles which is an isomorphism outside some compact subset of X.*

It follows from theorem 3.3.2 that any element in $K^0(X)$ can be represented by a quasi-isomorphism (E_0, E_1, σ) over X.

This slightly different point of view on K-theory with compact support allows to describe a multiplication in $K^0(X)$ by using the following heuristic formula:

$$[(E_0, E_1, \sigma)] \otimes [(F_0, F_1, \tau)] = \\ [((E_0 \otimes F_0) \oplus (E_1 \otimes F_1), (E_0 \otimes F_1) \oplus (E_1 \otimes F_0), \sigma \widehat{\otimes} 1 + 1 \widehat{\otimes} \tau)],$$

where $\sigma \widehat{\otimes} 1 + 1 \widehat{\otimes} \tau = \begin{pmatrix} \sigma \otimes 1 & -1 \otimes \tau^* \\ 1 \otimes \tau & \sigma^* \otimes 1 \end{pmatrix}$ (sharp product).
This product can be used to prove the Thom isomorphism for complex vector bundles. Let $\pi : E \longrightarrow X$ be a complex hermitian vector bundle over a compact space X, and consider the triple:

$$\Lambda_{-1}(E) = (E_0 = \pi^*(\Lambda_{\mathbb{C}}^{ev} E), E_1 = \pi^*(\Lambda_{\mathbb{C}}^{odd} E), \sigma),$$

where $\Lambda_{\mathbb{C}}^{ev} E = \bigoplus_p \Lambda_{\mathbb{C}}^{2p} E$ and $\sigma : E_0 \longrightarrow E_1$ is the morphism of bundles (over the total space of E) given by $\sigma(x, \zeta)(\omega) = \zeta \wedge \omega - \zeta^* \lfloor \omega$. Although $\Lambda_{-1}(E)$ is not a quasi-isomorphism over E, its sharp product $\pi_*(F_0, F_1, \varphi) \otimes \Lambda_{-1}(E)$ where (F_0, F_1, φ) is a quasi-isomorphism over X, yields an element in $K^0(E)$ which only depends on $[(F_0, F_1, \varphi)]$. Let us denote by $\pi_*([F_0, F_1, \varphi]) \otimes [\Lambda_{-1}(E)]$ this element. We have:

Theorem 3.3.4.(Thom isomorphism for complex hermitian bundles). *For any complex hermitian vector bundle $\pi : E \longrightarrow X$ over a compact space X, the map:*

$$[(F_0, F_1, \varphi)] \in K^0(X) \longrightarrow \pi_*([F_0, F_1, \varphi]) \otimes [\Lambda_{-1}(E)] \in K^0(E)$$

induces an isomorphism of K-theory groups.

For a proof of this result, see [10].

Exercise 3.3.5. *Show that $K^0(\mathbb{R}^{2n}) \cong \mathbb{Z}$.*

3.4 The Chern Character

For any locally compact space X, denote by $H^*(X, \mathbb{Q})$ the rational Čech chomology of X with compact support. We have by setting $H^{ev}(X, \mathbb{Q}) = \sum_k H^{2k}(X, \mathbb{Q})$:

Theorem 3.4.1. *There exists a natural homomorphism $ch : K^0(X) \longrightarrow H^{ev}(X, \mathbb{Q})$, called the Chern character, which satisfies the following properties:*

(i) $ch(f^*(x)) = f^*(ch(x))$ for any proper map $f : X \longrightarrow Y$ and $x \in K^0(Y)$;
(ii) $ch(x + y) = ch(x) + ch(y)$ for $x, y \in K^0(X)$;
(iii) $ch([L]) = e^{c_1(L)}$ for any complex line bundle L over X, where $c_1(L)$ is the first Chern class of L, i.e. the image of the 1-cocycle associated with the $\mathbb{S}^1$-bundle L by the natural isomorphism $c_1 : H^1(X, \mathbb{S}^1) \xrightarrow{\cong} H^2(X, \mathbb{Z})$;
(iv) $ch([E \otimes F]) = ch([E])ch([F])$ for any pair (E, F) of complex vector bundles over X;
(v) If X is compact, the Chern character extends to an isomorphism:

$$ch : K^0(X) \otimes \mathbb{Q} \xrightarrow{\cong} H^{ev}(X, \mathbb{Q}).$$

Let us give a construction of the Chern character.

3.4.2 Construction of the Chern character.

Assume for simplicity that X is a compact manifold M. Let E be a k-dimensional complex vector bundle over M and choose a connexion ∇ on E. To any polynomial function $P : M_k(\mathbb{C}) \longrightarrow \mathbb{C}$ such that $P(XY) = P(YX)$ for any $X, Y \in M_k(\mathbb{C})$ (one say that P is an *invariant polynomial*), we can associate a closed differential form $P(E)$ on M by the formula $P(E) = P(\Omega)$, where Ω is the curvature of the connexion ∇, which is a 2-form on M with values in $End(E)$. Choosing a local framing for E, we may identify Ω with a matrix of ordinary 2-forms. Since P is an invariant polynomial, one can check that $P(\Omega)$ is a well defined differentiable form (independent of the choice of the local framing) whose cohomology class, again denoted by $P(E)$, does not depend on the connexion ∇ in E (see for instance [18], prop. 10.5, p. 112). The Chern character of E is defined by:

$$ch(E) := \sum_{k \geq 0} s_k(E) \in H^{ev}(M),$$

where s_k is the invariant polynomial $s_k(X) = \frac{1}{k!} Tr(\left(\frac{X}{4i\pi}\right)^k)$. We thus have formally:

$$ch(E) = [Tr(exp(\frac{\Omega}{4i\pi}))] \in H^{ev}(M),$$

where Ω is the curvature of some connexion ∇ on E. One can prove that the Chern character only depends on the K-theory class of E, and extends to a homomorphism $ch : K^0(X) \longrightarrow H^{ev}(X, \mathbb{Q})$, which is the Chern character of theorem 3.4.1.

3.4.3 Computation of the Chern character.

Since the ring of invariant polynominals on $M_k(\mathbb{C})$ is generated by the polynomials $c_k(X) = \frac{Tr(\Lambda^k X)}{(4i\pi)^k}$, we can express $ch(E)$ from the corresponding *Chern classes* $c_k(E) \in H^{2k}(X, \mathbb{Q})$. We get:

$$ch(E) = dim(E) + c_1(E) + \frac{1}{2}(c_1^2(E) - 2c_2(E)) + ... \quad \text{(see below)}.$$

Exercise. *Show that* $c_i(\overline{E}) = (-1)^i c_i(E)$ *where* $\overline{E}$ *is the conjugate bundle of* E.

Let us denote by (x_i) the eigenvalues of $\frac{\Omega}{4i\pi}$. We have:

$$\begin{cases} x_1 + ... + x_m = c_1 \\ x_1 x_2 + ... + x_{m-1} x_{m-2} = c_2 \\ .... \\ x_1 ... x_m = c_m \end{cases}$$

where $m = dim(E)$, so that the Chern classes are the elementary symetric functions of the x_i. It follows that any symmetric formal power series in the x_i, which can therefore be expressed in terms of the elementary symmetric functions of the x_i, yields a cohomology class in $H^*(M, \mathbb{Q})$. For instance, any function $f(z)$ holomorphic near $z = 0$ gives rise to a cohomology class by the formula:

$$f(E) = \prod_{i=1}^{m} f(x_i).$$

When $E = \bigoplus_{i=1}^{m} L_i$ is a sum of complex line bundles L_i, we can choose $x_i = c_1(L_i) \in H^2(M, \mathbb{Z})$ and we get from theorem 3.4.1:

$$ch(E) = \sum_{i+1}^{m} ch(L_i) = \sum_{i=1}^{m} e^{x_i} = m + \sum_{i=1}^{m} x_i + \frac{1}{2} \sum_{i=1}^{m} x_i^2 + ...$$
$$= dim(E) + c_1(E) + \tfrac{1}{2}(c_1^2(E) - 2c_2(E)) + ...$$

For instance, it follows from the relation $\Lambda^k(E \oplus F) = \sum_{i+j=k} \Lambda^i(E) \otimes \Lambda^j(F)$ that:

$$ch([\Lambda^{even}(E)] - [\Lambda^{odd}(E)]) = \prod_{i=1}^{m} (1 - e^{x_i}).$$

Of course, a complex vector bundle E over M is not always isomorphic to a sum of complex line bundles. However, if we just want to identify $ch(E)$ with some naturally defined cohomology class (for instance, with $f(E)$ for some power series $f(z)$), we may use the following *splitting principle* (see [9], proposition 5.2, p. 237 for a proof):

Splitting principle. *For any complex vector bundle over a manifold M, there exists a smooth fibration $f : N \longrightarrow M$ such that:*

(i) $f^*(E)$ splits into a direct sum of complex line bundles;

(ii) $f^* : H^*(M, \mathbb{Q}) \longrightarrow H^*(N, \mathbb{Q})$ *is injective.*

To define the higher K-theory groups $K^n(X)$ $(n \geqq 1)$ of a locally compact space X, we shall directly define the K-groups $K_n(A)$ of a C^*-algebra A and set $K^n(X) = K_n(C_0(X))$, where $C_0(X)$ is the C^*-algebra of continuous functions on X vanishing at the infinity.

