DEFORMATION QUANTIZATION

A mini lecture

Martin Bordemann
Laboratoire des Mathématiques et Applications
Faculté des Sciences et Techniques
Université de Haute Alsace, Mulhouse
4, rue des Frères Lumière
68093 Mulhouse, France
e–mail: Martin.Bordemann@uha.fr

Contents

Introduction

Since the seminal article by Bayen, Flato, Frønsdal, Lichnerowicz et Sternheimer in 1978 [7] deformation quantization has become a large research area covering several algebraic theories like the formal deformation theoriy of associative algebras and the more recent theory of operades, as well as geometric theories like the theory of symplectic and (more generally) Poisson manifolds), and of physical theories like string theory and noncommutative gauge theory. In this theory, the noncommutative associative multiplication of operators in quantum mechanics is considered as a formal associative deformation of the pointwise multiplication of the 'algebra of symbols of these operators': in physical terms, this means the 'algebra of classical quantities' which are given by the algebra of all complex-valued $\mathcal{C}^\infty$-functions on a Poisson manifold, the 'phase space' of classical mechanics. The formal parameter is an interpretation of Planck's constant $\hbar$ in convergent situations. The advantage of this method is its universality: according to a theorem by Kontsevich [63] this construction is possible for any Poisson manifold. Moreover, geometric intuition is quite useful in concrete situations since everything is formulated in geometrical terms on a differentiable manifold in contrast to the usual formulation of quantum mechanics whre one has to specify a Hilbert space. The price to pay is the fact that complex numbers are replaced by the ring of all formal complex power series whose convergence is a case-by-case study.

The main objective of this mini-lecture is a more pedagogical introduction to this subject: I have not included in detail all the existence and classification proofs which are quite technical. Neither do I speak about the theory of operades which has become the algebraic framework of this theory since Kontsevich. I'd rather want to underline some motivations from physics, discuss concrete examples and talk -at the end- about the still open theory of modules and reduction.

In the first two chapters, I have given a little sketch of the relations between classical mechanics and symplectic and Poisson geometry and usual quantum mechanics. The central motivation for the deformed multiplication, the star-product, will be the symbol calculus of differential operators in Chapter 3 for several ordering prescriptions used in quantum physics. Chapter 4 is devoted to fix the notation of formal power series and to present Gerstenhaber's deformation formula which gives a more uniform view on the star-products introduced in Chapter 3. The definition and existence and classification theorems are given in Chapter 5, whereas certain explicit examples other than $\mathbb{R}^{2n}$ are discussed in Chapter 6. A preliminary discussion of representations of star-products is done in Chapter 7 where the analogues with C^*-algebras (positive linear functionals, GNS construction) are discussed.

After a second discussion of Poisson geometry, namely Poisson maps and co-isotropic maps, in Chapter 8 I have included some new results on the deformation of certain Poisson maps and coistropic (in physics: first class) submanifolds.

Je me dois excuser d'avance pour la liste des références: je n'avais pas beaucoup de temps pour profondément rechercher tous les articles importants dans le domaine, alors je les ai ramassés d'une façon hâtive, même pas brassensienne (les copains d'abord), et je prie le lecteur de me pardonner des omissions et de regarder des résumés comme [81], [75] ou les comptes rendus de la conférence Moshé Flato 1999 [35] pour plus de références.

Acknowledgements

I'd like to thank the organizers of the summer school in Villa de Leyva for the friendly *acceuil* and the perfect organization.

1 Classical mechanics, symplectic geometry and Poisson geometry

1.1 Classical Mechanics

Classical Mechanics is governed by *Newton's equations*: the motion of a particle of strictly positive mass $m \in \mathbb{R}$ in an open set of $\mathbb{R}^n$ is described by the following system of second order differential equations:

$$m\frac{d^2x_i}{dt^2} = F_i(x_1, \ldots, x_n) \tag{1.1.1}$$

where $x := (x_1, \ldots, x_n)$ denote the n coordinates of the particle and $F = (F_1, \ldots, F_n) : U \to \mathbb{R}^n$ is a $\mathcal{C}^2$-map, the force field. In case there are several particles of masses $m_1, \ldots, m_N$ the system (1.1.1) is generalized in an obvious fashion to Nn coordinates, namely $(x_{11}, \ldots, x_{nN}$ where the force field is now a $\mathcal{C}^2$-map $\mathbb{R}^{Nn} \supset U \to \mathbb{R}^{Nn}$ with U an open set in $\mathbb{R}^{Nn}$. The canonical example is the solar system with 1 particles, namely the sun and the nine planets, where the open set U is given by $\mathbb{R}^{30}$ minus the union of all the vector subspaces describing all collision situations where any two positions in $\mathbb{R}^3$ of two distinct particles coincide, and where F is the Newtonian gravitational field. The latter is an example of a conservative field, i.e. for which there is a $\mathcal{C}^\infty$-function $V : U \to \mathbb{R}$, the potential energy, such that

$$F_i = -\frac{\partial V}{\partial x_i} \quad \text{for all } 1 \leq i \leq n. \tag{1.1.2}$$

The principal idea of Hamiltonian mechanics is the transformation of the second ordre system in n variables (1.1.1) to a first order system in $2n$ variables

$$(q,p) := (q^1, \ldots, q^n, p_1, \ldots, p_n) := (x_1, \ldots, x_n, m\frac{dx_1}{dt}, \ldots, m\frac{dx_n}{dt}); \tag{1.1.3}$$

upon introducing the *Hamilton function* (sum of kinetic and potential energy)

$$H(q,p) := \sum_{i=1}^{n} \frac{{p_i}^2}{2m} \ + \ V(q), \tag{1.1.4}$$

such that the system (1.1.1) is rewritten in the following way ($1 \leq i \leq n$):

$$\begin{aligned} \frac{dq^i}{dt} &= \frac{p_i}{m} &&= \frac{\partial H}{\partial p_i}(q,p) &&=: X_{H q^i}(q,p) \\ \frac{dp_i}{dt} &= -\frac{\partial V}{\partial q^i}(q) &&= -\frac{\partial H}{\partial q^i}(q,p) &&=: X_{H p_i}(q,p) \end{aligned} \tag{1.1.5}$$

where $X_H := (X_{Hq}, X_{Hp}) := (X_{Hq^1}, \ldots, X_{Hq^n}, X_{Hp_1}, \ldots, X_{Hp_n}) : U \times \mathbb{R}^n \to \mathbb{R}^{2n}$ is called the *Hamitlonian vector field associated to the function H*. More generally such a vector field may be associated to an arbitrary $\mathcal{C}^\infty$-function $H : U \to \mathbb{R}$. It is clear that the Hamilton function H is always a first integral of the equation (1.1.5), i.e. $H\big(q(t), p(t)\big) = H\big(q(0), p(0)\big)$, which expresses the conservation of energy. The vector space $\mathcal{C}^\infty(U, \mathbb{R}) := \{f : U \to \mathbb{R} | f \ \mathcal{C}^\infty\}$ is called the set of *classical observables*, whereas the set of all points $(q, p) \in U \times \mathbb{R}^n$ is called the *phase space* or equivalently the set of *pure states* of the system.

1.2 Symplectic Geometry I

Symplectic geometry is the direct differential geometric generalization of the local concepts of the preceding subsection to the to differentiable manifolds.

Let M be a differentiable manifold. A 2-forme ω is called *symplectic* iff it is closed (i.e. $d\omega = 0$) and nondegenerate (i.e. for all $m \in M$ the fact that $\omega_m(X, Y) = 0$ for all $Y \in T_m M$ implies that $X = 0$), and the pair (M, ω) is called a *symplectic manifold*. The standard example is the vector space $\mathbb{R}^{2n}$ with coordinates $(q, p) := (q^1, \ldots, q^n, p_1, \ldots, p_n))$ equipped with the 2-form

$$\omega := \sum_{i=1}^{n} dq^i \wedge dp_i. \tag{1.2.1}$$

Other examples are all orientable manifolds of dimension 2 (e.g. all the orientable Riemann surfaces). In general, for a $\mathcal{C}^\infty$-function $H : M \to \mathbb{R}$ one defines its *Hamiltonian vector field X_H* by

$$dH =: \omega(X_H, \ ) \tag{1.2.2}$$

which is well-defined thanks to the nondegenerescence of ω. Moreover

$$L_{X_H}\omega = di_{X_H}\omega + i_{X_H}d\omega = dd\omega + 0 = 0, \tag{1.2.3}$$

and each Hamiltonian vector field is an *infinitesimal symmetry* of ω. The triple (M, ω, H) is called a *Hamiltonian system* and the function H is called the *Hamiltonian function* of the system. The first-order equation

$$\frac{dc}{dt}(t) = X_H\big(c(t)\big). \tag{1.2.4}$$

is called the *Hamiltonian equations* (of motion) corresponding to H. *Darboux's Theorem* ensures the existence of particular coordinates $(q, p) := (q^1, \ldots, q^n, p_1, \ldots, p_n)$ around each point of M in which the symplectic form takes the standard form (1.2.1), see e.g.[1], p.175, thm 3.2.2. for a proof. The fact that the symplectic form is antisymmetric immediately implies that the Hamiltonian function is always a *first integral* for ist Hamiltonian system, i.e.

$$\frac{dH\big(c(t)\big)}{dt} = dH\big(c(t)\big)X_H\big(c(t)\big) = \omega_{c(t)}\Big(X_H\big(c(t)\big), X_H\big(c(t)\big)\Big) = 0.$$

which again is the conservation of energy in physics.

We recall the definition of the *Poisson bracket* for two $\mathcal{C}^\infty$-functions $f, g : M \to \mathbb{R}$:

$$\{f, g\} := \omega(X_f, X_g) = df\ X_g = -dg\ X_f, \tag{1.2.5}$$

and we have the following

Proposition 1.1 *Let $f, g, h : M \to \mathbb{R}$ be three $\mathcal{C}^\infty$-functions on a symplectic manifold (M, ω). Then:*

1. $\{f, g\} = -\{g, f\}$ *(antisymmetry).*
2. $\{f, gh\} = \{f, g\}h + g\{f, h\}$ *(Leibniz rule).*
3. $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ *(Jacobi identity).*
4. $[X_f, X_g] = -X_{\{f,g\}}$

Proof: The first two assertions are trivial and the third follows from the fourth upon aplying it to h. The foruth statement is due to the fact that $d\omega = 0$, i.e.

$$\begin{aligned} i_{[X_f,X_g]}\omega &= [L_{X_f}, i_{X_g}]\omega \overset{\text{eqn(1.2.3)}}{=} L_{X_f} i_{X_g}\omega = di_{X_f} i_{X_g}\omega + i_{X_f} di_{X_g}\omega \\ &\overset{d\omega=0}{=} d\{g, f\} + i_{X_f}(di_{X_g} + i_{X_g}d)\omega \overset{\text{eqn(1.2.3)}}{=} -i_{X_{\{f,g\}}}\omega + 0. \end{aligned}$$

□

It follows that the Poisson bracket equips the space of classical observables $\mathcal{C}^\infty(M)$ with the structure of a Lie algebra.

Definition 1.1 *Let (M, ω) symplectic manifold. A connection ∇ in the tangent bundle TM is called* symplectic *iff*

$$\nabla_X \omega = 0 \quad \text{for each vector field } X \text{ on } M.$$

The following Proposition (due to Heß, Lichnerowicz, Tondeur, see e.g. [55]) is an analogue of the Levi-Civita Theorem in (semi)riemannian geometry although the uniqueness of the symplectic connection does no longer hold:

Theorem 1.1 *Let* (M, ω) *be a symplectic manifold and* $\tilde{\nabla}$ *a torsionfree connection in the tangent bundle* TM *of* M.
1. Then the following formula defines a symplectic torsionfree connection ∇ *in* TM*:*

$$\omega(\nabla_X Y, Z) := \omega(\tilde{\nabla}_X Y, Z) + \frac{1}{3}(\nabla_X \omega)(Y, Z) + \frac{1}{3}(\nabla_Y \omega)(X, Z)$$

for all vector fields X, Y, Z *on* M.
2. The difference between two torsionfree symplectic connections ∇ *and* ∇' *defines a totally symmetric tensor field* S *of rank* 3 *in* $\Gamma(M, S^3T^*M)$*:*

$$\omega(\nabla'_X Y, Z) - \omega(\nabla_X Y, Z) =: S(X, Y, Z),$$

for all vector fields X, Y, Z *on* M.
3. For any totally symmetric tensor field S *of rank* 3 *in* $\Gamma(M, S^3T^*M)$ *and every torsion free symplectic connection* ∇ *in* TM *the connection* ∇' *defined by*

$$\omega(\nabla'_X Y, Z) := \omega(\nabla_X Y, Z) + S(X, Y, Z), \quad \forall\ X, Y, Z \in \Gamma(M, TM)$$

is a torsion free symplectic connection in TM.

The proof is a direct verification.