3.5 Topological K-theory for C*-algebras

3.5.1 C*- algebras. Recall that a C^*-algebra is a complex Banach algebra A with involution $x \in A \longrightarrow x^* \in A$ whose norm satisfies:

$$\|x^*x\| = \|x\|^2 \text{ for any x} \in \text{A}.$$

By Gelfand theory, any commutative C^*-algebra is isometrically isomorphic to the C^*-algebra $C_0(X)$ of complex continuous functions vanishing at infinity on some locally compact space X (the spectrum of A). If H is a Hilbert space, a closed $*$-subalgebra of $B(H)$ is a C^*-algebra, and any C^*-algebra can be realized as a closed $*$-subalgebra of $B(H)$ for some Hilbert space H. For instance, the algebra $K(H)$ of all compact operators on a separable Hilbert space H is a C^*-algebra. C^*-algebras naturally appear in non-commutative topology to describe "quantum spaces" like the quotient of a locally compact space X by a non proper action of a discrete group Γ For instance, the "dual" of a discrete groupe Γ. is described by the C^*-algebra $C^*(\Gamma)$ generated in $B(l^2(\Gamma))$ by the left regular representation λ defined by:

$$[\lambda(g)\xi](h) = \xi(g^{-1}h), \text{ where } g, h \in \Gamma \text{ and } \xi \in l^2(\Gamma).$$

Another example is the crossed product C^*-algebra $A \times_\alpha G$ of a C^*-algebra A by a continuous action $g \in G \longrightarrow \alpha_g \in Aut(A)$ of a locally compact group G acting on A by automorphisms. Here, we assume that $g \in G \longrightarrow \alpha_g(x) \in A$ is continuous for any $x \in A$ and we denote by Δ_G the modular function of G. The vector space $C_c(G, A)$ of continuous compactly supported functions on G with values in A has a natural structure of $*$-algebra. To describe this structure, it is convenient to write any element $a \in C_c(G, A)$ as a formal integral $a = \int a(g)U_g dg$, where U_g is a letter satisfying:

$$U_{gh} = U_g U_h, \; U_g^* = U_g^{-1} = U_{g^{-1}}, \text{ and } U_g x U_g^{-1} = \alpha_g(x) \text{ for any } x \in A.$$

Then, the product and the involution on $C_c(G,A)$ are given by:

$$\left(\int a(g)U_g dg\right)\left(\int b(g)U_g dg\right) = \int c(g)U_g dg,$$

$$\text{where } c(g) = \int a(h)\alpha_h(b(h^{-1}g))dh$$

$$\text{and } (\int a(g)\,U_g\,dg)^* = \int b(g)\,U_g\,dg, \text{ where } b(g) = \Delta_G(g)\alpha_g(a(g^{-1})^*).$$

There are two natural ways of completing $C_c(G,A)$ to get a crossed product C^*-algebra $A \times_\alpha G$; they coincide when G is amenable. For more information on this subject, we refer to [17], p. 240.

Exercise 3.5.2. *Let θ be an irrational number and consider the action α of $\mathbb{Z}$ on $C(\mathbb{S}^1)$ defined by: $\alpha(f)(z) = f(e^{-2i\pi\theta}z)$. Show that $A_\theta = C(S^1)\times_\alpha \mathbb{Z}$ is the C^*-algebra generated by two unitaries U and V satisfying the commutation relation:*

$$UV = e^{2i\pi\theta}VU.$$

(non commutative 2-torus).

3.5.3 $\mathbf{K_0}$ of a $\mathbf{C^*}$-algebra.

Let A be a unital C^*-algebra. Recall that a finitely generated right module $\mathbf{E}$ over A is called *projective* if there exists a right module $\mathbf{F}$ over A such that $\mathbf{E}\oplus\mathbf{F} \cong A^n$. For instance, if $e \in M_n(A)$ is an idempotent, $\mathbf{E} = eA^n$ is a finitely generated projective module over A. Conversely, any finitely generated projective right A-module is of the above form.

Exercise 3.5.4. *Let X be a compact space. For any complex vector bundle E on X, denote by $\mathbf{E}$ the module of continuous sections of E. Show that $\mathbf{E}$ is a finitely generated projective module over $C(X)$.*

Let A be a unital C^*-algebra, and denote by $\mathbf{K}_0(A)$ the set of isomorphism classes of finitely generated projective A-modules. The direct sum of modules induces a commutative and associative sum on $\mathbf{K}_0(A)$.

Definition 3.5.5. *The group of formal differences $[\mathbf{E}] - [\mathbf{F}]$ of elements in $\mathbf{K}_0(A)$ is denoted by $K_0(A)$.*

Exercise 3.5.6. *Show that* $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$.

Note that any unital $*$-homomorphism $\pi : A \longrightarrow B$ between unital C^*-algebras induces a group homorphism $\pi_* : K_0(A) \longrightarrow K_0(B)$ by $\pi_*([\mathbf{E}]) = [\mathbf{E} \otimes_A B]$.
For a non unital C^*-algebra A, the unital morphism

$$\varepsilon : (a, \lambda) \in \widetilde{A} = A \oplus \mathbb{C} \longrightarrow \lambda \in \mathbb{C}$$

from the algebra $\widetilde{A}$ obtained by adjoining a unit to A to the scalars induces a K-theory map $\varepsilon_* : K_0(\widetilde{A}) \longrightarrow K_0(\mathbb{C})$. By definition, $K_0(A)$ is the kernel of ε_*.

3.5.7 $\mathbf{K_n}$ of a $\mathbf{C^*}$-algebra $(\mathbf{n \leq 1})$.
To define $K_n(A)$ $(n \in \mathbb{N} - \{0\})$ for a C^*-algebra A with unit, consider the group $GL_k(A)$ of invertible elements in $M_k(A)$. Let $GL_\infty(A)$ be the union of the $GL_k(A)$'s, where $GL_k(A)$ embeds in $GL_{k+1}(A)$ by the map $X \longrightarrow \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}$. Note that $GL_\infty(A)$ is a topological group for the inductive limit topology.

Definition 3.5.8. For $n \geqq 1$, we set: $K_n(A) := \pi_{n-1}(GL_\infty(A))$.

We thus define a group which is abelian for $n \geqq 2$, since the homotopy group π_n of a topological group is abelian for $n \geqq 1$.

Exercise 3.5.9. *Show that* $\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}$ *are in the same connected component of* $GL_{2n}(A)$ *for any* $X \in GL_n(A)$. *Deduce that* $K_1(A)$ *is abelian.*

Exercise 3.5.10. *Show that* $K_1(M_n(\mathbb{C})) = \{0\}$.

Any unital $*$-homomorphism $\pi : A \longrightarrow A$ between unital C^*-algebras yields a group homorphism $\pi_* : K_n(A) \longrightarrow K_n(B)$. For a non unital C^*-algebra A, the group $K_n(A)$ will be the kernel of the map $\varepsilon_* : K_n(\widetilde{A}) \longrightarrow K_n(\mathbb{C})$.

3.6 Main properties of the topological K-theory for C*-algebras

The following theorem summarizes the main properties of the K-theory for C^*-algebras:

Theorem 3.6.1. *The covariant functor* $A \longrightarrow K_n(A)$ *satisfies the following properties:*

(i) (Homotopy invariance). *We have* $(\pi_0)_* = (\pi_1)_* : K_n(A) \longrightarrow K_n(B)$ *for any path* $\pi_t : A \longrightarrow B$ $(t \in [0,1])$ *of unital* $*$*-homomorphisms from* A *to* B *such that* $t \longrightarrow \pi_t(x)$ *is norm continuous for any* $x \in A$*;*

(ii) (Stability). *There is a natural isomorphism* $K_n(A) \cong K_n(A \otimes K(H))$*, where* $K(H)$ *is the algebra of compact operators on a separable infinite dimensional Hilbert space* H*;*

(iii) For any $n \geqq 0$, there exists a natural isomorphism $\beta_n : K_n(A) \longrightarrow K_{n+2}(A)$;

(iv) (Six terms exact sequence).*Any short exact sequence of* C^**-algebras* $0 \longrightarrow J \xrightarrow{i} A \xrightarrow{p} B \longrightarrow 0$ *yields a cyclic exact sequence in* K*-theory:*

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{i_*} & K_0(A) & \xrightarrow{p_*} & K_0(B) \\ \uparrow \delta & & & & \delta \downarrow \\ K_1(B) & \xleftarrow{p_*} & K_1(A) & \xleftarrow{i_*} & K_1(J) \end{array}$$

(v) (Bott periodicity). *For any* C^**-algebra* A*, there exists a natural isomorhism* $K_{i+n}(A) \xrightarrow{\approx} K_i(A \otimes C_0(\mathbb{R}^n))$*;*

(vi) (Thom isomorphism). *For any continuous action* α *of* $\mathbb{R}^n$ *by automorphisms of the* C^**-algebra* A*, there exists a natural isomorphism*

$$K_{i+n}(A) \xrightarrow{\approx} K_i(A \times_\alpha \mathbb{R}^n),$$

where $A \times_\alpha \mathbb{R}^n$ *is the crossed product* C^**-algebra of* A *by the action* α *of* $\mathbb{R}^n$*.*

Let us make some comments on the proofs.

Property (i) is obvious for $n \geq 1$. For $n = 0$, it follows from the fact that two nearby projections e, f in a C^*-algebra A are equivalent, i.e. there exists $u \in A$ such that $u^* = e$ and $u^*u = f$ (henceforth, $u : eA \longrightarrow fA$ is an A-module isomorphism).

Property (ii) is an immediate corollary of (i).

Property (iii) is a theorem, originally proved by Bott. It implies that the K-theory of a C^*-algebra A reduces to the groups $K_0(A)$ and $K_1(A)$. For $n = 0$, the isomorphism $\beta_0 : K_0(A) \longrightarrow K_2(A)$ is easy to describe: it sends the class of the module $eA^n (e = e^* = e^2 \in M_n(A))$ to the class of the loop:

$$z \in U(1) \longrightarrow ze + (1-e) \in GL_n(A).$$

Property (iv) is a consequence of the long exact sequence for the homotopy groups of a fibration, which reduces here to a cyclic exact sequence in view of (iii).
Property (v) follows from (iv) for the exact sequence

$$0 \longrightarrow C_0(]0,1[,A) \xrightarrow{i} C_0(]0,1],A) \xrightarrow{p} A \longrightarrow 0,$$

where p is the evaluation at 1, since we have $K_n(C_0(]0,1],A)) = 0$.

Exercise 3.6.2. *Show that the path of $*$-morphisms $\pi_t : C_0(]0,1],A) \longrightarrow C_0(]0,1],A)$ ($t \in [0,1]$) defined by $\pi_t(f)(s) = \begin{cases} f(s-t) & if\ 0 \leq t < s \leq 1 \\ 0 & if\ 0 \leq s \leq t \leq 1 \end{cases}$ yields a homotopy between 0 and Id. Deduce that $K_n(C_0(]0,1],A)) = 0$.*

Property (vi) was originally proved by A. Connes [5], and can be reduced to Bott periodicity (see for instance [7]).

3.7 Kasparov's picture of $\mathbf{K_0(A)}$

In analogy with Atiyah's description of $K^0(X)$ for a compact space X, it is possible to describe $K_0(A)$ for any unital C^*-algebra A from *generalized A-Fredholm operators.* The main change consists in replacing the notion of Hilbert space by that of *Hilbert C^*-module.*

Definition 3.7.1. *Let A be a C^*-algebra. We call Hilbert C^*-module over A (or Hilbert A-module) any right A-module $\mathbf{E}$ equipped with an A-valued scalar product $< .,. >$ satisfying the following conditions:*

(i) $< \xi, \lambda\eta > = \lambda < \xi, \eta >$ and $< \xi, \eta a > = < \xi, \eta > a$ for any $\xi, \eta \in \mathbf{E}, a \in A$;
(ii) $< \xi, \eta > = < \eta, \xi >^*$ for any $\xi, \eta \in \mathbf{E}$;
(iii) $< \xi, \xi > \in A^+$ for any $\xi \in \mathbf{E}$ and ($< \xi, \xi > = 0 \Longrightarrow \xi = 0$);
(iv) $\mathbf{E}$ is complete for the norm $\| \xi \| = \| < \xi, \xi > \|_A^{1/2}$.

A basic example of Hilbert A-module is the completion H_A of the algebraic direct sum $A \oplus A \oplus A \oplus ...$ for the norm associated with the A-valued scalar product

$< (x_n), (y_n) >= \sum_{n \geqq 1} x_n^* y_n \in A$. In fact, one can show in analogy with the Hilbert space theory [11] that the sum of any countably generated Hilbert A-module with H_A is isomorphic to H_A.

Exercise 3.7.2. *Let X be a compact space. Show that the Hilbert $C(X)$-modules are exactly the spaces of continuous sections of continuous fields of Hilbert spaces over X.*

Definition 3.7.3. *Let $\mathbf{E}$ be a Hilbert A-module. We shall call endomorphism of $\mathbf{E}$ any map $T : \mathbf{E} \longrightarrow \mathbf{E}$ such that there exists $T^* : \mathbf{E} \longrightarrow \mathbf{E}$ satisfying:*

$$< T\xi, \eta >=< \xi, T^*\eta > \;\; for\ any\ \xi, \eta \in \mathbf{E}.$$

An endomorphism of $\mathbf{E}$ is automatically A-linear and bounded. We denote by $B(\mathbf{E})$ the space of all endomorphisms of $\mathbf{E}$; it is a C^*-algebra for the operator norm. The closed ideal of $B(\mathbf{E})$ generated by the "rank one" operators $\xi \longrightarrow \xi_2 < \xi_1, \xi >$ $(\xi_1, \xi_2 \in \mathbf{E})$ is denoted by $K(\mathbf{E})$; we call it the algebra of *compact operators of the Hilbert A-module* $\mathbf{E}$.

Definition 3.7.4. *Let A be a unital C^*-algebra and $\mathbf{E}$ a countably generated Hilbert A-module. A generalized A-Fredholm operator on $\mathbf{E}$ is by definition an endomophism $P \in B(\mathbf{E})$ such that there exists $Q \in B(\mathbf{E})$ with $R = I - PQ \in K(\mathbf{E})$ and $S = 1 - QP \in K(\mathbf{E})$.*

3.7.5 Generalized Fredholm A-index and Kasparov's definition of $\mathbf{K_0(A)}$.

Let A be a unital C^*-algebra and consider a *generalized A-Fredholm operator* $P \in B(\mathbf{E})$. In analogy with the description of the index of a Fredholm operator given in proposition 1.3.6, we shall define a *generalized A-index* $Ind_A(P) \in K_0(A)$. Since $\mathbf{E} \oplus H_A$ is isomorphic to H_A, we may assume, replacing P by $\begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}$ if necessary, that $\mathbf{E} = H_A$. In analogy with proposition 1.3.6, the generalized A-index $Ind_A(P) \in K_0(A) = K_0(K \otimes A)$ is defined by the formula:

$$Ind_A(P) = \left[X \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X^{-1}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in K_0(K(H_A)) = K_0(K \otimes A),$$

where $X \in B(H_A \oplus H_A)$ is some invertible lift of:

$$\begin{pmatrix} \dot{P} & 0 \\ 0 & \dot{P}^{-1} \end{pmatrix} \in B(H_A \oplus H_A)/K(H_A \oplus H_A).$$

Here, $[e]$ is a shorthand for $[e\widetilde{B}^n]$ for any idempotent e of $M_n(\widetilde{B})$. Since the K-theory class of $\left[X \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X^{-1}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$ does not depend on the choice of such an invertible lift X, the only point that we need to check is the existence of such a lift. With this aim in mind, consider the element:

$$X = \begin{pmatrix} P + (I - PQ)P & PQ - I \\ I - QP & Q \end{pmatrix} \in B(H_A \oplus H_A).$$

It is equal to $\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$ modulo $K(H_A \oplus H_A)$ and the formula:

$$\begin{pmatrix} P + (I - PQ)P & PQ - I \\ I - QP & Q \end{pmatrix} = \begin{pmatrix} I & P \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -Q & I \end{pmatrix} \begin{pmatrix} I & P \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

shows that it is invertible, with inverse:

$$X^{-1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & -P \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix} \begin{pmatrix} I & -P \\ 0 & I \end{pmatrix} = \begin{pmatrix} Q & I - QP \\ PQ - I & P + (I - PQ)P \end{pmatrix}.$$

Since we have $X \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X^{-1} = \begin{pmatrix} I - R^2 & (I + R)PS \\ SQ & S^2 \end{pmatrix}$, the generalized A-index $Ind - A(P) \in K_0(A)$ will be finally defined by:

$$Ind_A(P) = \left[\begin{pmatrix} I - R^2 & (I + R)PS \\ SQ & S^2 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in K_0(A).$$

One can show that the index map $P \in Fred_A(H_A) \longrightarrow Ind_A(P) \in K_0(A)$ from the space $Fred_A(H_A)$ of generalized A-Fredholm operators to $K_0(A)$ induces a group isomorphism from $\pi_0(Fred_A(H_A))$ to $K_0(A)$. This leads Kasparov [12] to define $K_0(A)$ as the set of homotopy classes of triples

$(\mathbf{E}^0, \mathbf{E}^1, F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix})$ where $\mathbf{E} = \mathbf{E}^0 \oplus \mathbf{E}^1$ is a graded Hilbert A-module and F an element in $B(\mathbf{E}^0 \oplus \mathbf{E}^1)$ such that:

$$F^2 - I = \begin{pmatrix} QP - I & 0 \\ 0 & PQ - I \end{pmatrix} \in K(\mathbf{E}^0 \oplus \mathbf{E}^1).$$

This will be used in section 5.