Remark: there are topological obstructions for compact manifolds to admit a symplectic form ω: the class of each of the following closed $2k$-forms $\omega^{\wedge k}$, $1 \leq k \leq n := \dim M/2$ has to be non zero: in fact, if there was a $k-1$-form θ with $\omega^{\wedge k} = d\theta$, the the volume form $\omega^{\wedge n}$ would be equal to $d\theta \wedge \omega^{\wedge(n-k)} = d(\theta \wedge \omega^{\wedge(n-k)})$ which would be absurd since the total volume $\int_M \omega^{\wedge n}$ of M would be zero by Stokes's Theorem. For example, the spheres S^{2n} do not admit any symplectic structure for all $n \geq 2$.

1.2.1 Cotangent bundles

Let Q be a differentiable manifold, T^*Q its cotangent bundle, and $\tau_Q^* : T^*Q \to Q$ the canonical bundle projection. *The canonical* 1*-form* θ_0 on the manifold T^*Q is defined in the following manner: let $q \in Q$, $\alpha \in T_qQ^*$, and $W_\alpha \in T_\alpha T^*Q$, then

$$\theta_0(\alpha)\big(W_\alpha\big) := \alpha\big(T_\alpha \tau_Q^*\ W_\alpha\big). \tag{1.2.6}$$

Let $\big((U, (q^1, \ldots, q^n)\big)$ be a chart of Q, and $\big(T^*U, (q^1, \ldots, q^n, p_1, \ldots, p_n)\big)$ the corresponding canonical chart of T^*Q (i.e. $q^k(\alpha) := q^k\big(\tau_Q^*(\alpha)\big)$ and $p_l(\alpha) := \alpha(\partial/\partial q^l))$, then θ_0 takes the form

$$\theta_0 := \sum_{k=1} p_k dq^k \tag{1.2.7}$$

whence the fact that the *canonical* 2*-form*

$$\omega_0 := -d\theta_0 \tag{1.2.8}$$

is nondegenerate, hence a symplectic form on T^*Q. The cotangent bundles generalize the phase spaces in physics where Q is a configuration space and the fibres represent the cojugate momenta.

1.2.2 Complex Projective Space

Apart from the tori of even dimension, the complex projective spaces are the simplest compact symplectic manifolds:

Consider the complex manifold $\mathbb{C}^{n+1} \setminus \{0\}$ equipped with complex coordinates $z := (z_1 := q_1 + ip_1, \ldots, z_{n+1} := q_{n+1} + ip_{n+1})$ and with the standard symplectic form

$$\omega_0 := \frac{i}{2} \sum_{k=1}^{n+1} dz_k \wedge d\bar{z}_k = \sum_{k=1}^{n+1} dq_k \wedge dp_k. \tag{1.2.9}$$

Comnplex projective space $\mathbb{C}P^n$ is defiend by the following equivalence relation

$$z \sim z' \text{ si et seulement si } \exists \alpha \in \mathbb{C} \setminus \{0\} \text{ such that } z' = \alpha z, \tag{1.2.10}$$

and $\mathbb{C}P^n := \mathbb{C}^{n+1} \setminus \{0\}/\sim$. Let

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n : z \mapsto [z] \tag{1.2.11}$$

be the canonical projection canonique whose fibres obviously are the complex lines in $\mathbb{C}^{n+1} \setminus \{0\}$ passing through the origine There are $n+1$ complex charts (U_k, v) defined by

$$\begin{aligned} U_k &:= \{[z] \in \mathbb{C}P^n | z_k \neq 0\} \\ v &:= \left(v_1 := \frac{z_1}{z_k}, \ldots, v_{k-1} := \frac{z_{k-1}}{z_k}, v_{k+1} := \frac{z_{k+1}}{z_k}, \ldots, v_{n+1} := \frac{z_{n+1}}{z_k} \right). \end{aligned}$$

the *Fubini-Study* 2*-form* ω is defined in each chart (U_k, v) (where we set $|v|^2 := \sum_{\substack{l=1,\\ l\neq k}}^{n+1} |v_l|^2$):

$$\omega|_{U_k} := \frac{i}{2(1+|v|^2)} \left(\sum_{\substack{l=1,\\ l\neq k}}^{n+1} dv_l \wedge d\bar{v}_l - \frac{1}{(1+|v|^2)} \sum_{\substack{l,l'=1,\\ l,l'\neq k}}^{n+1} \bar{v}_l dv_l \wedge v_{l'} d\bar{v}_{l'} \right) \tag{1.2.12}$$

It can be shown that these locally defined closed two-forms $\omega|_{U_k}$ are well-behaved under the change of charts and thus define a global 2-form ω. Moreover, the map

$$\Phi : \mathbb{C}P^1 \to S^2 : [z_1, z_2] \mapsto \frac{1}{|z_1|^2 + |z_2|^2} \big(z_1 z_2 + \bar{z}_1 \bar{z}_2, -i(z_1 z_2 - \bar{z}_1 \bar{z}_2), |z_1|^2 - |z_2|^2 \big)$$

is easily computed to be a diffeomorphism.

1.3 Poisson Geometry I

Poisson geometry is a generalization of symplectic geometry in the following sens:

Let M a differentiable manifold and $H : M \to \mathbb{R}$ a $\mathcal{C}^\infty$-function. If one likes to defines a vector field associated to H which depends in a $\mathcal{C}^\infty(M, \mathbb{R})$-linear way on dH and for which H is a first integral, one needs a *bivector field* P, i.e. a $\mathcal{C}^\infty$-section of the fibre bundle $\Lambda^2 TM$. In a local chart $\big(U, (x^1, \ldots, x^n)\big)$ this bivector field P takes the form

$$P = \frac{1}{2} \sum_{i,j=1}^{n} P^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}. \tag{1.3.1}$$

The *rank of a bivector field* P in $m \in M$ is defined by the rank of the matrix $P^{ij}(m)$ in an arbitrary chart.

Each bivector field can canonically be considered as an antisymmetric bilinear form on the cotangent bundle T^*M by using the natural pairing: let α, β two 1-forms on M

$$P(\alpha, \beta) = i_\beta i_\alpha P = \sum_{i,j=1}^{n} P^{ij} \alpha_i \beta_j \tag{1.3.2}$$

Using P in this way, one defines the *Hamiltonian vector field* X_H in a fashion analogous to (1.2.2):

$$X_H := P(\ , dH) = \sum_{i,j=1}^{n} P^{ij} \frac{\partial H}{\partial x^j} \frac{\partial}{\partial x^i}. \tag{1.3.3}$$

Since $dH(X_H) = P(dH, dH) = 0$ it is clear that H is always a first integral of the dynamical system defined by X_H. Moreover, one may define a *Poisson bracket* for $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$ in a way analogous to (1.2.5) by

$$\{f, g\} := P(df, dg) = df\ X_g = -dg\ X_f. \tag{1.3.4}$$

In general, the analogue of Proposition 1.1 is no longer true, concerning the Jacobi identity and has to be demanded as an additional condition on P. In order to study that condition one introduces the following algebraic structure on the space of $\mathcal{C}^\infty$-sections of the vector bundle ΛTM, the space of *multivector fields*:

1.3.1 Schouten bracket

Definition 1.2 *Let $X_1, \dots, X_k, Y_1, \dots, Y_l$ vector fields onf the differentiable manifold M and $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$. The* Schouten bracket[1] *on M by:*

$$
\begin{aligned}
&[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l]_S := \\
&\quad \sum_{i=1}^{k} \sum_{j=1}^{l} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge X_{i-1} \wedge X_{i+1} \wedge \cdots \wedge X_k \\
&\qquad\qquad \wedge Y_1 \wedge \cdots \wedge Y_{j-1} \wedge Y_{j+1} \wedge \cdots \wedge Y_l, \\
&[X_1 \wedge \cdots \wedge X_k, f]_S := \\
&\quad \sum_{i=1}^{k} (-1)^{k-i} X_i(f) X_1 \wedge \cdots \wedge X_{i-1} \wedge X_{i+1} \wedge \cdots \wedge X_k \\
&\quad =: -(-1)^{k-1} [f, X_1 \wedge \cdots \wedge X_k]_S, \\
&[f, g]_S := 0
\end{aligned}
$$

By means of the following proposition one easily sees that this bracket does not depend on the decomposition of a multivector field in exterior products of vector fields:

Proposition 1.2 *Let ∇ be a torsion-free connection in the tangent bundle TM of M, let $X \in \Gamma(\Lambda^k TM)$ and $Y \in \Gamma(\Lambda^l TM)$ be multivector fields, and let $\big(U, (x^1, \dots, x^n)\big)$ be a local chart.*
Then the Schouten bracket of X and Y can be computed by the following formula

$$
\begin{aligned}
[X, Y]_S &= \sum_{i=1}^{n} (-1)^{k-1} i_{dx^i}(X) \wedge \nabla_{\frac{\partial}{\partial x^i}} Y \\
&\qquad - (-1)^{(k-1)(l-1)} \sum_{i=1}^{n} (-1)^{l-1} i_{dx^i}(Y) \wedge \nabla_{\frac{\partial}{\partial x^i}} X
\end{aligned}
$$

which does not depend on the chosen connection nor on the chosen chart.

[1] The pronunciation is “S-khaouten”.

The space of all smooth multivector fields, $\Gamma(M, \Lambda TM)$, equipped with the Schouten bracket, is a *Lie superalgebra*, i.e.:

Proposition 1.3 *Let $X \in \Gamma(\Lambda^k TM)$, $Y \in \Gamma(\Lambda^l TM)$ and $Z \in \Gamma(\Lambda^r TM)$. Then:*

1. $[X, Y]_S = -(-1)^{(k-1)(l-1)}[Y, X]_S$ *(graded antisymmetry).*
2. $[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(k-1)l} Y \wedge [X, Z]$ *(graded Leibniz identity).*
3. $$(-1)^{(k-1)(r-1)}[[X, Y], Z] + (-1)^{(l-1)(k-1)}[[Y, Z], X] + (-1)^{(r-1)(l-1)}[[Z, X], Y] = 0.$$
(graded Jacobi identity).

The proof is a direct computation. Note that the $\mathbb{Z}$-grading for the Schouten bracket $[\ ,\ ]_S$ is shifted by -1 to the usual grading of the exterior multiplication $\wedge$, e.g. vector fields are of degree 1 in the Grassmann algebra and of degree 0 in Schouten Lie superalgebra. In general, a $\mathbb{Z}$-graded vectorspace $\mathfrak{g} = \oplus_{k \in \mathbb{Z}} \mathfrak{g}^k$ equipped with a graded commutative multiplication and a graded Lie bracket on the shifted space $\mathfrak{g}[1]$ (where $\mathfrak{g}[1]^k := \mathfrak{g}^{k+1}$) such that both are compatible by the graded Leibniz identity, is called a *Gerstenhaber algebra.*

1.3.2 Poisson Structures

Definition 1.3 *A bivector field P on a differentiabl manifold M is called a* Poisson structure *iff*

$$[P, P] = 0 \quad \text{or} \quad \sum_{r=1}^{n} \left(\frac{\partial P^{ij}}{\partial x^r} P^{rk} + \frac{\partial P^{jk}}{\partial x^r} P^{ri} + \frac{\partial P^{ki}}{\partial x^r} P^{rj} \right) = 0.$$

If this condition is satisfied the pair (M, P) is called a Poisson manifold.

The following two propositions are proved by a direct computation:

Proposition 1.4 *Let P be a bivector field on the differentiable mamnifold M. The Poisson bracket of two real-valued smooth functions on M defined by (1.3.4) satisfies the Jacobi identity iff P is a Poisson structure.*

Proposition 1.5 *Let (M, P) and (M', P') two Poisson manifolds and $s, t \in \mathbb{R}$. By means of the idsentification $T_{(m,m')}(M \times M') = T_m M \times T_{m'} M'$ for all $m \in M$ and $m' \in M'$ and the canonical injections $i_{(m,m')} : T_m M \to T_m M \times T_{m'} M' : v \mapsto (v, 0)$ and $i'_{(m,m')} : T_{m'} M' \to T_m M \times T_{m'} M' : w \mapsto (0, w)$ one writes $P_{(1)}(m, m') := (i_{(m,m')} \otimes i_{(m,m')})(P_m)$ and $P'_{(2)}(m, m') := (i'_{(m,m')} \otimes i'_{(m,m')})(P'_{m'})$.*
Then $sP_{(1)} + tP'_{(2)}$ is a Poisson structure on the product manifold $M \times M'$.

The case $P_{(1)} - P'_{(2)}$ is sometimes denoted by $M_1 \times \overline{M_2}$.

The canonical example is $\mathbb{R}^{2n}$ equipped with the symplectic Poisson structure

$$P := \sum_{k=1}^{n} \frac{\partial}{\partial q^k} \wedge \frac{\partial}{\partial p_k}. \tag{1.3.5}$$

More generally, any symplectic manifold (M, ω) is a Poisson manifold: using the musical isomorphism $\omega^\flat : TM \to T^*M$ given by $X \mapsto \omega(X, \ )$ the bivector field $P := \omega\big((\omega^\flat)^{-1}, (\omega^\flat)^{-1}\big)$ is a Poisson structure coinciding with (1.3.5) in any Darboux chart.