4 The Atiyah-Singer Index Theorem

4.1 Statement of the theorem

We are now in position to state the Atiyah-Singer index theorem, which computes the analytical index of an elliptic operator P on M from its principal symbol $\sigma(P)$. Note that $\sigma(P)$ is a purely topological data which can be viewed as an element in $K^0(T^*M)$ (see remark after definition 3.3.3).

Theorem 4.1.1 (Atiyah-Singer index theorem). *Let P be an elliptic pseudodifferential operator on an n-dimensional compact oriented manifold M without boundary. Denote by $\sigma(P)$ the principal symbol of P, viewed as an element of the K-theory (with compact support) group $K^0(T^*M)$. Let $\pi! : H^*(T^*M) \longrightarrow H^*(M)$ be the integration's map (in cohomology with compact support) on the fibre of the canonical projection $\pi : T^*M \longrightarrow M$. Then, we have:*

$$Ind(P) = (-1)^{\frac{n(n+1)}{2}} \int_M ch_M(\sigma(P)) Td_{\mathbb{C}}(TM \otimes \mathbb{C}),$$

where $ch_M(\sigma(P)) = \pi! ch(\sigma(P))$ is the image of $ch(\sigma(P)) \in H^*(T^*M)$ by $\pi!$

The main steps of the proof are the following:

(i) *Construction of an analytical map $Ind_a : K^0(T^*M) \longrightarrow \mathbb{Z}$*, called the *analytical index,* such that $Ind_a(\sigma(P)) = Ind(P)$ for any elliptic pseudodifferential operator P on M with principal symbol $\sigma(P) \in K^0(T^*M)$;

(ii) *Construction of a topological map $Ind_t : K^0(T^*M) \longrightarrow K^0(T^*\mathbb{R}^N) = \mathbb{Z}$*, called the *topological index*, by using an embedding $M \longrightarrow \mathbb{R}^N$;

(iii) Proof of the equality $Ind_a = Ind_t$;

(iv) *Computation of the topological index* Ind_t by using the Chern character, to get the cohomological formula:

$$Ind_t(x) = (-1)^{\frac{n(n+1)}{2}} \int_M ch_M(x) Td_{\mathbb{C}}(TM \otimes \mathbb{C}).$$

4.2 Construction of the analytical index map

Let $P : C^\infty(M, E) \longrightarrow C^\infty(M, F)$ be an elliptic operator of order m on M, and consider its principal symbol $p = \sigma(P) \in C^\infty(T^*M, Hom(\pi^*E, \pi^*F))$, which is a bounded function in $S^m(T^*M, Hom(\pi^*E, \pi^*F))$. By ellipticity, there exists a bounded map $q \in S^{-m}(T^*M, Hom(\pi^*F, \pi^*E))$ such that $pq - I$ and $qp - I$ are bounded functions in $S^{-1}(T^*M, Hom(\pi^*F, \pi^*F))$ and $S^{-1}(T^*M, Hom(\pi^*E, \pi^*E))$ respectively. Note that the index of P only depends on the homotopy class of $\sigma(P)$:

Proposition 4.2.1. *Let $P_0, P_1 : C^\infty(M, E) \longrightarrow C^\infty(M, F)$ be two elliptic operators of order m on M. Assume that there exists a homotopy inside the symbols of elliptic operators of order m between the principal symbols*

$$p_0 = \sigma(P_0), p_1 = \sigma(P_1) \in C^\infty(T^*M, Hom(\pi^*E, \pi^*F))$$

of P_0 and P_1. Then, we have: $Ind(P_0) = Ind(P_1)$.

Proof. Any homotopy $t \in [0, 1] \longrightarrow p(t) \in C^\infty(T^*M, Hom(\pi^*E, \pi^*F))$ inside the symbols of elliptic operators of order m between p_0 and p_1 yields by pseudo-differential calculus a continuous field of Fredholm operators $P(t) : H^{s+m}(M, E) \longrightarrow H^s(M, F)$ with $P(0) = P_0$ and $P(1) = P_1$. By homotopy invariance of the index, we get $Ind(P_0) = Ind(P_1)$. QED

Let $P : C^\infty(M, E) \longrightarrow C^\infty(M, F)$ be an elliptic operator of order m on M, and choose a homogeneous function h of degree one on T^*M which is positive and C^∞ outside the zero section. By using proposition 4.2.1 we can show that, for any pseudodifferential operator $P_\tau : C^\infty(M, E) \longrightarrow C^\infty(M, F)$ with principal symbol $p_\tau(x, \xi) = p(x, \tau \frac{\xi}{h(\xi)})$, we have:

$$Ind(P_\tau) = Ind(P) \text{ for any } \tau > 0 \text{ sufficiently large.}$$

In other words, to determine the index of elliptic operators, it suffices to study operators with polyhomogeneous symbols of order 0. This leads to the following definition of the *analytical index map.* We shall use the following notation:

Notation. If $p \in C(T^*M, Hom(\pi^*E, \pi^*F))$ is a continuous section of the bundle $Hom(\pi^*E, \pi^*F)$ such that the set $\{(x, \xi) \in T^*M; p(x, \xi) \text{ is not invertible}\}$ is compact, we shall set: $p_\tau(x, \xi) = p(x, \tau \frac{\xi}{h(\xi)})$. We thus define a homogeneous continuous symbol p_τ which is invertible for τ sufficiently large.

Proposition and definition 4.2.2. *Let $p \in C(T^*M, Hom(\pi^*E, \pi^*F))$ be such that the set of $(x, \xi) \in T^*M$ where $p(x, \xi)$ is not invertible is compact.*

(i) We have $Ind(P_1) = Ind(P_2)$ for any pair $P_1, P_2 : C^\infty(M, E) \longrightarrow C^\infty(M, F)$ of elliptic pseudodifferential with principal symbols $p_i(x, \xi)$ $(i = 1, 2)$ satisfying: $\underset{x,\xi}{Sup} \|p_\tau(x, \xi)^{-1} p_i(x, \xi) - I\| < 1$ for $\tau > 0$ large enough;

(ii) Set $Ind_a(p) := Ind(P)$, where $P : C^\infty(M, E) \longrightarrow C^\infty(M, F)$ is any elliptic pseudo-differential operator with principal symbol $p(x, \xi)$ satisfying $\underset{x,\xi}{Sup} \|p_\tau(x, \xi)^{-1}$
$p(x, \xi) - I\| < 1$ for $\tau > 0$ large enough. Then, $Ind_a(p)$ doesn't depend on the choice of h and τ. We call $Ind_a(p)$ the analytical index of p;

(iii) We have $Ind_a(p) = Ind(P)$ if P is an elliptic pseudodifferential operator of order 0 with polyhomogeneous principal symbol p of order 0;

(iv) If $t \in [0, 1] \longrightarrow p_t \in C(T^*M, Hom(\pi^*(E), \pi^*(F)))$ is a continuous path such that there exists a compact K in T^*M with $p_t(x, \xi)$ invertible for all t when $(x, \xi) \notin K$, then $Ind_a(p_t)$ is independent of t.

By (iv), the map $p \longrightarrow Ind_a(p)$ defined in (ii) yields a K-theory map $Ind_a : K^0(T^*M) \longrightarrow \mathbb{Z}$, called the *analytical index map.* Let us show that this K-theory map can be defined in a purely topological way.