As opposed to the case of a symplectic manifold where the existence of a symplectic structure can restrict the topology of the manifold, it is no longer true that the existence of a nonzero Poisson structure can influence the global nature of the manifold:

Theorem 1.2 *: Let M be a differentiable manifold of dimension $n \geq 2$, let p a point in M and k a nonnegative integer with $k \leq [n/2]$, the integer part of $n/2$. Then there always exists a Poisson structure P on M whose rank at p is equal to $2k$.*

Proof: (C.Nowak, J.Schirmer, 1996). Note first that on $\mathbb{R}^n$ the following vector fields $Z_1, \ldots, Z_n$ are independent at the origin, have compact support and commute.

$$\begin{aligned} Z_j(x) \ &:= \ \phi_j(x_1) \cdots \phi_j(x_{j-1}) \phi_1(x_j) \phi_{j+1}(x_{j+1}) \phi_{j+2}(x_{j+2}) \cdots \phi_n(x_n) \frac{\partial}{\partial x_j} \\ &\qquad \forall\ 1 \leq j \leq n \end{aligned}$$

where $\phi_j : \mathbb{R} \to \mathbb{R}$ is a nonnegative $\mathcal{C}^\infty$-function being zero for $|x| \geq j$ and equal to 1 for $|x| \leq j-(1/2)$. By means of a chart (U, ψ) of M with $p \in U$ and $\psi(p) = 0$ the Z_j are pulled back and extended by 0 outside U. These vector fields still commute. One then chooses $2k$ vector fields $X_1, \ldots, X_k$, $Y_1, \ldots, Y_k$ among them such that they are independant $p = \psi^{-1}(0)$. The following bivector field

$$P := \sum_{i=1}^{k} X_i \wedge Y_i$$

obviously is Poisson structure of rank $2k$ at p. □

1.3.3 The dual space of a Lie algebra

We obtain another very important example of a Poisson manifold in the following way: let $(\mathfrak{g}, [\ ,\ ])$ be an n-dimensional real Lie algebra and $M := \mathfrak{g}^*$

its dual space. Let $e_1, \dots, e_n$ be a base of $\mathfrak{g}$, let $e^1, \dots, e^n$ be the dual base, and $c_{lm}^k := e^k([e_l, e_m])$ the structure constants of $\mathfrak{g}$. Then for all $\xi \in \mathfrak{g}^*$ one defines on M the *linear Poisson structure* corresponding to $[\ ,\ ]$:

$$P_{\mathfrak{g}}(\xi) := \xi([\ ,\ ]) = \frac{1}{2} \sum_{k,l,m=1}^{n} \xi_k c_{lm}^k \frac{\partial}{\partial \xi_k} \wedge \frac{\partial}{\partial \xi_l}. \tag{1.3.6}$$

The Jacobi identity for this Poisson structure is a direct consequence of the Jacobi identity for the Lie bracket $[\ ,\ ]$ of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g} = \mathfrak{so}(3) \cong \mathbb{R}^3$ of all real 3×3 antisymmetric matrices with the bracket $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$ coming from the vector product is an important example for the dynamics of a *freely spinning top*: let Θ be a positive definite 3×3-matrix (the inertia tensor), and $H : \mathbb{R}^3 \to \mathbb{R}$ the real valued function $H(L) = \frac{1}{2} \sum_{i,j=1}^{3} L_i (\Theta^{-1})^{ij} L_j$. Then the dynamical system corresponding to H is the *Euler equation of a freely spinning top*

$$\frac{dL}{dt} = [\Theta^{-1} L, L]$$

where L is the angular momentum and $\Theta^{-1} L$ the angular velocity of the top.

2 Quantum Mechanics

In qunatum mechanics, the classical picture of observables, states and dynamical laws is replaced by more complicated structures along the following philosophy: according to de Broglie every particle (like e.g. an electron) always has a 'wave aspect', and to its energy E he associates Planck's formula $E = \hbar\omega$ (where ω is 2π times the frequency of the wave and $\hbar$ denotes Planck's constant) and to its momentum p he associates $\hbar k$ where the length of the vector k is given by $2\pi/\lambda$, λ being the wave length of the wave. A free particle (for which the potential energy is zero) is described by a plane wave

$$(t, q) \mapsto \psi(t, q) = \exp(-i\omega t + ik \cdot q) \tag{2.0.7}$$

where $k \cdot q := \sum_{j=1}^{n} k_j q_j$. The *wave function* ψ obviously is a solution to the equation

$$E\psi = i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi = \frac{p^2}{2m}\psi$$

where $\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial q_j^2}$ is the Laplace operator and $p^2 := p \cdot p$. Erwin Schrödinger has generalized this equation to include forces by his *Schrödinger equation*

$$i\hbar\frac{\partial\psi}{\partial t}(t, q) = -\frac{\hbar^2}{2m}\Delta\psi(t, q) + V(q)\psi(t, q) =: \big(\hat{H}\psi\big)(t, q) \tag{2.0.8}$$

where the differential operator $\hat{H}$ is called the *Hamilton operator* of the system by its evident analogy with a Hamilton function. Schrödinger's description had a big success for the nonrelativistic hydrogene atom for which $V(q) = -\alpha/|q|$, α being a constant: the set of eigenvalues of $\hat{H}$ exactly matched the experimental spectrum. Here $\hat{H}$ is considered to be a selfadjoint operator on a dense domain $D(\hat{H})$ in the *Hilbert space* $\mathcal{H} := L^2(\mathbb{R}^3, d^3q)$. The (classes of) non zero wave functions in $\mathcal{H}$ –up to a nonzero complex multiple– are interpreted as *pure states* of the quantum system, i.e. those which give a complete description of the system. the square of the absolute value of ψ, $|\psi|^2$, is then interpreted as a *probability density* for the position in case the norm of ψ is equal to 1. Of course, the wave function for the free particle is not an element of $\mathcal{H}$: seen as a tempered distribution (in the sense of Laurent Schwartz) it is an approximation (in the sense of distributions) by elements in $\mathcal{H}$ (which is dense in that distribution space). More generally, as already in classical mechanics, one may consider other dans la mécanique classique, on peut considérer d'autres *quantum observables* like for instance *position*

$$Q_k : \psi \mapsto \big(q \mapsto q_k\psi(q)\big) \tag{2.0.9}$$

or *momentum* (which is proportional to the speed for nonrelativistic systems)

$$P_l : \psi \mapsto \frac{\hbar}{i}\frac{\partial \psi}{\partial q_l}, \tag{2.0.10}$$

and in general all self-adjoint operators A defined on a dense domain $D(A) \subset \mathcal{H}$, i.e. for which one has a good definition of its spetrum. From the mathematical point of view it is more convenient to consider $B(\mathcal{H})$, the C^*-algebra of all bounded operators : $\mathcal{H} \to \mathcal{H}$ since there are no domains other than $\mathcal{H}$: the spectral projections of an unbounded self-adjoint operator are always in $B(\mathcal{H})$. Heisenberg observed that the dispersion effect of a wave translates into his famous *uncertainty relation* between the measurement of position and momentum. He deduced that the only experimental values of an observable which can sharply be measured are its eigenvalues (in general its spectral values).

There is the following synopsis

Elements	CLASSICAL MECHANICS	QUANTUM MECHANICS
{pure states}	Poisson (or symplectic) manifold (M, P)	projective space of a complex Hilbert space $\mathcal{H}$
{observables}	$\mathcal{C}^\infty(M, \mathbb{R})$	{selfadjoint operators in $\mathcal{H}$}
Algebraic structure of observables	Poisson algebra	associative algebra $B(\mathcal{H})$
Generator of a dynamical system	Hamilton function $H : M \to \mathbb{R}$	Hamilton operator $\hat{H} : D(\hat{H}) \subset \mathcal{H} \to \mathcal{H}$
Equation of motion	Hamilton's equation: $\frac{dc}{dt} = X_H(c)$	Schrödinger equation: $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$
Value of an observable in a state	value of the function f in m, $f(m)$	mean value of A in ψ, $\langle A\rangle_\psi := \langle\psi, A\psi\rangle/\langle\psi,\psi\rangle$
Exactly measurable value	value of the function f in m, $f(m)$	spectral value of A

Two systems: {states}	product manifold $(M_1 \times M_2, P_{(1)} + P_{(2)})$	projective space of the tensor product $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$
Two systems: {observables}	$\mathcal{C}^\infty(M_1, \mathbb{R}) \hat{\otimes} \mathcal{C}^\infty(M_2, \mathbb{R})$	$B(\mathcal{H}_1) \hat{\otimes} B(\mathcal{H}_2)$

The interpretation problems of quantum mechanics take their origin in the fact that the combination of the state spaces of two systems (for instance the measured system and the measuring system in a measurement), namely the projective space of the *tensor* product of two Hilbert spaces, contains 'more' states than the so-called *separated states* which are the elements of the *cartesian* product of the two projective spaces: for example, if $\mathcal{H}_1 = \mathbb{C}^{m+1}$ and $\mathcal{H}_2 = \mathbb{C}^{n+1}$, then the set of separated states, $\mathbb{C}P(m) \times \mathbb{C}P(n)$, is a 'tiny' submanifold of real dimension $2(m+n)$ of $2(m+n+mn)$-dimensional projective space $\mathbb{C}P\big((m+1)(n+1)-1\big)$ (the Segre embedding in algebraic geometry). These 'extra' states are unavoidable for each nontrivial interaction between the two systems and had been called *entangled states* (in German *verschränkte Zustände*) by Schrödinger. If the combined system is in an entangled state it is no longer possible to say that "system 1 is in state 1 and system 2 is in state 2" like in classical mechanics where every combined state is separated since the combined state space is always a cartesian product. Until now there is no commonly accepted satisfactory interpretation of these entangled states.

As opposed to the set of states there is no longer a conceptual difference between the structures of the observable spaces of combined systems: in both cases it is the (topological) tensor product of the two algebras. Therefore it seems to be more reasonable to consider the observables as more fundamental objects than the states –in contrast to the historical and intuitive physical approaches, but in agreement with the mathematics of C^*-algebras. For a detailed discussion of the interpretation problems of quantum mechanics see for example the book by d'Espagnat [30].

3 Symbol Calculus and Elementary Star-Products

In order to describe a quantum system it is necessary to know its Hamilton operator $\hat{H}$. In practise, the source of inspiration is the Hamilton function of the corresponding system of classical mechanics, and any 'reasonable' recipe of translating classical observables to quantum observables is called *quantization.*

According to P.A.M. Dirac, all quantizations should satisfy a *classical limit condition*, i.e. for all classical observables f, g

$$\hat{f}\hat{g} = \widehat{fg} + o(\hbar), \tag{3.0.11}$$

$$\hat{f}\hat{g} - \hat{g}\hat{f} = i\hbar\widehat{\{f,g\}} + o(\hbar^2). \tag{3.0.12}$$

In this section we shall discuss several possible quantizations in one degree of freedom which are used in quantum physics. We recall the differential operators Q (2.0.9) et P (2.0.10) in case $n = 1$: $(Q\psi)(q) := q\psi(q)$ and $P := (\hbar/i)\partial/\partial q$.

In the following section we denote by $\mathbb{C}[s_1, \ldots, s_N]$ the space of complex polynomials in N variables $s_1, \ldots, s_N$. Moreover, the symbol $\mathrm{Diffop}_{poly}(\mathbb{R})$ denotes the space of all differential operators with polynomial coefficients in the space $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$, i.e. an element D takes the following general form

$$\sum_{k=0}^{N} f_k \partial^k / \partial q^k \tag{3.0.13}$$

where $f_1, \ldots, f_N \in \mathbb{C}[q]$.

3.1 Standard Ordering

We shall consider the following linear map ρ_s of the space of all complex polynomials of two variables $\mathbb{C}[q,p]$ in the space $\mathrm{Diffop}_{poly}(\mathbb{R})$:

$$1 \mapsto \rho_s(1) := 1 \tag{3.1.1}$$

$$q \mapsto \rho_s(q) := Q \tag{3.1.2}$$

$$p \mapsto \rho_s(p) := P \tag{3.1.3}$$

$$q^m p^n \mapsto \rho_s(q^m p^n) := Q^m P^n \tag{3.1.4}$$

Since every differential operator in $\mathrm{Diffop}_{poly}(\mathbb{R})$ takes the form (3.0.13) is is obvious that this linear map is a bijection. The principal idea of star-products is to pull back the (noncommutative) associative multiplication of differential operators by the map ρ_s:

Proposition 3.1 *Let f, g be in $\mathbb{C}[q,p]$ and $\phi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C})$. Then*

$$\rho_s(f)\big(\phi\big) = \sum_{r=0}^{\infty} \frac{(\hbar/i)^r}{r!} \left.\frac{\partial^r f}{\partial p^r}\right|_{p=0} \frac{\partial^r \phi}{\partial q^r}. \tag{3.1.5}$$

Moreover

$$f *_s g := \rho_s^{-1}\big(\rho_s(f)\rho_s(g)\big) = \sum_{r=0}^{\infty} \frac{(\hbar/i)^r}{r!} \frac{\partial^r f}{\partial p^r} \frac{\partial^r g}{\partial q^r}$$

is well-defined associative noncommutative multiplicatiion on the space $\mathbb{C}[q,p]$ which satisfies the classical limit

$$f *_s g = fg - i\hbar \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} + o(\hbar^2).$$

Proof: The proof is direct computation; since $*_s$ is obviously isomorphic to the associative multiplication of differential operators, it is clear that $*_s$ is also associative. □

Note that for two given polynomials f, g the series in $\hbar$ is always a finite sum. Moreover, every term in that series is a *bidifferential operator* $\frac{(1/i)^r}{r!}\frac{\partial^r f}{\partial p^r}\frac{\partial^r g}{\partial q^r}$.

3.2 Weyl-Moyal ordering prescription

From the point of view of physics, standard ordering is not satisfactory: when considering the pre-Hilbert space

$$\mathcal{D}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} | f \text{ is } \mathcal{C}^\infty \text{ and } supp(f) \text{ is compact}\} \tag{3.2.1}$$

equipped with the scalar product que (for all $\phi, \psi \in \mathcal{D}(\mathbb{R})$):

$$\langle \phi, \psi \rangle := \int dq \ \overline{\phi(q)}\psi(q) \tag{3.2.2}$$

we quickly see that the two real-valued functions q and p correspond to *symmetric operators*, i.e. for $A = Q$ or $A = P$

$$\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle, \tag{3.2.3}$$

whereas the real-valued function qp corresponds to the operator QP whose adjoint in $\mathcal{D}(\mathbb{R})$ is equal to $PQ = QP - i\hbar 1$: hence $\rho_s(qp)$ is no longer symmetric which would be necessary to make it into a self-adjoint operator in the completion $L^2(\mathbb{R}, dq)$ of $\mathcal{D}(\mathbb{R})$. In order to avoid these problems, the *Weyl-Moyal ordering prescription* had been introduced: this uses a symmetrization of the monomials in Q and P.

We consider the following linear map ρ_w of the space of all complex polynomials of two variables $\mathbb{C}[q,p]$ in the space $\mathrm{Diffop}_{poly}(\mathbb{R})$:

$$1 \quad \mapsto \quad \rho_w(1) := 1 \tag{3.2.4}$$
$$q \quad \mapsto \quad \rho_w(q) := Q \tag{3.2.5}$$
$$p \quad \mapsto \quad \rho_w(p) := P \tag{3.2.6}$$
$$q^m p^n \quad \mapsto \quad \rho_s(q^m p^n) := \frac{1}{(m+n)!} \sum_{\sigma \in S_{m+n}} A_{\sigma(1)} \cdots A_{\sigma(m+n)} \tag{3.2.7}$$

where the operators $A_1, \cdots, A_{m+n}$ are given by

$$A_k := \begin{cases} Q & \text{if } 1 \leq k \leq m \\ P & \text{if } m+1 \leq k \leq m+n \end{cases}$$

For example, $\rho_w(qp) = (QP + PQ)/2$ and $\rho_w(q^2p) = (Q^2P + QPQ + PQ^2)/3$. By definition, the operator $\rho_w(f)$ is symmetric if f est réel, since you easily compute that

$$\rho_w(f)^\dagger = \rho_w(\bar{f})$$

where $A^\dagger$ is the adjoint operator of A in $\big(\mathcal{D}(\mathbb{R})), \langle\ ,\ \rangle\big)$.

For two formal parameters α, β the exponential function $\exp(\alpha q + \beta p)$ is mapped to $\rho_w\big(\exp(\alpha q + \beta p)\big) = \exp(\alpha Q + \beta P)$. Using the fact that $\rho_s\big(\exp(\alpha q + \beta p)\big) = \exp(\alpha Q)\exp(\beta P)$, the fact that $[Q,P] = i\hbar 1$,and the Baker-Campbell-Hausdorff formula, we compute

$$e^{(\alpha Q + \beta P)} = e^{\frac{\hbar\alpha\beta}{2i}} e^{\alpha Q} e^{\beta P}.$$

Since the exponential function $\exp(\alpha q + \beta p)$ is a generating function for all polynomials in q, p one realizes the following fundamental relation between standard and Weyl-Moyal ordering:

$$\rho_w(f) = \rho_s(Nf) \tag{3.2.8}$$

where the map $N : \mathbb{C}[q,p] \to \mathbb{C}[q,p]$ is defined by

$$N := e^{\frac{\hbar}{2i}\frac{\partial^2}{\partial q \partial p}}. \tag{3.2.9}$$

It is clear that N is well-defined and invertible, and one deduces that $\rho_w : \mathbb{C}[q,p] \to \mathrm{Diffop}_{poly}(\mathbb{R})$ isa a linear bijection. There is the following analogue to Proposition 3.1:

Proposition 3.2 *Let f, g be in $\mathbb{C}[q,p]$. Then*

$$f *_w g := \rho_w^{-1}\big(\rho_w(f)\rho_w(g)\big) = \sum_{r=0}^{\infty} \frac{(i\hbar/2)^r}{r!} \sum_{a=0}^{r} \binom{r}{a} (-1)^{r-a} \frac{\partial^r f}{\partial q^a p^{r-a}} \frac{\partial^r g}{\partial q^{r-a} p^a} \tag{3.2.10}$$

is a well-defined noncommutative associative multiplication on the space $\mathbb{C}[q,p]$ satisfying the classical limit

$$f *_w g = fg + \frac{i\hbar}{2}\{f,g\} + o(\hbar^2),$$

and is isomorphic to $*_s$ via N:

$$N(f *_w g) = (Nf) *_s (Ng).$$

Proof: The proof is a direct computation using the operator N. □

Again, it is easily seen that $*_w$ is a series of bidifferential operators.

3.3 Wick Ordering

There is a third quantization related to the harmonic oscillator which is very often used in quantum field theory: firstly, one forms the following complex variable variable

$$z := q + ip. \tag{3.3.1}$$

On the complex vector space

$$\bar{\mathcal{O}}(\mathbb{C}) := \{\phi : \mathbb{C} \to \mathbb{C} | \phi \text{ antiholomorphe } \}$$

one defines the following scalar product

$$\langle \phi, \psi \rangle := \frac{1}{4\pi\hbar} \int dz d\bar{z} e^{-\frac{|z|^2}{2\hbar}} \overline{\phi(\bar{z})} \psi(\bar{z}) \tag{3.3.2}$$

(which may still diverge), and finally the HIlbert space of all square integrable antiholomorphic functions

$$\mathcal{H} := \{\phi \in \bar{\mathcal{O}}(\mathbb{C}) | \langle \phi, \phi \rangle < \infty\}, \tag{3.3.3}$$

which is closed subspace of the Hilbert space $L^2\big(\mathbb{R}^2, e^{-(q^2+p^2)/(2\hbar)} dqdp/(2\pi\hbar)\big)$. the space of polynomials in the variable $\bar{z}$, $\mathbb{C}[\bar{z}]$, is a dense subspace of $\mathcal{H}$. Partial integration yields the fact that the operator A which multiplies by the variable z induces *the annihilation operator*

$$A := 2\hbar \frac{\partial}{\partial \bar{z}} \tag{3.3.4}$$

on $\mathbb{C}[\bar{z}]$. By a second partial integration $\mathbb{C}[\bar{z}]$ we see that its adjoint $A^\dagger$ (*the creation operator*) is the operator

$$(A^\dagger \phi)(\bar{z}) := \bar{z}\phi(\bar{z}). \tag{3.3.5}$$

It follows that we can –in a manner completely analogous to standard ordering– consider the following linear map ρ_{wick} of the space of all complex polynomials in two variables $\mathbb{C}[z, \bar{z}]$ in the space $\mathrm{Diffop}_{poly}(\bar{z})$ of all differential operators having polynomial coefficients and acting in the space of polynomials $\mathbb{C}[\bar{z}]$:

$$\begin{aligned} 1 &\mapsto \rho_{wick}(1) := 1 && (3.3.6)\\ z &\mapsto \rho_{wick}(z) := A && (3.3.7)\\ \bar{z} &\mapsto \rho_{wick}(\bar{z}) := A^\dagger && (3.3.8)\\ \bar{z}^m z^n &\mapsto \rho_{wick}(\bar{z}^m z^n) := A^{\dagger m} A^n && (3.3.9) \end{aligned}$$

It is obvious that this linear map is a bijection.

Proposition 3.3 *Let f, g be in $\mathbb{C}[q,p]$ and $\phi \in \mathbb{C}[\bar{z}]$. Then*

$$\rho_{wick}(f)(\phi) = \sum_{r=0}^{\infty} \frac{(2\hbar)^r}{r!} \left.\frac{\partial^r f}{\partial z^r}\right|_{z=0} \frac{\partial^r \phi}{\partial \bar{z}^r}.$$

Moreover

$$f *_{wick} g := \rho_{wick}^{-1}\big(\rho_{wick}(f)\rho_{wick}(g)\big) = \sum_{r=0}^{\infty} \frac{(2\hbar)^r}{r!} \frac{\partial^r f}{\partial z^r} \frac{\partial^r g}{\partial \bar{z}^r}$$

is a well-defined noncommutative associative multiplication on the space $\mathbb{C}[q,p]$ satisfying the classical limit

$$f *_s g = fg + 2\hbar \frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}} + o(\hbar^2).$$

Proof: The proof is completely analogous to the proof of Proposition 3.1. Note that

$$\frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p} = \frac{2}{i}\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}.$$

□

As for the relation between standard ordering and Weyl ordering there is also an analogue of the operator N (3.2.9): one defines

$$\Delta' := \frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}$$

and

$$N' := e^{\frac{\hbar}{4}\Delta'}.$$

Then, for any $f, g \in \mathbb{C}[z, \bar{z}]$:

$$N'(f *_w g) = (N'f) *_{wick} (N'g).$$

Remark: The use of antiholomorphic functions instead of holomorphic functions is a quantum mechanical tradition: the creation (i.e. increase of the degree of the polynomial) is historical related to $A^\dagger$.

4 Formal Deformations

4.1 Formal Power series

In this section I shall recall several elementary about formal power series which I shall need later on, for more details see e.g. the book by Ruiz [74]. Let R be a ring (always with unit element, for instance a field) and M a left module over R (for example R-vector space). We shall write a map $a : \mathbb{N} \to M$ in the form of a *formal power series with coefficients in M*

$$a =: \sum_{r=0}^{\infty} \lambda^r a_r$$

where $a_r := a(r)$ is called the *rth component* of a, and the symbol λ is called the *formal parameter*. The set of all formal power series with coefficients in M is denoted by $M[[\lambda]]$. The sets $M[[\lambda]]$ and $R[[\lambda]]$ carry the structure of an abelian group in the canonical way ($b = \sum_{r=0}^{\infty} \lambda^r b_r$ where $b_r \in M$):

$$a + b := \sum_{r=0}^{\infty} \lambda^r (a_r + b_r).$$

Furthermore $R[[\lambda]]$ carries the structure of ring via ($\alpha = \sum_{r=0}^{\infty} \lambda^r \alpha_r, \beta = \sum_{r=0}^{\infty} \lambda^r \beta_r$, $\alpha_r, \beta_r \in R$)

$$\alpha\beta := \sum_{r=0}^{\infty} \lambda^r \sum_{s=0}^{r} \alpha_s \beta_{r-s},$$

and $M[[\lambda]]$ becomes a left $R[[\lambda]]$-module via

$$\alpha b := \sum_{r=0}^{\infty} \lambda^r \sum_{s=0}^{r} \alpha_s b_{r-s}.$$

The *order* of a power series a, $o(a)$, is defined by the minimum of the set of all nonnegative integers r such that $a_r \neq 0$ in case $a \neq 0$ and is defined to be $+\infty$ in case $a = 0$. It can be shown that the function

$$d : M[[\lambda]] \times M[[\lambda]] \to \mathbb{R} : (a, b) \mapsto d(a, b) := \begin{cases} 2^{-o(a-b} & \text{si } a \neq b \\ 0 & \text{si } a = b \end{cases}$$

defines a metric on $M[[\lambda]]$ qui induces a Hausdorff topology called *the λ-adic topology* of $M[[\lambda]]$.

The following Lemma turns out to be useful:

Lemma 4.1 *Let* M_1 *and* M_2 *two* R*-modules and* $\Phi : M_1[[\lambda]] \to M_2[[\lambda]]$ *an* $R[[\lambda]]$*-linear map.*
Then for each nonnegative integer r *there is a unique* R*-linear map* $\Phi_r : M_1 \to M_2$ *such that*

$$\Phi(a) = \sum_{r=0}^{\infty} \lambda^r \sum_{s=0}^{r} \Phi_s(a_{r-s}) \qquad (4.1.1)$$

for all $a = \sum_{r=0}^{\infty} \lambda^r a_r \in M_1[[\lambda]]$.

Proof: The restriction of Φ to M_1 is a R-linear map into $M_2[[\lambda]]$. The components of this map are R-linear maps $\Phi_r : M_1 \to M_2$. The right hand side of (4.1.1) – which we shall call $\hat{\Phi}$ is a $R[[\lambda]]$-linear map of $M_1[[\lambda]]$ in $M_2[[\lambda]]$. By its definition, hte difference $\Phi - \hat{\Phi}$ is zero on all those formal series whose nonzero components form a finite set. Let us suppose that there is formal series $a \in M_1[[\lambda]]$ such that que $b := (\Phi - \hat{\Phi})(a)$ is not equal to 0. Let k be the order of b. Since b is also given by $b = (\Phi - \hat{\Phi})(a - a_0 - \lambda^1 a_1 - \cdots - \lambda^k a_k)$ and $a - a_0 - \lambda^1 a_1 - \cdots - \lambda^k a_k =: \lambda^{k+1} a'$ it would follow that the order of b soud be grater or equal to $k+1$ which is absurd. Hence $\Phi = \hat{\Phi}$, and the Lemma is proved. □

In case the ring R is commutative, one can easily generalize this Lemma to the case of R-multilinear maps.