4.3 Construction of the topological index map

Let M be a compact oriented manifold without boundary of dimension n. There is a natural way to send $K^0(T^*M) \cong K^0(TM)$ to $K^0(pt) = \mathbb{Z}$ that we shall now describe. Choose an imbedding $i : M \longrightarrow \mathbb{R}^N$ of M into

$\mathbb{R}^N$ (such an embedding always exist) and denote by $di : TM \longrightarrow T\mathbb{R}^N$ the corresponding proper imbedding of TM into $T\mathbb{R}^N$. The normal bundle to this embedding identifies wit the pull-back to TM of $N \oplus N$, where N is the normal bundle to the imbedding $i : M \longrightarrow \mathbb{R}^N$. Let us identify $N \oplus N$ with a tubular neighbourhood W of TM in $T\mathbb{R}^N$. Then, the Thom isomorphism for hermitian complex vector bundles (cf. [10]) yields a map $K^0(TM) \longrightarrow K^0(N \oplus N) \cong K^0(W)$.
Since W is an open subset of $T\mathbb{R}^N$, the natural inclusion $C_0(W) \longrightarrow C_0(T\mathbb{R}^N)$ yields a map $K^0(W) \longrightarrow K^0(T\mathbb{R}^N)$ and hence, by composition, a map:

$$i! : K^0(TM) \longrightarrow K^0(T\mathbb{R}^N) = K^0(\mathbb{R}^{2N}).$$

Note that any smooth proper embedding $i : M \longrightarrow V$ of M into a smooth manifold V yields in the same way a natural map $i! : K^*(TM) \longrightarrow K^*(TV)$ which does not depend on the factorization of $di : TM \longrightarrow TV$ through the zero section associated with a tubular neighbourhood of TM into TV. Since $\mathbb{R}^{2N} = \mathbb{R}^N \oplus \mathbb{R}^N = \mathbb{C}^N \longrightarrow pt$ can be considered as a complex vector bundle over a point, we have a Thom isomorphism $K^0(pt) \longrightarrow K^0(\mathbb{R}^{2N})$ whose inverse is just the Bott periodicity isomorphism: $\beta : K^0(\mathbb{R}^{2N}) \longrightarrow K^0(pt) = \mathbb{Z}$. Taking $V = \mathbb{R}^{2N}$ for some large enough N, the composition map:

$$Ind_t = \beta \circ i! : K^0(T^*M) \cong K^0(TM) \longrightarrow \mathbb{Z}$$

is called the *topological index.* One can prove that it does not depend on the choice of the imbedding $i : M \longrightarrow \mathbb{R}^N$. The main content of the Atiyah-Singer index theorem is in fact the equality:

$$Ind_a = Ind_t : K^0(T^*M) \longrightarrow \mathbb{Z},$$

which allows computing the analytical index of an elliptic operator by a cohomological formula.

4.4 Coincidence of the analytical and topological index maps

The proof of the equality $Ind_a = Ind_t$ is based on the following two properties of the analytical index:

Property 1. For $M = pt$, the analytical index $Ind_a : K^0(pt) \longrightarrow \mathbb{Z}$ is the identity.

Property 2. For any smooth embedding $i : M \longrightarrow V$ between compact smooth manifolds, the following diagram is commutative:

$$\begin{array}{ccc} K^*(TM) & \xrightarrow{i!} & K^*(TV) \\ Ind_a \searrow & & \swarrow Ind_a \\ & \mathbb{Z} & \end{array}$$

To check that these properties imply the equality $Ind_a = Ind_t$, choose an embedding $i : M \longrightarrow \mathbb{R}^N \subset S^N = \mathbb{R}^N \cup \{\infty\}$ and denote by $j : \{\infty\} \longrightarrow S^N$ the inclusion of the point ∞. By property 2, we have for any $x \in K^0(TM)$:

$$Ind_a(x) = Ind_a(i!x) = Ind_a(j!^{-1}i!x)$$

and, since $Ind_a \circ j!^{-1} = j!^{-1}$ by property 1, and $j!^{-1}$ is just the Bott periodicity isomorphism on $K^0(\mathbb{R}^N)$, we get:

$$Ind_a(x) = j!^{-1} \circ i!(x) = Ind_t(x).$$

Exercise 4.4.1. *Check directly property 1.*

To prove property 2, consider a tubular neighbourhood of M in V, which is diffeomorphic to the normal bundle N of M in V. To prove that

$$Ind_a(x) = Ind_a(i!x) \text{ for any } x \in K^0(TM),$$

one can show that it suffices to prove that $Ind_a(x) = Ind_a(j!x)$, where $j : M \longrightarrow N$ is the inclusion of the zero section. We may also replace (cf. [16]) the principal O_k-bundle $N = P \times_{O_k} \mathbb{R}^k$ by the associated sphere bundle $S_N = P \times_{O_k} S^k$, where O_k acts on S^k by trivially extending the natural representation on $\mathbb{R}^k$ to $\mathbb{R}^{k+1} = \mathbb{R}^k \times \mathbb{R}$ and then restricting to the unit sphere. In other word, we may compactify the fibre of N. If $j : M \longrightarrow S_N$ denotes the natural inclusion, we have for any $x \in K^0(TM) \cong K^0(T^*M)$:

$$j!(x) = x \otimes [D],$$

where $[D] \in K_{O_k}(T^*S^k)$ is the equivariant K-theory class of the de Rham-Hodge O_k-operator $d + d^* : \Lambda^{even} \longrightarrow \Lambda^{odd}$. Here, the product:

$$K(T^*M) \otimes K_{O_k}(T^*S^k) \longrightarrow K(T^*S_N)$$

is naturally defined by using a splitting $T^*S_N = \pi^*(T^*M) \oplus T(S_N/M)$, where $T(S_N/M) = T^*S_N/\pi^*(T^*M)$ denotes the tangent spaces along the fibres of the projection $\pi : S_N \longrightarrow M$, the obvious inclusion

$$K_{0_k}(T^*S^k) \longrightarrow K_{0_k}(P \times T^*S^k) \longrightarrow K(P \times_{O_k} T^*S^k) = K(T(S_N/M)),$$

and the external product: $K(T^*M) \otimes K(T(S_N/M)) \longrightarrow K(T^*S_N)$.
In this setting, property 2 follows from the multiplicativity property for sphere bundles:

Proposition 4.4.2. *Let S be an S^k-bundle over a compact manifold M. For any $x \in K(T^*M)$ and $[P] \in K_{O_k}(T^*S^k)$, we have:*

$$Ind_a(x.[P]) = Ind_a(x.Ind_{O_k}(P)),$$

where $Ind_{O_k}(P) \in R(O_k)$ is the equivariant index of the O_k-operator P. Here, the $R(O_k)$-module structure on $K(T^*M)$ (which is a $K(M)$-module in an obvious way) comes from the natural morphism $R(O_k) \longrightarrow K(M)$.

The equivariant index of an O_k-operator P is heuristically defined as the difference $Ind_{0_k}(P) = [Ker\ P] - [Ker\ P^*]$ of O_k-representations.

Since the equivariant index of the de Rham-Hodge O_k-operator $d + d^* : \Lambda^{even} \longrightarrow \Lambda^{odd}$ on S^k is equal to $1 \in R(O_k)$ (see for instance [16], p. 253), we get from proposition 4.4.2:

$$Ind_a(j!(x)) = Ind_a(x \otimes [D]) = Ind_a(x.Ind_{O_k}(D)) = Ind_a(x)$$

for any $x \in K^0(TM) \cong K^0(T^*M)$, and property 2 is proved. The technical proof of proposition 4.4.2 is modelled on the proof of the *multiplicativity of the analytical index:*

$$Ind_a([P] \otimes [Q]) = Ind_a(P)Ind_a(Q), \tag{1}$$

for any pair of first order elliptic operators P and Q on compact manifolds M and N. Since $Ind_a([P] \otimes [Q]) = Ind_a(D)$ where D is the sharp product:

$$D = \begin{pmatrix} P \otimes 1 & -1 \otimes Q^* \\ 1 \otimes Q & P^* \otimes 1 \end{pmatrix},$$

this multiplicative property (1) is straightforward.