4.2 Formal Deformations of Associative Algebras

Let $(\mathcal{A}_0, \mu_0)$ be an associative algebra with unit 1 over a commutative ring R.

Definition 4.1 *A formal associative deformation of the associative algebra with unit* 1*,* $(\mathcal{A}_0, \mu_0)$*, is given by a sequence of* R*-bilinear maps* $\mu_1, \mu_2, \ldots$*:* $\mathcal{A}_0 \times \mathcal{A}_0 \to \mathcal{A}_0$ *such that*

1. $$\sum_{s=0}^{r} \big(\mu_s\big(\mu_{r-s}(a,b), c\big) - \mu_s\big(a, \mu_{r-s}(b,c)\big)\big) = 0 \qquad (4.2.1)$$
   for all $r \in \mathbb{N}$ *and* $a, b, c \in \mathcal{A}_0$.
2. $\mu_r(1, a) = 0 = \mu_r(a, 1)$ *for all* $r \in \mathbb{N}$*,* $r \geq 1$ *and* $a \in \mathcal{A}_0$.

The following Proposition is obvious:

Proposition 4.1 *The space* $\mathcal{A} := \mathcal{A}_0[[\lambda]]$ *equipped with the* $R[[\lambda]]$*-bilinear multiplication* $\mu := \sum_{r=0}^{\infty} \lambda^r \mu_r$*, i.e.*

$$\mu(a, b) := \sum_{r=0}^{\infty} \lambda^r \sum_{s+t+u=0} \mu_s(a_t, b_u)$$

for all $a = \sum_{t=0}^{\infty} \lambda^t a_t$ *and* $b = \sum_{u=0}^{\infty} \lambda^u b_u$ *dans* $\mathcal{A}$*, is an associative algebra with unit element* 1 *over the ring* $R[[\lambda]]$.

For the case $r = 1$ of equation (4.2.1) wer get (writing $\mu_0(a, b) =: ab$):

$$0 = a\mu_1(b, c) - \mu_1(ab, c) + \mu_1(a, bc) - \mu_1(a, b)c =: (\delta_H\mu_1)\big(a, b, c\big)$$

where δ_H is the *Hochschild coboundary operator* defined on Kest l'opérateur cobord de Hochschild défini sur

$$C(\mathcal{A}_0, \mathcal{A}_0) := \bigoplus_{k=0}^{\infty} C^k(\mathcal{A}_0, \mathcal{A}_0) := \bigoplus_{k=0}^{\infty} Hom_R(\mathcal{A}_0 \otimes_R \cdots \otimes_R \mathcal{A}_0, \mathcal{A}_0)$$

by

$$\begin{aligned}(\delta_H f)(a_1 \otimes \cdots \otimes a_{k+1}) &:= a_1 f(a_2 \otimes \cdots \otimes a_{k+1}) \\ &+ \sum_{r=1}^{k} (-1)^r f(a_1 \otimes \cdots \otimes a_{r-1} \otimes a_r a_{r+1} \otimes \cdots \otimes a_{k+1}) \\ &+ (-1)^{k+1} f(a_1 \otimes \cdots \otimes a_k) a_{k+1}.\end{aligned}$$

It is well-known that $\delta_H^2 = 0$, hence that operator defines a cohomology theory called the *Hochschild cohomology*:

$$\begin{aligned}Z^k(\mathcal{A}_0, \mathcal{A}_0) &:= Ker\big(\delta_H : C^k(\mathcal{A}_0, \mathcal{A}_0) \to C^{k+1}(\mathcal{A}_0, \mathcal{A}_0)\big) \\ B^k(\mathcal{A}_0, \mathcal{A}_0) &:= Im\big(\delta_H : C^{k-1}(\mathcal{A}_0, \mathcal{A}_0) \to C^k(\mathcal{A}_0, \mathcal{A}_0)\big) \\ HH^k(\mathcal{A}_0, \mathcal{A}_0) &:= Z^k(\mathcal{A}_0, \mathcal{A}_0)/B^k(\mathcal{A}_0, \mathcal{A}_0)\end{aligned}$$

The elements of $Z^k(\mathcal{A}_0, \mathcal{A}_0)$ are called *Hochschild k-cocycles* of $\mathcal{A}_0$, the elements $B^k(\mathcal{A}_0, \mathcal{A}_0)$ are called *HOchschild k-coboundaries* of $\mathcal{A}_0$ (where $B^0(\mathcal{A}_0, \mathcal{A}_0) := 0$), and $HH^k(\mathcal{A}_0, \mathcal{A}_0)$ is called the k^{eme} *Hochschild cohomology group of* $\mathcal{A}_0$ (with values in $\mathcal{A}_0$).

It follows that for any formal deformation the term μ_1 is always a HOchschild-2-cocycle . In the more general case where μ is not necessarily associative it is easily computed that the *associator* of μ

$$A(a, b, c) := \mu\big(\mu(a, b), c\big) - \mu\big(a, \mu(b, c)\big)$$

satisfies the following identity:

$$\begin{aligned}0 \;=\; &\mu\big(a, A(b, c, d)\big) - A\big(\mu(a, b), c, d\big) + A\big(a, \mu(b, c), d\big) \\ &- A\big(a, b, \mu(c, d)\big) + \mu\big(A(a, b, c), d\big)\end{aligned}$$

for all $a, b, c, d \in \mathcal{A}_0$. For an associative formal deformation we demand that $A = \sum_{r=0}^{\infty} \lambda^r A_r = 0$. Let us suppose that $A_0, A_1, \ldots, A_k$ are all zero. Thanks to the preceding identity we get at order $r+1$ in λ:

$$\delta_H A_{r+1} = 0.$$

Since

$$A_{r+1} = \delta_H \mu_{r+1} + A'_{r+1}$$

where the rest A'_{r+1} contains only the terms $\mu_0, \cdots, \mu_r$ it follows that

$$\begin{array}{lccccc} \text{We have :} & \delta_H A'_{r+1} = 0 & \Longrightarrow & A'_{r+1} & \in & Z^3(\mathcal{A}_0, \mathcal{A}_0) \\ \text{We want :} & A'_{r+1} \stackrel{!}{=} -\delta_H \mu_{r+1} & \Longrightarrow & A'_{r+1} & \stackrel{!}{\in} & B^3(\mathcal{A}_0, \mathcal{A}_0) \end{array}$$

Consequently, the *recursive obstructions* to continue the construction of the term μ_{r+1} of a formal associative deformation of μ_0 are contained at each stage in

$$HH^3(\mathcal{A}_0, \mathcal{A}_0).$$

For the very important particular case where $\mathcal{A}_0$ is given by $= C^\infty(M, \mathbb{C})$ (equipped with the pointwise multiplication. Here one considers Hochschild cochains which are given by multidifferential operators. This subspace of the HOchschild complex $C\big(C^\infty(M, \mathbb{C}), C^\infty(M, \mathbb{C})\big)$ (denoted by by $C_{\text{diff}}\big(C^\infty(M, \mathbb{C}), C^\infty(M, \mathbb{C})\big)$) is a subcomplex with respcet to the Hochschild coboundary. Its cohomology is called the *differential Hochschild cohomology of* $C^\infty(M, \mathbb{C})$ and is denoted by $HH_{\text{diff}}\big(C^\infty(M, \mathbb{C}), C^\infty(M, \mathbb{C})\big)$. The computation of this cohomology is due Hochschild-Kostant-Rosenberg [56], Cahen-DeWilde-Gutt [22] et DeWilde-Lecomte [31]:

Theorem 4.1

$$HH_{\text{diff}}\big(C^\infty(M, \mathbb{C}), C^\infty(M, \mathbb{C})\big) \cong \Gamma(M, \Lambda TM).$$

A generalization of this result had been obtained by A.Connes in 1985 (see [27], p.207-210) who had replaced the differential cochains by cochains which are continuous with respect to the standard Fréchet topology of this space. Pflaum [72] and Nadaud [64] have shown that on may drop Connes's hypothesis that the Euler characteristic of the manifold is zero. In this cases the resulting Hochschild cohomology is isomorphic to the right hand side of the HKR-theorem 4.1, i.e. the space of all multivector fields on M.

4.3 Gerstenhaber's Formula

The explicit formulas for the associative multiplications $*_s$, $*_w$ et $*_{wick}$ have a common algebraic feature: the following theorem is due to M.Gerstenhaber [47], p.13, Thm.8:

Theorem 4.2 *Let (A, μ_0) be an associative algebra with unit 1 over a commutative ring k which contains the rationals $\mathbb{Q}$ where $\mu_0 : A \otimes A \to A$ denotes the (not necessarily commutative) multiplication of A. Let $D_1, \ldots, D_n, E_1, \ldots, E_n$ $2n$ be derivations of (A, μ_0) which all pairwise commute, i.e. $D_k\mu_0 = \mu_0(D_k \otimes 1 + 1 \otimes D_k)$, $E_l\mu_0 = \mu_0(E_l \otimes 1 + 1 \otimes E_l)$, $D_kD_l = D_lD_k$, $D_kE_l = E_lD_k$ and $E_kE_l = E_lE_k$ for all integers $1 \leq k, l \leq n$. Let $r := \sum_{k=1}^{n} D_k \otimes E_k$. Then on the $\mathsf{k}[[\lambda]]$-module $A[[\lambda]]$ there is a $\mathsf{k}[[\lambda]]$-bilinear associative multiplication μ defined by*

$$\mu := \mu_0 \circ e^{\lambda r}. \tag{4.3.1}$$

Proof: The following elegant reasoning has been found by A.Dimakis and F.Müller-Heussen in [34] for a particular case: one defines the following three linear maps: $A \otimes A \otimes A \to A \otimes A \otimes A$ (where $\mathbf{1}$ denotes the identity map $A \to A$) $r_{12} := r \otimes \mathbf{1}$, $r_{23} := \mathbf{1} \otimes r$ and $r_{13} := \sum_{k=1}^{n} D_k \otimes \mathbf{1} \otimes E_k$. Since the derivations commute we have $[r_{12}, r_{13}] = 0$, $[r_{12}, r_{23}] = 0$ and $[r_{13}, r_{23}] = 0$. Thanks to the derivation identity it follows that

$$\begin{aligned} r\ \mu_0 \otimes \mathbf{1} &= \mu_0 \otimes \mathbf{1}\ (r_{13} + r_{23}) \quad \text{and} \\ r\ \mathbf{1} \otimes \mu_0 &= \mathbf{1} \otimes \mu_0\ (r_{12} + r_{13}), \end{aligned}$$

hence

$$\begin{aligned} e^{\lambda r}\ \mu_0 \otimes \mathbf{1} &= \mu_0 \otimes \mathbf{1}\ e^{(r_{13}+r_{23})} \quad \text{and} \\ e^{\lambda r}\ \mathbf{1} \otimes \mu_0 &= \mathbf{1} \otimes \mu_0\ e^{(r_{12}+r_{13})}, \end{aligned}$$

therefore, as the r_{ij} commute:

$$\mu\ \mu \otimes \mathbf{1} = \mu_0\ e^{\lambda r}\ \mu_0 \otimes \mathbf{1}\ e^{\lambda r_{12}} = \mu_0\ \mu_0 \otimes \mathbf{1}\ e^{\lambda(r_{12}+r_{13}+r_{23})}.$$

Analogously:

$$\mu\ \mathbf{1} \otimes \mu = \mu_0\ e^{\lambda r}\ \mathbf{1} \otimes \mu_0\ e^{\lambda r_{23}} = \mu_0\ \mathbf{1} \otimes \mu_0\ e^{\lambda(r_{12}+r_{13}+r_{23})}.$$

Since μ_0 is associative we have $\mu_0\ \mu_0 \otimes \mathbf{1} = \mu_0\ \mathbf{1} \otimes \mu_0$, whence the associativity of the multiplication μ. □

4.4 Standard, Weyl-Moyal and Wick ordering in $\mathbb{R}^{2n}$

It is easily seen that the multiplications $*_s$ et $*_w$ are particular cases of the Gerstenhaber formula if the real number $\hbar$ is replaced by the formal parameter λ and if we set

$$r_s := \frac{1}{i}\frac{\partial}{\partial p} \otimes \frac{\partial}{\partial q}$$

to find $*_s$,

$$r_w := \frac{i}{2}\left(\frac{\partial}{\partial q} \otimes \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \otimes \frac{\partial}{\partial q}\right)$$

for $*_w$ and

$$r_{wick} := 2\frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}}$$

for $*_{wick}$. In this fashion we obtain the associative multiplications on the $\mathbb{C}[[\lambda]]$-module $\mathcal{C}^\infty(\mathbb{R}^2, \mathbb{C})[[\lambda]]$: in this framework of formal power series the multiplications $*_s$ et $*_w$ remain well-defined although we can no longer set $\lambda = \hbar$ because the formal series do no longer converge for general smooth functions.