4.5 Cohomological formula for the topological index

Let $\pi : E \longrightarrow M$ be a complex vector bundle of rank n over a smooth manifold M. Denote by i the zero section and consider the following diagram:

$$\begin{array}{ccc} K_0(M) & \xrightarrow{i!=Thom\ iso,\ in\ K-theory} & K_0(E) \\ ch\downarrow & & \downarrow ch \\ H^{ev}(M) & \xrightarrow{i!=Thom\ iso,\ in\ cohomology} & H^{ev}(E) \end{array}$$

where $i! : H^{ev}(M) \longrightarrow H^{ev}(E)$ is the inverse of the *"integration on the fibres"* $\pi! : H^{ev}(E) \longrightarrow H^{ev}(M)$, which is an isomorphism in cohomology. It turns out that this diagram is not commutative, since the cohomology class

$$\tau(E) = \pi!\, ch(i!(1)) \in H^{ev}(M)$$

is not trivial in general. This cohomology class really measures the defect of commutativity in the above diagram, since:

Proposition 4.5.1. *For any $x \in K^0(M)$, we have: $ch(i!(x)) = i!(ch(x)\tau(E))$.*

Exercise 4.5.2. *Check proposition 4.5.1.*

The computation of the obstruction class $\tau(E)$ follows from the formula:

$$\chi(E)\tau(E) = i_* i! \tau(E) = ch([\Lambda^{even} E] - [\Lambda^{odd} E]),$$

where $\chi(E)$ is the Euler class of E. If E is a complex bundle of dimension k over M, we get from a formal splitting $E \cong L_1 \oplus ... \oplus L_k$ of E into line bundles:

$$\chi(E)\tau(E) = \left(\prod_{i=1}^{k} c_1(L_i)\right)\tau(E) = \left(\prod_{i=1}^{k} x_i\right)\tau(E),$$

where $x_i = c_1(L_i)$. On the other hand, since $\Lambda^p(E \oplus F) = \bigoplus_{i+j=p} \Lambda^i(E) \otimes \Lambda^j(F)$, we get from the multiplicativity of the Chern character:

$$ch([\Lambda^{even} E] - [\Lambda^{odd} E]) = \prod_{i=1}^{k} (1 - e^{x_i}).$$

We deduce that: $\tau(E) = \prod_{i=1}^{k} \dfrac{1 - e^{x_i}}{x_i} = (-1)^k \prod_{i=1}^{k} \left(\dfrac{1 - e^{-(-x_i)}}{-x_i} \right) = (-1)^{dim(E)} Td_{\mathbb{C}}(\overline{E})^{-1}$, where $\overline{E}$ is the conjugate of E and the Todd class $Td_{\mathbb{C}}(E)$ is defined by the formal power series $\frac{x}{1-e^{-x}}$, i.e. $Td_{\mathbb{C}}(E) = \prod_{i=1}^{k} \left(\frac{x_i}{1-e^{-x_i}} \right)$.

We are now in position to give a cohomological formula for the topological index:

Theorem 4.5.3. *Let M be an n-dimensional compact oriented manifold without boundary. Denote by $\pi! : H^*(T^*M) \longrightarrow H^*(M)$ the integration's map (in cohomology with compact support) on the fiber of the canonical projection $\pi : T^*M \longrightarrow M$. Then, we have for any $x \in K^0(T^*M)$:*

$$Ind_t(x) = (-1)^{\frac{n(n+1)}{2}} \int_M ch_M(x) Td_{\mathbb{C}}(TM \otimes \mathbb{C}),$$

where $ch_M(x) = \pi! ch(x)$ is the image of $ch(x) \in H^*(T^*M)$ by $\pi!$.

Proof. In the following diagram, where N is the normal bundle to some inclusion $M \hookrightarrow \mathbb{R}^n$ as before:

$$\begin{array}{ccccccc}
K^0(TM) & \xrightarrow{i!} & K^0(N \oplus N) \cong K^0(W) & \longrightarrow & K^0(\mathbb{R}^{2N}) & \longrightarrow & K^0(p^t) = \mathbb{Z} \\
\downarrow ch & & \downarrow ch & & \downarrow ch & & id \downarrow \\
H^{ev}(TM) & \xrightarrow{i!} & H^{ev}(N \oplus N) \cong H^{ev}(W) & \longrightarrow & H^{ev}(\mathbb{R}^{2N}) & \xrightarrow{q!} & H^0(p^t) = \mathbb{Z}
\end{array}$$

the two squares on the right commute since $\tau(\mathbb{C}^N) = 1$ so that we have $Ind_t(x) = q!i!(ch(x)\tau(N \oplus N))$ for any $x \in K^0(T^*M)$. Since $TM \oplus N$ is trivial and $TM \otimes \mathbb{C}$ is self-conjugate, we get:

$$\tau(N \otimes \mathbb{C}) = \tau(TM \otimes \mathbb{C})^{-1} = (-1)^n Td_{\mathbb{C}}(TM \otimes \mathbb{C})$$

and hence:

$$Ind_t(x) = (-1)^n \int_{TM} ch(x) Td_{\mathbb{C}}(TM \otimes \mathbb{C}).$$

Taking into account the difference between the orientation of TM induced by the one of M and the "almost complex" orientation of $T(TM)$, we get from the above formula by using the Thom isomorphism in cohomology:

$$Ind_t(x) = (-1)^{\frac{n(n+1)}{2}} \int_M ch_M(x) Td_{\mathbb{C}}(TM \otimes \mathbb{C}). \text{ QED}$$

One obtain various index theorems by applying the above formula to $x = \sigma$, the symbol of an elliptic pseudodifferential operator.

Exercise 4.5.4. *Show that the index of any elliptic differential operator P on an odd-dimensional compact manifold M is zero (Hint: use the Atiyah-Singer index theorem together with the formulas: $c_*[TM] = -[TM], c^*(\sigma(P)) = \sigma(P) \in K^0(T^*M)$ where c is the diffeomorphism $\xi \in TM \longrightarrow -\xi \in TM$).*

Exercise 4.5.5. *By using the Atiyah-Singer index formula for the de Rham operator on a compact oriented manifold M, prove the equality:*

$$\sum_{i=0}^{dim(M)} (-1)^i dim H^i(M, \mathbb{R}) = \int_M \chi(T_{\mathbb{C}} M).$$

4.6 The Hirzebruch signature formula

From the Atiyah-Singer formula, we get the Hirzebruch formula:

Theorem 4.6.1. *The signature $\sigma(M)$ of any $4k$ oriented compact smooth manifold M is given by:*

$$\sigma(M) = \int_M L(M),$$

where $L(M)$ is the Hirzebruch-Pontrjagin class defined from a formal splitting $TM \otimes \mathbb{C} = \bigoplus_{i=1}^{2k} (L_i \oplus \overline{L}_i)$ into complex line bundles by: $L(M) = 2^{2k} \prod_{i=1}^{2k} \frac{c(L_i)/2}{th(c(L_i)/2)}$.

Proof. Let D_+ be the signature operator on M (cf. 2.3.2.). Since we have:

$$[\sigma(D_+)] = [(\Lambda^+(T^*M \otimes \mathbb{C}), \Lambda^-(T^*M \otimes \mathbb{C}), i(ext(\xi) - int(\xi)))],$$

we get from a formal splitting $TM \otimes \mathbb{C} = \bigoplus_{i=1}^{2k} (L_i \oplus \overline{L}_i)$ of $TM \otimes \mathbb{C}$ into complex line bundles, by setting $x_i = c(L_i)$:

$$\begin{aligned} ch_M(\sigma(D_+)) &= \frac{ch([\Lambda^+(T^*M\otimes\mathbb{C})]-[\Lambda^-(T^*M\otimes\mathbb{C})])}{\chi(TM)} \\ &= \prod_{i=1}^{2k}\frac{ch([\overline{L}_i]-[L_i])}{x_i} = \prod_{i=1}^{2k}\frac{e^{-x_i}-e^{x_i}}{x_i} = 2^{2k}\prod_{i=1}^{2k}\frac{e^{x_i}-e^{-x_i}}{2x_i} \\ &= 2^{2k}\prod_{i=1}^{2k}\frac{x_i/2}{th(x_i/2)}\left(\prod_{i=1}^{2k}\frac{x_i/2}{sh(x_i/2)}\right)^{-2}, \end{aligned}$$

and hence:

$$\begin{aligned} &ch_M(\sigma(D_+))Td_{\mathbb{C}}(TM\otimes\mathbb{C}) = \\ &\qquad = 2^{2k}\prod_{i=1}^{2k}\frac{x_i/2}{th(x_i/2)}\left(\prod_{i=1}^{2k}\frac{x_i/2}{sh(x_i/2)}\right)^{-2}\left(\prod_{i=1}^{2k}\frac{x_i/2}{sh(x_i/2)}\right)^{2} \\ &\qquad = 2^{2k}\prod_{i=1}^{2k}\frac{x_i/2}{th(x_i/2)} = L(M). \end{aligned}$$

Since the signature of M is equal to $Ind(D_+)$ by theorem 2.3.6, we get from the Atiyah-Singer index formula:

$$\sigma(M) = \int_M ch_M(x)Td_{\mathbb{C}}(TM\otimes\mathbb{C}) = \int_M L(M).\ \text{QED}$$

5 The index theorem for foliations

5.1 Index theorem for elliptic families

5.1.1 Elliptic families. Consider a smooth fibration $p : M \longrightarrow B$ with connected fiber F on a compact manifold M. For each $y \in B$, set $F_y = p^{-1}(y)$ and denote by $T_F^*(M)$ the bundle dual to the bundle $T_F(M)$ of vectors tangent to the fibres of the fibration. Let $q : T_F^*(M) \longrightarrow M$ be the projection map and consider a family $P = (P_y)_{y\in B}$ of zero order pseudodifferential operators:

$$P_y : C^\infty(F_y, E^0) \longrightarrow C^\infty(F_y, E^1)$$

on the fibres of the fibration $p : M \longrightarrow B$, where $E = E^0 \oplus E^1$ is a $\mathbb{Z}/2\mathbb{Z}$ graded hermitian vector bundle over M.

Definition 5.1.2. *The family $(P_y)_{y\in B}$ is said to be continuous if the map P defined on $C^\infty(M, E^0)$ by $Pf(x) = (P_y f_y)(x)$ $(y = p(x), f_y = f|_{F_y})$ sends $C^\infty(M, E^0)$ into $C^\infty(M, E^1)$.*

The *principal symbol* of such a continuous family $P = (P_y)_{y\in B}$ is by definition the family $\sigma(P) = (\sigma(P_y))_{y\in B}$ of the symbols of the P_y's. It can be viewed as a vector bundle morphism $\sigma(P) : q^*(E^0) \longrightarrow q^*(E^1)$. The family $P = (P_y)_{y\in B}$ is said to be *elliptic* if all the P_y's are elliptic. In this case, the principal symbol $\sigma(P)$ yields a K-theory class $[\sigma(P)] \in K^0(T_F^*M)$.

5.1.3 Analytical index of a family of elliptic operators. Let P be as above. By working locally as in the case of an elliptic operator, we can prove the existence of a continuous family $Q = (Q_y)_{y\in B}$ of zero order pseudodifferential operators such that $P_yQ_y - I = R_y$ and $Q_yP_y - I = S_y$ are continuous families of infinitely smoothing operators. In particular, the family $P = (P_y)_{y\in B}$ gives rise to a continuous field of Fredholm operators $P_y : L^2(F_y, E^0) \longrightarrow L^2(F_y, E^1)$, and the index $Ind(P) \in K_0(B)$ of this family of Fredholm operators makes sense by theorem 3.2.1.

5.1.4 Topological index of a family of elliptic operators. On the other hand, the principal symbol $\sigma(P)$ yields a K-theory class $[\sigma(P)] \in K^0(T_F^*M)$. To define a *topological index* $Ind_{top} : K^0(T_F^*M) \longrightarrow K^0(B)$, let us choose a smooth map $f : M \longrightarrow B \times \mathbb{R}^N$ which reduces for any $y \in B$ to a smooth embedding $f_y : F_y \longrightarrow \{y\} \times \mathbb{R}^N$. Such a map gives rise to a smooth embedding $f_* : T_FM \longrightarrow B \times T\mathbb{R}^N$ with normal bundle $N \oplus N$, where N_y is the normal bundle of $f_y(F_y)$ in $\{y\} \times \mathbb{R}^N$, pulled back to M. Since $N_y \oplus N_y \simeq N_y \otimes \mathbb{C}$, we have a well defined Gysin map:

$$p! : K^0(T_F^*M) \longrightarrow K^0(N\otimes\mathbb{C}) \longrightarrow K^0(B\times T\mathbb{R}^N) = K^0(B\times \mathbb{R}^{2N}) \longrightarrow K^0(B),$$

where the last map on the right is the Bott periodicity isomorphism. We call *topological index* the map:

$$Ind_{top} = p! : K^0(T_F^*M) \longrightarrow K^0(B),$$

which is well defined and does not depend on the choice of $f : M \longrightarrow B\times \mathbb{R}^N$. At this point, it is almost clear that the proof of the Atiyah-Singer index theorem for elliptic operators on compact manifolds extends to the framework of fibrations to give the following index theorem:

Theorem 5.1.5. *Let $p : M \longrightarrow B$ be a smooth fibration with fiber F on a compact manifold M, and $P = (P_y)_{y\in B}$ be a continuous family of elliptic* *zero order pseudodifferential operators on the fibres* $F_y = p^{-1}(y)$*. Then, we have:*

(i) $Ind(P) = Ind_{top}([\sigma(P)]) \in K^0(B)$;
(ii) $ch(Ind(P)) = (-1)^{\frac{n(n+1)}{2}} \pi!(ch([\sigma(P)])Td_{\mathbb{C}}(TM)) \in H^*(B)$,

where $n = dim(F)$ *is the dimension of the fibre and* $\pi : T_F^*M \longrightarrow B$ *is the natural projection.*

For a detailed proof of this result, see [2].

5.2 The index theorem for foliations

5.2.1 Foliations. Let M be a smooth n-dimensional compact manifold. Recall that a smooth p-dimensional subbundle F of TM is called *integrable* if every $x \in M$ is contained in the domain U of a submersion $p : U \longrightarrow \mathbb{R}^{n-p}$ such that $F_y = Ker(p_*)_y$ for any $y \in U$. A p-dimensional *foliation* F on M is given by an integrable p-dimensional subbundle F of TM. We call *leaves* of the foliation (M, F) the maximal connected submanifolds L of M such that $T_x(L) = F_x$ for any $x \in L$. In any foliation (M, F), the equivalence relation on M corresponding to the partition into leaves is locally trivial, i.e. every point $x \in M$ has a neighborhood U with local coordinates $(x^1, ..., x^n) : U \longrightarrow \mathbb{R}^n$ such that the partition of U into connected components of leaves corresponds to the partition of $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ into the plaques $\mathbb{R}^p \times \{y\}$. We shall called $p = dim(F)$ the *dimension* of the foliation, and $q = n - dim(F) = codim(F)$ the *codimension* of F. For instance, any smooth fibration $p : M \longrightarrow B$ with connected fiber on a compact manifold defines a foliation on M whose leaves are the fibres $p^{-1}(y), y \in B$. If θ is an irrational number, the flow of the differential equation $dy - \theta dx = 0$ on the two-dimensional torus $M = \mathbb{R}^2/\mathbb{Z}^2$ defines a codimension 1 foliation F_θ on M called the irrational *Kronecker foliation.*

Exercise 5.2.2. *Show that each leaf of the irrational Kronecker foliation* F_θ *on the* 2*-dimensional torus* $M = \mathbb{R}^2/\mathbb{Z}^2$ *is non compact and dense. Let* M/F *be the quotient of* M *by the equivalence relation corresponding to the partition into leaves. Show that the quotient topology on* M/F *is trivial (the only open subsets are* M/F *and* $\varnothing$*).*

5.2.3 Holonomy of a foliation. Let $\gamma : [0,1] \longrightarrow M$ be a continuous path on a leaf L of F, and consider two q-dimensional submanifolds T, T' transverse to the foliation and whose interiors contain respectively the source $x = \gamma(0)$ and the range $y = \gamma(1)$ of γ. By "following the leaves" through some small enough tubular neighborhood of $\gamma([0,1])$, we get from γ a local diffeomorphism $\varphi_\gamma : Dom(\varphi_\gamma) \subset T \longrightarrow T'$ with $x = s(\gamma) \in Dom(\varphi_\gamma)$. The *holonomy germ* of γ is by definition the germ $h(\gamma) = [\varphi_\gamma]_x$ of φ_γ at $x = s(\gamma)$. Two paths $\gamma_1, \gamma_2 : [0,1] \longrightarrow L$ having the same source $x = s(\gamma_1) = s(\gamma_2)$ and the same range $y = r(\gamma_1) = r(\gamma_2)$ are said holonomy equivalent (and we write $\gamma_1 \sim \gamma_2$) if there exist transverse submanifolds T at x and T' at y such that $h(\gamma_1) = h(\gamma_2)$. We thus define an equivalence relation on the set of all paths drawn on the leaves. The *holonomy groupoid* of the foliation is by definition the set G of all equivalence classes. Any $\gamma \in G$ is thus the holonomy class of a path on some leaf, with source x (denoted by $s(\gamma)$) and range y (denoted by $r(\gamma)$). For any $x \in M$, we shall set $G_x = \{\gamma \in G | r(\gamma) = x\}$. The composition of paths induces a natural structure of groupoid on G, and it can be shown that G has the structure of a smooth (possibly non Hausdorff) manifold. For more information on the holonomy groupoïd G of (M, F), see [3].