The generalisations of these formulas $*_s$, $*_w$ et $*_{wick}$ to the case $\mathbb{R}^{2n} = \mathbb{C}^n$ are now completely clear:

$$r_s := \frac{1}{i}\sum_{k=1}^{n} \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial q_k} \tag{4.4.1}$$

$$r_w := \frac{i}{2}\sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k} \otimes \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial q_k}\right) \tag{4.4.2}$$

$$r_{wick} := 2\sum_{k=1}^{n} \frac{\partial}{\partial z_k} \otimes \frac{\partial}{\partial \bar{z}_k} \tag{4.4.3}$$

and we get

$$f *_s g = \sum_{r=0}^{\infty} \frac{(\lambda/i)^r}{r!} \sum_{k_1,\ldots,k_r=1}^{n} \frac{\partial^r f}{\partial p_{k_1}\cdots\partial p_{k_r}}\,\frac{\partial^r g}{\partial q_{k_1}\cdots\partial q_{k_r}} \tag{4.4.4}$$

for the standard product, and (writing $\sum_{k=1}^{n}\frac{\partial}{\partial q_k}\wedge\frac{\partial}{\partial p_k} = \frac{1}{2}\sum_{k,l=1}^{2n} P^{kl}\frac{\partial}{\partial x_k}\wedge\frac{\partial}{\partial x_l}$ with $(q,p) = x$) for the Weyl-Moyal product:

$$f *_w g = \sum_{r=0}^{\infty} \frac{(i\lambda/2)^r}{r!} \sum_{k_1,\ldots,k_r,l_1,\ldots,l_r=1}^{2n} P^{k_1 l_1}\cdots P^{k_r l_r}\frac{\partial^r f}{\partial x_{k_1}\cdots\partial x_{k_r}}\,\frac{\partial^r g}{\partial x_{l_1}\cdots\partial x_{l_r}}. \tag{4.4.5}$$

and

$$f *_{wick} g = \sum_{r=0}^{\infty} \frac{(2\lambda)^r}{r!} \sum_{k_1,\dots,k_r=1}^{n} \frac{\partial^r f}{\partial z_{k_1} \cdots \partial z_{k_r}} \frac{\partial^r g}{\partial \bar{z}_{k_1} \cdots \partial \bar{z}_{k_r}}. \tag{4.4.6}$$

for the Wick product.

4.5 Standard symbols of multidifferential operators

The formulas of $*_w$ et $*_s$ of the preceding section converge in the variable λ in the case where the functions f and g are polynomials in the momentum variables $p_1, \dots, p_n$ with coefficients in $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$, and we call this space $\mathcal{C}^\infty_{pp}(\mathbb{R}^{2n}, \mathbb{C})$. For $\alpha \in \mathbb{R}^{2n*}$ let e_α be the *exponential function*

$$e_\alpha(x) := e^{\alpha(x)}.$$

Let D be a differential operator in $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$, i.e. there is a nonnegative integer N and functions $D^{a;i_1\dots i_a} \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ such that

$$D = \sum_{a=0}^{N} \sum_{i_1,\dots,i_a=1}^{n} D^{a;i_1\dots i_a} \frac{\partial^a}{\partial x^{i_1} \cdots \partial x^{i_a}}. \tag{4.5.1}$$

Its *standard symbol* is defined by

$$\check{D}(q,\alpha) := (De_\alpha)(q)e_{-\alpha}(q) = \sum_{a=0}^{N} \sum_{i_1,\dots,i_a=1}^{n} D^{a;i_1\dots i_a} \alpha_{i_1} \cdots \alpha_{i_k}, \tag{4.5.2}$$

and it is immediate that

$$\check{\rho}_s(f)(q, \frac{ip}{\hbar}) = f(q,p). \tag{4.5.3}$$

It is also not hard to see that the standard symbol defines a linear bijection between the space $\mathrm{Diffop}(\mathbb{R}^n)$ of all differential operators in $\mathbb{R}^n$ and the space $\mathcal{C}^\infty_{pp}(\mathbb{R}^{2n}, \mathbb{C})$.

Let k be a nonnegative integer and D be a k-differential operator, i.e. there is a nonnegative integer N and functions $D^{(a_1,\dots,a_k),I_1,\dots,I_k}$ where $I_1 := (i_{11}, \dots, i_{1a_1})$, $I_2 := (i_{21}, \dots, i_{2a_2})$,. . .,$I_k := (i_{k1}, \dots, i_{ka_k})$ are blocks of indices (the i_{bc} always vary between 1 et n) such that

$$D(f_1, \dots, f_k) = \sum_{a_1,\dots,a_k=1}^{N} \sum_{I_1,\dots,I_k} D^{(a_1,\dots,a_k),I_1,\dots,I_k} \frac{\partial^{a_1} f_1}{\partial x^{I_1}} \cdots \frac{\partial^{a_k} f_k}{\partial x^{I_k}} \tag{4.5.4}$$

where for example $\partial^{a_1}/\partial x^{I_1}$ is short for $\partial^{a_1}/(\partial x^{i_{11}} \cdots \partial x^{i_{1a_1}})$. One defines the *standard symbol of a k-differential operator D* for $\alpha^{(1)}, \ldots, \alpha^{(k)} \in \mathbb{R}^{n*}$ by

$$\check{D}(q, \alpha^{(1)}, \ldots, \alpha^{(k)}) := \big(D(e_{\alpha^{(1)}}, \ldots, e_{\alpha^{(k)}})\big)(q)e^{-(\alpha^{(1)}+\cdots+\alpha^{(k)})(q)}. \qquad (4.5.5)$$

As in the case of differential operators this is equivalent to saying that one replaces the partial derivatives in (4.5.4) by the nk additional variables $\alpha^{(1)}, \ldots, \alpha^{(k)}$, on which $\check{D}$ depends in a polynomial way.
The following Lemma is obvious:

Lemma 4.2 *Each k-differential operator D in $\mathbb{R}^n$ est uniquely determined by its standard symbol $\check{D}$ or, equivalently, by its values on exponential functions.*

5 Star-produits

In the preceding chapter we have seen that on can construct noncommutative or "quantum" associative multiplications $*$ on $\mathbb{C}[q,p]$ and even on $\mathcal{C}^\infty(\mathbb{R}^{2n},\mathbb{C})[[\lambda]]$ by using symbol calculus, i.e. by using a linear bijection between $\mathbb{C}[q,p]$ and an already given associative algebra, namely the algebra of all differential operators with polynomial coefficients acting on $\mathcal{C}^\infty(\mathbb{R},\mathbb{C})$. The principal idea of star products is to construct such an associative multiplication $*$ directly on the space of classical observables, i.e. on $\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$ (where M is a given Poisson manifold) without referring to a 'representation' in a differential operator algebra: for most of the Poisson manifolds it is not at all clear how such a differential operator algebra could be chosen. From the point of view of physics this means that the construction of the the quantum system starts with the observable algebra (unlike the usual approach), whereas the construction of the Hilbert space is postponed.

5.1 Definition

The following definition had been given by F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz et D. Sternheimer in 1978 [7]:

Definition 5.1 *Let (M,P) be a Poisson manifold. The structure of a* star-produit *on M or a* deformation quantization *on M is defined by the following sequence of $\mathbb{C}$-bilinear maps*

$$C_r : \mathcal{C}^\infty(M,\mathbb{C}) \times \mathcal{C}^\infty(M,\mathbb{C}) \to \mathcal{C}^\infty(M,\mathbb{C})$$

for all $r \in \mathbb{N}$ subject to the following conditions ($f,g,h \in \mathcal{C}^\infty(M,\mathbb{C})$):

1. *Every C_r is a bidifferential operator, i.e. there is a nonnegative integer N_r such that in each chart $\big(U,(x^1,\ldots,x^n)\big)$ there are smooth functions $C_r^{(a,b),i_1\cdots i_a,j_1\cdots j_b} : U \to \mathbb{C}$ such that C_r takes the following form in U:*

$$C_r(f,g) = \sum_{a,b=0}^{N_r} \sum_{i_1,\ldots,i_a,j_1,\ldots,j_b=1}^{n} C_r^{(a,b),i_1\cdots i_a,j_1\cdots j_b} \frac{\partial^a f}{\partial x^{i_1}\cdots\partial x^{i_a}} \frac{\partial^b g}{\partial x^{j_1}\cdots\partial x^{j_b}}.$$

2. *$C_0(f,g) = fg$ (classical limit).*

3. *$C_1(f,g) - C_1(g,f) = i\{f,g\} := iP(df,dg)$ (classical limit).*

4. *$C_r(1,g) = 0 = C_r(f,1)$ for all $r \geq 1$ (the constant function 1 remains the unit element).*

5. $\sum_{s=0}^{r}\big(C_s(C_{r-s}(f,g),h\big) = \sum_{s=0}^{r}\big(C_s(f,C_{r-s}(g,h)\big)$ *for all* $r\in\mathbb{N}$ *(associativity).*

The formal series

$$*:=\sum_{r=0}^{\infty}\lambda^r C_r$$

is called a star-product *on* M.
Moreover, if for all $r\in\mathbb{N}$ *and* $f,g\in\mathcal{C}^\infty(M,\mathbb{C})$

$$\overline{C_r(f,g)} = C_r(\bar{g},\bar{f})$$

(where $^{-}$ *denotes pointwise complex conjugation) the star-product is called* symmetric *or* hermitean.

For example $*_w$ and $*_{wick}$ are hermitean.
The following corollary is obvious:

Corollary 5.1 *Let* $*$ *be a star-product on the Poisson manifold* (M,P). *Then the* $\mathbb{C}[[\lambda]]$*-module* $\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$ *becomes an associative algebra over the ring* $\mathbb{C}[[\lambda]]$ *via (*$F=\sum_{r=0}^{\infty}\lambda^r F_r, G=\sum_{r=0}^{\infty}\lambda^r G_r\in\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$*)*

$$F*G:=\sum_{r=0}^{\infty}\lambda^r\sum_{s+t+u=0}C_s(F_t,G_u).$$

If moreover the star-product is hermitean, then the pointwise complex conjugation $^{-}$ *becomes an antiautomorphism of the algebra* $(\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]],*)$, *i.e.*

$$\overline{F*G}=\overline{G}*\overline{F}.$$

We list further properties of star-products:

Definition 5.2 *A hermitean star-product is called* of Weyl-Moyal type *iff*

$$C_r(g,f)=(-1)^r C_r(f,g).$$

If the order N_r *of the bidifferential operator* C_r *is equal to* r *the star-product is called* natural *by S.Gutt and J.Rawnsley. Finally, if* (M,P) *is a* semi-Kähler mainifold *(i.e.* (M,ω) *is symplectic and admits a complex structure* J *(i.e.* $J\in\Gamma(M,Hom(TM,TM))$, $J^2=-\mathbf{1}$, *and* J *having vanishing Nijenhuis torsion) such that* $\omega(JX,JY)=\omega(X,Y)$ *for any vector field* X,Y*) the star-product is called* of type Wick *or* admitting separation of variables *iff in each complex chart* $\big(U,(z^1,\ldots,z^n)\big)$ *there are* $\mathcal{C}^\infty$*-functions* $C_r^{(a,b),i_1\cdots i_a,j_1\cdots j_b}:U\to\mathbb{C}$ *such that in* U:

$$C_r(f,g)=\sum_{a,b=0}^{N_r}\sum_{i_1,\ldots,i_a,j_1,\ldots,j_b=1}^{n}C_r^{(a,b),i_1\cdots i_a,j_1\cdots j_b}\frac{\partial^a f}{\partial z^{i_1}\cdots\partial z^{i_a}}\frac{\partial^b g}{\partial\bar{z}^{j_1}\cdots\partial\bar{z}^{j_b}}.$$

For example, $*_s$, $*_w$, and $*_{wick}$ are natural, $*_w$ are $*_{Wick}$ are hermitean, $*_w$ is of Weyl-Moyal type, and $*_{wick}$ is of Wick type.

For two star-products $*$ and $*'$ there is the following notion of formal isomorphy which had already encountered for $*_s$ and $*_w$:

Definition 5.3 *Let (M, P) a Poisson manifold and $*, *'$ two star-products. We say that $*$ is* equivalent to $*'$ *iff there is a formal series of differential operators, called an* equivalence transformation

$$S = id + \sum_{r=1}^{\infty} \lambda^r S_r$$

(where $S_r : \mathcal{C}^\infty(M, \mathbb{C}) \to \mathcal{C}^\infty(M, \mathbb{C})$ are differential operators vanishing on the constants) such that

$$F *' G = S^{-1}\big((SF) * (SG)\big)$$

for all $F, G \in \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]]$.

Since the operator series N (see eq. (3.2.9)) takes the form $1 + \frac{\lambda}{2i}\frac{\partial}{\partial q}\frac{\partial}{\partial p} + o(\lambda^2)$, it defines an equivalence transformation between the star-products $*_w$ et $*_s$.