Exercise 5.2.4. *Show that the holonomy groupoid of the Kronecker foliation* F_θ *on the* 2*-dimensional torus* $M = \mathbb{R}^2/\mathbb{Z}^2$ *identifies with* $M \times \mathbb{R}$*. Describe its groupoid structure and its smooth structure.*

5.2.5 C^*-algebra of a foliation. For a fibration, the space of leaves is a nice compact space which identifies with the base of the fibration. However, for a foliation with dense leaves such as the Kronecker foliation F_θ, the space of leaves can be very complicated although the local picture is that of a fibration. A. Connes [3] suggested describing the topology of the "leafspace" M/F of any foliation (M, F) by a C^*-algebra $C^*(M, F)$ whose noncommutativity tells us how far the foliation lies from a fibration. This C^*-algebra, which describes the non commutative space M/F, is obtained by quantization of the holonomy groupoid. More precisely, $C^*(M, F)$ is defined as the minimal C^*-completion of the algebra of continuous compactly supported sections $C_c(G, \Omega^{1/2})$ of the bundle $\Omega^{1/2}_\gamma = \Omega^{1/2}_{s(\gamma)} \otimes \Omega^{1/2}_{r(\gamma)}$ of half densities along the leaves of the foliation, endowed with the following laws:

$$\begin{cases} (f*g)(\gamma) = \int\limits_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2) \\ f^*(\gamma) = \overline{f(\gamma^{-1})}. \end{cases}$$

When the foliation comes from a fibration $p : M \longrightarrow B$, the C^*-algebra $C^*(M,F)$ identifies with $C(B)\otimes K(L^2(F))$, where $K(H)$ denotes the algebra of compact operators on the Hilbert space H. In the case of the Kronecker foliation F_θ, we have $C^*(\mathbf{T}^2, F_\theta) \simeq A_\theta \otimes K(H)$ where H is a separable infinite dimensional Hilbert space and A_θ the irrational rotation algebra generated by two unitaries U and V in H satisfying the commutation relation $VU = exp(2i\pi\theta)UV$.

5.2.6 Elliptic operators along the leaves of a foliation. Let (M,F) be a smooth foliation on a compact manifold M and E^0, E^1 two smooth complex vector bundles over M. A *differential operator elliptic along the leaves* of (M,F) acting from the sections of E^0 to the sections of E^1 is a differential operator $D : C^\infty(M,E^0) \longrightarrow C^\infty(M,E^1)$ which restricts to the leaves and is elliptic when restricted to the leaves. Its principal symbol $\sigma(D)(x,\xi) \in Hom(E^0_x, E^1_x)$ is thus invertible for any non zero $\xi \in F^*_x$ and yields a K-theory class

$$[\sigma(D)] \in K^0(F^*).$$

Since a foliation is locally a fibration, the notion of elliptic pseudodifferential operator along the leaves of (M,F) can be defined in a natural way. As in the case of fibrations, it generalizes the notion of differential elliptic operator along the leaves.

5.2.7 Analytical index of an operator elliptic along the leaves. Assume for simplicity that the foliation (M,F) has no holonomy and consider an elliptic pseudodifferential operator P of order zero along the leaves of (M,F), acting from the sections of E^0 to the sections of E^1. The restriction P_L of P to the leaf L is a bounded operator in the Hilbert space $H_L = L^2(L, E^0 \oplus E^1)$. Moreover, the family $(P_L)_{L\in M/F}$ yields in a natural way an endomorphism of a Hilbert $C^*(M,F)$-module that we now describe. Let $\mathbf{E}$ be the Hilbert completion of the linear span of the 1/2 sections of the field $H_x = L^2(G_x, E^0 \oplus E^1)$ that have the from $x \longrightarrow \int\limits_{G_x} (\xi\circ\gamma)f(\gamma)$ where ξ is a basic 1/2 section of H and $f \in C^*(M,F)$ an element with a square integrable restriction to G_x for any $x \in M$. There is an obvious structure of $C^*(M,F)$-module on $\mathbf{E}$. Moreover, since the coefficient $< \xi, \eta > (\gamma) =< \xi_{r(\gamma)} \circ \gamma, \eta_{s(\gamma)} >$ is required to be in $C^*(M,F)$ for any pair ξ, η of basic 1/2 sections of H, the $C^*(M,F)$-valued scalar product $< \xi, \eta >$ of two elements in $\mathbf{E}$ is well defined, and it is straightforward to check that $\mathbf{E}$ is a Hilbert $C^*(M,F)$-module (see [4] for more details).

Exercise 5.2.8. *Show that the family $(P_L)_{L \in M/F}$ yields an endomorphism P of the $C^*(M,F)$-module $\mathbf{E}$.*

By using the local construction of a parametrix for families of elliptic operators, one can show as in the case of families the existence of an endomorphism Q of $\mathbf{E}$ such that $PQ - I \in K(\mathbf{E})$ and $QP - I \in K(\mathbf{E})$. It follows that P is a generalized $C^*(M,F)$-Fredholm operator, and hence has an analytical index:

$$Ind_{C^*(M,F)}(P) \in K_0(C^*(M,F)).$$

5.2.9 The index theorem for foliations. To compute $Ind_{C^*(M,F)}(P)$ we shall define, as in the case of fibrations, a *topological index:*

$$Ind_t : K^0(F^*) \longrightarrow K_0(C^*(M,F))$$

by choosing an auxiliary embedding $i : M \longrightarrow \mathbb{R}^{2m}$. Let N be the total space of the normal bundle to the leaves (i.e. $N_x = i_*(F_x)^\perp$ for the Euclidean metric) and consider the product manifold $M \times \mathbb{R}^{2m}$ foliated by the $L \times \{t\}$'s (L = leaf of $F, t \in \mathbb{R}^{2m}$). The map $(x, \xi) \in N \longrightarrow (x, i(x) + \xi) \in M \times \mathbb{R}^{2m}$ sends a small neighborhood of the zero section of N into an open transversal T to the foliation $\widetilde{F}$ on $M \times \mathbb{R}^{2m}$. Putting T inside a small open tubular neighborhood Ω in $M \times \mathbb{R}^{2m}$ we get, from the inclusion of $C^*(\Omega, \widetilde{F}) \cong C_0(T) \otimes K(H)$ into $C^*(M \times \mathbb{R}^{2m}, \widetilde{F})$, a K-theory map:

$$K_0(C_0(T)) = K_0(C_0(T) \otimes K(H)) = K_0(C^*(\Omega, \widetilde{F})) \longrightarrow K_0(C^*(M \times \mathbb{R}^{2m}, \widetilde{F})).$$

Since we have, by Bott periodicity:

$$K_0(C^*(M \times \mathbb{R}^{2m}, \widetilde{F})) = K_0(C^*(M,F) \otimes C_0(\mathbb{R}^{2m})) \cong K_0(C^*(M,F)),$$

we get by composition a K-theory map:

$$Ind_t : K^0(F^*) \cong K^0(N) \cong K^0(T) = K_0(C_0(T)) \longrightarrow K_0(C^*(M,F)).$$

This map, which does not depend on the choices made, is called the *topological index.*

Theorem 5.2.9. (Index theorem for foliations). *For any zero order elliptic pseudodifferential operator P along the leaves of a foliation (M,F) with principal symbol $\sigma(P) \in K^0(F^*)$, we have:*

$$Ind_{C^*(M,F)}(P) = Ind_t(\sigma(P)) \in K_0(C^*(M,F)).$$

For a proof of this result, see [6]. When the foliation (M,F) has an invariant transverse measure Λ, there exist a trace τ_Λ on $C^*(M,F)$ which yields a map from the finite part of $K_0(C^*(M,F))$ generated by trace-class projections in $M_n(C^*(M,F))$ to $\mathbb{R}$. This trace is given by:

$$\tau_\Lambda(k) = \int_{M/F} Trace(k_L)d\Lambda(L) \ (k \in C_c^\infty(G, \Omega^{1/2})),$$

where $Trace(k_L)$ is viewed as a measure on the leaf manifold. For a zero order elliptic pseudodifferential operator P along the leaves of (M,F), one can show that $Ind_{C^*(M,F)}(P)$ belongs to the finite part of $K_0(C^*(M,F))$. In this case, we get from theorem 5.2.9 (see [3] for the original proof):

Theorem 5.2.10. (Measured index theorem for foliations). *Let (M,F) be a p-dimensional smooth foliation on a compact manifold M. Assume that (M,F) has a holonomy invariant transverse measure Λ and denote by $[\Lambda]$ the associated Ruelle-Sullivan current. For any zero order elliptic pseudodifferential operator P along the leaves with principal symbol $\sigma(P) \in K^0(F^*)$, we have:*

$$\tau_\Lambda(Ind_{C^*(M,F)}(P)) = (-1)^{\frac{p(p+1)}{2}} < ch(\sigma(P))Td_{\mathbb{C}}(F_{\mathbb{C}}), [\Lambda] > .$$

For two-dimensional leaves, this theorem gives in the case of the leafwise de Rham operator:

$$\beta_0 - \beta_1 + \beta_2 = \frac{1}{2\pi} \int K d\Lambda,$$

where the β_i are the Betti numbers of the foliation and K is the Gaussian curvature of the leaves. If the set of compact leaves is negligible, we have $\beta_0 = \beta_2 = 0$, and the above relation shows that $\int K d\Lambda \leq 0$. It follows that the condition $\int K d\Lambda > 0$ implies the existence of compact leaves.

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