5.2 Existence

5.2.1 Symplectic manifolds

After some important results for special cases (like symplectic manifolds whose third de Rham-cohomology group vanishes [65] and cotangent bundles of parallelisable manifolds [23]) the first complete existence result had been shown by M.DeWilde et P.Lecomte in 1983, [32]:

Theorem 5.1 (DeWilde,Lecomte 1983) *On any symplectic manifold (M, ω) there is a star-product.*

The proof was based on explicit computations of the differential Hochschild cohomology of $\big(\mathcal{C}^\infty(M, \mathbb{C}), \cdot\big)$ and of the second and third Chevalley-Eilenberg cohomology of the Lie algebra $\mathcal{C}^\infty(M, \mathbb{C})$ equipped with the Poisson bracket $\{\ ,\ \}$, see [33] and the use of a local homogeneity by means of local Euler field in a Darboux chart.

Independantly of this result, B.Fedosov has given a proof of Thorem 5.1 in 1985, [40]. HIs proof is remakable since it rather uses symplectic connections than local charts: therefore his method allows to construct directly in tensorial terms, which sometimes is more adapted to the implementation of symmetries.

5.2.2 Poisson manifolds

The main obstacle to translate (even locally) the methods of the previous section to a general Poisson manifold was the fact that there is in general no connection in the tangent bundle leaving invariant the Poisson structure (if this is the case it must have constant rank). In the deformation community it came out as a big sensational surprise in 1997 when the following result was announced by Maxim Kontsevitch:

Theorem 5.2 (Kontsevitch 1997) *On any Poisson manifold* (M, P) *there is a star-product.*

For the algebraic framework of operades and L_∞ structures, see the original article [63] and the article [3] for more details. Cattaneo and Felder have retraced the quuntum-fieldtheoretic roots of Kontsevitch's construction in the theory of Poisson-Sigma models, see [25], and have given a globalisation in the spirit of Fedosov in [26].

Pour the Poisson manifold $(\mathbb{R}^n, \frac{1}{2}\sum_{a,b=1}^n P^{ab}\partial_a \wedge \partial_b)$ Kontsevitch uses the following ansatz for the bidifferential operators C_r of the star-product: let $f, g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$, let $2r = n_1 + \cdots + n_r + M + N$ a partition of the nonnegative integer $2r$ as a sum of nonnegative integers, and let σ be permutation of $\{1, 2, \ldots, 2r\}$. Let us denote the pair $\big((n_1, \ldots, n_r, M, N), \sigma\big) =: \Gamma_r$, and one defines the bidifferential operator

$$\begin{aligned} C_{\Gamma_r}(f,g) \;:=\; & \sum_{a_1,\ldots,a_{2r}=1}^{n} \left(\frac{\partial^{n_1} P^{a_{\sigma(1)}a_{\sigma(2)}}}{\partial x_{a_1}\cdots \partial x_{a_{n_1}}} \cdots \frac{\partial^{n_r} P^{a_{\sigma(2r-1)}a_{\sigma(2r)}}}{\partial x_{a_{n_1+\cdots+n_{r-1}+1}}\cdots \partial x_{a_{n_1+\cdots+n_r}}} \right. \\ & \left. \frac{\partial^M f}{\partial x_{a_{n_1+\cdots+n_r+1}}\cdots \partial x_{a_{n_1+\cdots+n_r+M}}} \; \frac{\partial^N g}{\partial x_{a_{n_1+\cdots+n_r+M+1}}\cdots \partial x_{a_{2r}}} \right). \end{aligned} \tag{5.2.1}$$

The operator C_r is obtained by a particular linear combination of the preceding operators parametrised by all possible pairs Γ_r with weights w_Γ which are at the heart of its construction: Kontsevitch represents Γ_r by graphs having $r+2$ vertices (corresponding to r Poisson structures and two functions) and $2r$ edges (corresponding to $2r$ partial derivatives) in the upper half plane, and the weights w_{Γ_r} are obtained by an integration related to the geometric image of the graph:

$$C_r(f,g) = \sum_{\Gamma_r} w_{\Gamma_r} C_{\Gamma_r}(f,g). \tag{5.2.2}$$

5.2.3 Semi-Kähler manifolds

As symplectic manifolds semi-Kähler manifolds are equipped with star-products. A more interesting question is whether the star-product can always be chosen to be of Wick type, which has to be proved separately:

Theorem 5.3 *On any semi-Kähler manifold* (M, ω, I) *there is a star-product of Wick type.*

This Theorem is due to A.Karabegov [57] and later – but independantly– by S.Waldmann and the author [18]. Karabegov has glued local differential operators on local holomorphic functions while Bordemann-Waldmann were using an evident modification of Fedosov's method.

5.2.4 Even symplectic supermanifolds

Let $\tau : E \to M$ be a real vector bundle over a symplectic mainifold (M, ω) equipped with a fibre metric q and a connection ∇^E in E compatible with q. The space of all $\mathcal{C}^\infty$-sections of the bundle ΛE^*, $\mathcal{C}_0 := \Gamma(\Lambda E^*)$ is a graded commutative associative algebra with respect to the point-wise exterior multiplication. $\mathcal{C}_0$ is called the algebra of superfunctions. This structure is called a *split supermanifold*, see [5], [38, 39], [49] for maore details. There is a graded Poisson bracket $\{\ ,\ \}_R : \mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{C}_0$ on this algebra due to M.Rothstein [73] and is defined in the following way: let U be the domain of a chart of M which trivializes the vector bundle E where: $(x^1, \ldots, x^m)$ denotes the chart of M, $\partial_1, \ldots, \partial_m$ denotes a local base of vector fields on M corresponding to that chart, let P^{kl} $(1 \leq k, l \leq m)$ be the components of the Poisson structure on M in that chart, let $e_1, \ldots, e_n$ be a base of local sections of E, $e^1, \ldots, e^n$ its dual base, let q_{AB} $(1 \leq A, B \leq m)$ be the components of the fibre metric q with respect to the preceding base, let q^{AB} (the inverse matrix of q_{AB}) the components of the fibre metric q^{-1} on the dual bundle E^* induced by q, and let $R^{(E)}{}^A_{Bkl}$ be the components of the curvature tensor of the connection ∇^E. One first forms the tensor field $\hat{R}^{(E)} \in \Gamma\big(Hom(TM) \otimes \Lambda^2 E^*\big)$ by

$$\big(\hat{R}^{(E)}\big)^k{}_{lAB} := -\frac{1}{2} \sum_{r=1}^{m} \sum_{C=1}^{n} q_{AC} P^{kr} R^{(E)}{}^C_{Brl}. \tag{5.2.3}$$

The space of smooth sections $\Gamma\big(Hom(TM) \otimes \Lambda E^*\big)$ is an associative graded algebra in the natural way, and the element $\hat{R}^{(E)}$ is obviously nilpotent. Hence the geometric series

$$(1 - 2\hat{R}^{(E)})^{-1} := \sum_{a=0}^{m} \big(2\hat{R}^{(E)}\big)^a \tag{5.2.4}$$

is well-defined. Let $\phi \in \Gamma(\Lambda^s E^*)$ and $\psi \in \Gamma(\Lambda E^*)$ be two superfunctions. Hence the Rothstein superbracket is defined in the following way:

$$\begin{aligned}\{\phi,\psi\} \;:=\; & \sum_{i,j,k=1}^{m} P^{ij}\big((1-2\hat{R}^{(E)})^{-1}\big)_i^k \wedge \nabla^E_{\partial_k}\phi \wedge \nabla^E_{\partial_j}\psi \\ & + \sum_{A,B=1}^{n} q^{AB}(-1)^{s-1}(i_{e_a}\phi)\wedge(i_{e_B}\psi). \end{aligned} \tag{5.2.5}$$

Upon using some elements of Fedosov's construction I have shown in [21] (voir aussi [10]) the following result:

Theorem 5.4 *The graded commutative associative algebra $\mathcal{C}_0$ admets a graded formal associative deformation such that the term of order 1 is equal to $\frac{i\lambda}{2}$ times the Rothstein superbracket.*

R.Eckel has formulated a full Fedosov construction in the framework of supermanifolds in his PhD-thesis [39].

5.3 Equivalence

5.3.1 Symplectic manifolds

Twelve years after the first star-product existence proof the classification of equivalence classes of star-products on a symplectic manifold has been achieved my Deligne [29], Nest-Tsygan [66, 67] and Bertelson-Cahen-Gutt [9]:

Theorem 5.5 *Let (M,ω) a symplectic manifold. Hence the equivalence classes of star-products on (M,ω) are in bijection with the formal series having coefficients in $H^2_{\mathrm{dR}}(M)$, the second de Rham cohomology group on the manifold M.*

The above bijection is given explicitly and is called the *Deligne class* $[*]$ *of* $*$ (see e.g. the excellent review [53]). In Fedosov's construction one could equally well introduce a formal series of closed 2-forms: Neumaier has shown that they coincide with the representatives of the Deligne class, see [68].

5.3.2 Poisson manifolds

In the case of a Poisson manifold the classification result proved to be much more difficult and had also been done by Kontsevitch [63]: a *formal Poisson* *structure* P on a differentiable manifold M is a formal series $P = \sum_{r=0}^{\infty} \lambda^r P_r$ where the coefficients P_r are bivector fields in $\Gamma(\Lambda^2 TM)$ such that

$$[P,P]_S = 0 \quad \Longleftrightarrow \quad \sum_{a=0}^{r} [P_a, P_{r-a}]_S = 0 \;\text{ quel que soit } r \in \mathbb{N}$$

where $[\ ,\ ]_S$ denotes the Schouten bracket. Likewise, a *formal vector field* X is a formal series $X = \sum_{r=0}^{\infty} \lambda^r X_r$ where the coefficiens are vector fields on M. The Lie derivative of P with respect to X is defined in a natural way via the Schouten bracket

$$L_X(P) := [X,P]_S = \sum_{r=0}^{\infty} \lambda^r \sum_{a=0}^{r} [X_a, P_{r-a}].$$

Two formal Poisson structures P and P' are said to be *formally diffeomorphic* iff there exists a formal vector field X sucht that

$$P' = e^{\lambda L_X}(P),$$

where equivalence is seen upon using the Baker-Campbell-Hausdorff series. By means of these structures the star-products are calssified as follows:

Theorem 5.6 (Kontsevitch 1997) *Let (M, P_0) be a Poisson manifold. Hence the equivalence classes of star-products on (M, P_0) are in bijection with the formal diffeomorphism classes of formal Poisson structures whose zeroth order terem is equal to P_0.*

6 Explicit Examples

6.1 Cotangent bundle of S^n

This example is due to F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz et D. Sternheimer [7]:

Consider the symplectic manifold $M' := T^*(\mathbb{R}^{n+1} \setminus \{0\}) = (\mathbb{R}^{n+1} \setminus \{0\}) \times \mathbb{R}^{n+1}$ equipped with canonical coordinates (q,p) and the canonical symplectic form $\sum_{k=1}^{n+1} dq_k \wedge dp_k$. The following two functions

$$H_1(q,p) := \sum_{k=1}^{n+1} q_k p_k =: q \cdot p \tag{6.1.1}$$

$$H_2(q,p) := \sum_{k=1}^{n+1} (q_k)^2 =: |q|^2 \tag{6.1.2}$$

span the two-dimensional non abelian Lie algebra (with respect to the Poisson bracket, and we have

$$\{H_1, H_2\} = -2H_2. \tag{6.1.3}$$

Moreover, the flows of H_1 and of H_2 take the form

$$\Phi_s^1(q,p) = (e^s q, e^{-s} p) \tag{6.1.4}$$

$$\Phi_t^2(q,p) = (q, p - 2tq) \tag{6.1.5}$$

and they generate the action of the Lie group

$$G := \{(\alpha, t) \in \mathbb{R}^2 | \alpha > 0\} \tag{6.1.6}$$

on M' given by

$$(\alpha, t).(q,p) := (\alpha q, -2tq + \alpha^{-1} p). \tag{6.1.7}$$

Let T^*S^n be defined by

$$M := T^*S^n := \{(q,p) \in M' | q \cdot p = 0 \text{ et } |q|^2 = 1\}. \tag{6.1.8}$$

It is easily seen that this definition gives the tangent bundle of the n-sphere S^n which is isomophic to its cotangent bundle via the canonical metric on S^n. There is a projection

$$\pi : M' \to M : (q,p) \mapsto \left(\frac{q}{|q|}, |q|p - \frac{q \cdot p}{|q|} q\right) \tag{6.1.9}$$

which is a surjective submersion. The fibres of the projection are the orbits of the group G. Therefore there is the following

Lemma 6.1 *Let $F \in \mathcal{C}^\infty(M', \mathbb{C})$. Hence there is a function $f \in \mathcal{C}^\infty(M, \mathbb{C})$ such that $F = f \circ \pi$ if and only if F is G-invariant, i.e $F\big((\alpha,t).(q,p)\big) = F\big(q,p\big)$.*

Since G is connected, it follows that F is G-invariant iff

$$\{F, H_1\} = 0 = \{F, H_2\}. \tag{6.1.10}$$

Using the star-product $*_w$ (4.4.5) on M' one sees that for every quadratic polynomial F and every $\tilde{F} \in \mathcal{C}^\infty(M', \mathbb{C})$ there is the important formula

$$F *_w \tilde{F} - \tilde{F} *_w F = i\lambda\{F, \tilde{F}\} \tag{6.1.11}$$

where the terms of higher order vanish. If this formula is applied to $F = H_1$ or $F = H_2$ one directly sees –using (6.1.10)– that a function $\tilde{F}$ is G-invariant iff it commutes with H_1 and H_2 with respect to $*_w$. It follows that the space of all G-invariant functions is an associative subalgebra of $(\mathcal{C}^\infty(M', \mathbb{C})[[\lambda]], *_w)$. Therefore one has the following

Theorem 6.1 *There exists a star-product $*_{BFFLS}$ on M for which one has the following explicit formula:*

$$f *_{BFFLS} g\big(\pi(q,p)\big) = (\pi^* f) *_w (\pi^* g)(q,p).$$

6.2 Complex projective space

The following explicit formula described further down for a star-product on complex projective space $\mathbb{C}P^n$ has been found in [20] where one may find the details of its deduction:

Let

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n \tag{6.2.1}$$

be the canonical projection whose fibres are the complex lines in $\mathbb{C}^{n+1} \setminus \{0\}$ passing through the origin. As in the preceding example, the fibres are obtained by the action of a two-dimensional Lie group, namely the multiplicatif group of all nonzero complex numbers. Unfortunately, this group does no longer preserve the star-product of Wick type on $\mathbb{C}^{n+1} \setminus \{0\}$ which renders the deduction more difficult. By means of the complex coordinates $z := (z_1, \ldots, z_{n+1})$ on $\mathbb{C}^{n+1} \setminus \{0\}$ we define

$$x := \sum_{k=1}^{n+1} |z_k|^2. \tag{6.2.2}$$

By modifying the usual star-product of Wick type $*_{Wick}$ on $\mathbb{C}^{n+1} \setminus \{0\}$ by an explicit equivalence transformation we get the following

Theorem 6.2 *Let $f, g \in \mathcal{C}^\infty(\mathbb{C}P^n, \mathbb{C})$. Hence the following formula defines a star-produit $*$ of Wick type on the Kähler mainifold $\mathbb{C}P^n$:*

$$\begin{aligned} \pi^*(f * g)(z) \; &:= \; \pi^*(fg)(z) \\ &+ \sum_{r=1}^{\infty} \frac{(2\lambda)^r}{r!} \frac{x^r}{(1+\lambda)\cdots(1+r\lambda)} \sum_{k_1,\ldots,k_r=1}^{n+1} \frac{\partial^r \pi^* f}{\partial z_{k_1}\cdots\partial z_{k_r}}(z) \frac{\partial^r \pi^* g}{\partial \bar{z}_{k_1}\cdots\partial \bar{z}_{k_r}}(z). \end{aligned}$$

In [19] we have shown that this star-product converges on all representative functions for the canonical action of the unitary group $U(n+1)$ for certain values of λ.

6.3 The dual space of a Lie algebra

This very important star-product has been found independantly by V.Drinfel'd et S.Gutt, [36], [52], in 1983:

Let $(\mathfrak{g}, [\ ,\ ]$ be a finite-dimensional real Lie algebra and $\mathfrak{g}^*$ its dual space which is a Poisson manifold (see 1.3.6)). Here we use the formal parameter $\nu := i\lambda$. Let $H : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}[[\nu]]$ be the formal group law by *Baker-Campbell-Hausdorff*:

$$H(x, y) := x + y + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \sum_{\substack{k_1,\ldots,k_n \geq 0 \\ l_1,\ldots,l_n \geq 0 \\ k_i + l_i \geq 1}} \nu^{\sum_{i=1}^n k_i + l_i} \frac{\big(ad(x)\big)^{k_1}\big(ad(y)\big)^{l_1} \cdots \big(ad(x)\big)^{k_n}\big(ad(y)\big)^{l_n}}{(k_1 + \cdots + k_n + 1) k_1! \cdots k_n! l_1! \cdots l_n!} x. \tag{6.3.1}$$

It is easily seen that one can extend H to $\mathfrak{g}[[\nu]] \times \mathfrak{g}[[\nu]]$. By its definition, H is equal to the logarithm of a product of two exponential functions in the completed free algebra generated by the two letters x and y,

$$H\big(H(x, y), z\big) = H\big(x, H(y, z)\big) \quad \forall x, y, z \in \mathfrak{g}. \tag{6.3.2}$$

One defines the standard symbol of $*$: $x, y \in \mathfrak{g} \cong \mathfrak{g}^{**}$ by

$$e_x * e_y := e_{H(x,y)} \tag{6.3.3}$$

Since it is evident from (6.3.1) that $H(x, y) - x - y$ is a multiple of ν the standard symbol of $*$ is a formal series in the parameter ν. Moreover, for each power of ν there is only a finite number of summands in $H(x, y)$ (6.3.1): this implies that the standard symbol of $*$ is polynomial in (x, y) for each power of ν. Therefore the formula (6.3.3) is well-defined. The star-product is associative because $(x, y, z \in \mathfrak{g})$

$$\begin{aligned}(e_x * e_y) * e_z = e_{H(x,y)} * e_z &= e_{H\left(H(x,y),z\right)} \\ = e_{H\left(x,H(y,z)\right)} &= e_x * e_{H(y,z)} = e_x * (e_y * e_z)\end{aligned}$$

thanks to (6.3.2). Hence the two tridifferential operators defined by their standard symbols as $(e_x * e_y) * e_z$ and $e_x * (e_y * e_z)$ coincide, hence they are equal thanks to Lemma 4.2. The formal series of the standard symbol of $*$ has the following terms of order zero and one:

$$\check{*}(\xi, x, y) = e^{\xi(H(x,y)-x-y)} = 1 + \frac{\nu}{2}\xi([x, y]) + o(\nu^2)$$

whence the classical limit of $*$ is readily deduced. Finally, the Euler-like operator

$$\nu\frac{\partial}{\partial\nu} + \sum_{k=1}^{n} \xi_k \frac{\partial}{\partial \xi_k}$$

which counts the sum of the degree in ν and the degree in $\xi \in \mathfrak{g}^*$ is a derivation of de $*$ by (6.3.1), hence the bidifferential operator C_r of $*$ (which has degree r in ν) has at leasts r partial derivatives with respect to ξ distributed on the two functions f et g. It follows that $f * g$ is a polynomial in ν if f and g are polynomials on $\mathfrak{g}^*$. Hence one may set $\nu = 1$ on polynomials. This latter complex associative algebra is isomorphic to the complexified universal envelopping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ (see [52]).

Theorem 6.3 *Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. Then there exists a star-product $*$ (called BCH (Baker-Campbell-Hausdorff) defined by (6.3.1) on the Poisson manifold $(\mathfrak{g}^*, P_{\mathfrak{g}})$ which converges (in ν) on the subspace of all polnomials on $\mathfrak{g}^*$ where the induced multiplication is ismorphic to the complexified universal envelopping algebra of $\mathfrak{g}$.*
In particular, for $\xi, \eta \in \mathfrak{g}$ it follows that for the two linear functions $\tilde{\xi}$ and $\tilde{\eta}$ defined on $\mathfrak{g}^*$ by $\tilde{\xi}(\alpha) := \langle \alpha, \xi \rangle$:

$$\tilde{\xi} * \tilde{\eta} - \tilde{\eta} * \tilde{\xi} = i\lambda \widetilde{[\xi, \eta]}.$$

6.4 L'espace dual d'une algèbre associative

Cet exemple est une version simplifiée de l'exemple précédent qui est due à l'auteur.

Soit A une algèbre associative réelle de dimension finie n. Dans une base $e_1, \ldots, e_n$ de A (et la base duale $e^1, \ldots, e^n$ de A^*) on peut exprimer les constantes de structure

$$m^i_{jk} := e^i(e_j e_k) \in \mathbb{R} \tag{6.4.1}$$

On définit le star-produit $*$ suivant sur A^* ($f, g \in \mathcal{C}^\infty(A^*, \mathbb{C})$):

$$\begin{aligned} f * g\ (\xi) \ &:= \\ \sum_{r=0}^{\infty} \frac{\nu^r}{r!} & \sum_{\substack{1 \le i_1, \ldots, i_r \le n \\ 1 \le j_1, \ldots, j_r \le n \\ 1 \le k_1, \ldots, k_r \le n}} m^{i_1}_{j_1 k_1} \cdots m^{i_r}_{j_r k_r} \xi_{i_1} \cdots \xi_{i_r} \frac{\partial^r f}{\partial \xi_{j_1} \cdots \partial \xi_{j_r}}(\xi) \frac{\partial^r g}{\partial \xi_{k_1} \cdots \partial \xi_{k_r}}(\xi) \end{aligned} \tag{6.4.2}$$

Pour vérifier l'associativité on calcule le symbole standard de $*$: soient $x, y, z \in A \cong A^{**}$, alors

$$e_x * e_y = e_{x+y+\nu xy}$$

et par conséquent

$$\begin{aligned} (e_x * e_y) * e_z = e_{x+y+\nu xy} * e_z \ &= \ e_{x+y+z+\nu(xy+xz+yz)+\nu^2 xyz} \ \text{ et} \\ e_x * (e_y * e_z) = e_x * e_{y+z+\nu yz} \ &= \ e_{x+y+z+\nu(xy+xz+yz)+\nu^2 xyz}, \end{aligned}$$

ce qui prouve l'associativité. On peut démontrer le

Theorem 6.4 *Soit A une algèbre associative réelle de dimension finie. Alors il existe un star-produit $*$ défini par (6.4.2) sur la variété de Poisson A^* (munie de la structure (1.3.6) pour le crochet de Lie $(x, y) \mapsto [x, y] := xy - yx$) qui converge (en ν) sur le sous-espace des polynômes sur A^* où la multiplication est isomorphe à l'algèbre enveloppante complexifiée de $(A, [\ ,\ ])$.*

7 Représentations des star-produits I

7.1 Définition

Soit (M, P) une variété de Poisson et C une variété différentiable. On considère l'espace Diffop(C) de tous les opérateurs différentiels sur l'espace $\mathcal{C}^\infty(C, \mathbb{C})$. Ceci est une algèbre associative munie de la multiplication usuelle d'opérateurs différentiels. Il en est de même avec Diffop$(C)[[\lambda]]$. Soit $*$ un star-produit sur M.

Definition 7.1 *Une* représentation du star-produit $*$ dans C *est un homomorphisme d'algèbres associatives $\rho : \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]] \to$* Diffop$(C)[[\lambda]]$ *sur* $\mathbb{C}[[\lambda]]$.

Les applications ρ_s et ρ_w de la section **??** (étendues aux séries formelles) sont des exemples des représentations des star-produits. On obtient un autre exemple en choisissant $M = C$ et en définissant $\rho(f) := L_f : g \mapsto f * g$ comme la multiplication gauche. On va étudier d'autres propriétés des réprésentations en paragraphe 9.2.

7.2 Représentations GNS

Une classe de représentations particulières s'obtient par un procédé analogue à celui qu'on utilise pour les représentations de Gel'fand, Naimark et Segal (GNS) des algèbres stellaires et a été étudiée par [17], [78]:

Supposons que le star-produit $*$ sur la variété de Poisson (M, P) soit symétrique, c.-à-d. $\overline{f * g} = \overline{g} * \overline{f}$ quelles que soient $f, g \in \mathcal{A} := \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]]$ où $\overline{(\ )}$ désigne la conjugaison point-par-point. L'anneau $C := \mathbb{C}[[\lambda]]$ s'écrit visiblement de façon $C = R \oplus iR$ où $R := \mathbb{R}[[\lambda]]$. On observe que R est un *anneau commutatif unitaire ordonné*: ceci veut dire que R se décompose en trois parties $R = R^+ \cup \{0\} \cup (-R^+)$ où R^+ désigne l'ensemble des *éléments* *strictement positifs* dont la définition est la suivante:

$$\alpha = \sum_{r=0}^{\infty} \lambda^r \alpha_r \quad \begin{cases} > 0 \text{si } \alpha_{o(\alpha)} > 0 \\ < 0 \text{si } \alpha_{o(\alpha)} < 0 \end{cases} \tag{7.2.1}$$

et on a $R^+ + R^+ \subset R^+$ et $R^+R^+ \subset R^+$. On a la conjugaison complexe $\overline{r_1 + ir_2} = r_1 - ir_2$ dans C et on écrit $|c|^2$ pour $\bar{c}c$. Soit maintenant $\mathcal{A}'$ un idéal bilatère de $\mathcal{A}$ stable par la conjugaison complexe (par exemple $\mathcal{A}' = \mathcal{A}$ ou $\mathcal{A}' := \mathcal{C}_0^\infty(M, \mathbb{C})[[\lambda]]$, l'espace des séries formelles à coefficients dans l'espace des fonctions de classe $\mathcal{C}^\infty$ à valeurs complexes à support compact), et $\omega : \mathcal{A}' \to C$ une fonction C-linéaire. ω est dite *réelle* ssi $\omega(\overline{f}) = \overline{\omega(f)}$ et *positive* ssi

$$\omega(\overline{f} * f) \geq 0 \quad \text{quel que soit } f \in \mathrm{A}' \tag{7.2.2}$$

où la relation $\geq$ est celle dans $R \subset C$. Grâce à l'inégalité de Cauchy-Schwarz pour une forme linéaire réelle positive,

$$\overline{\omega(\overline{f} * g)}\ \omega(\overline{f} * g) \leq \omega(\overline{f} * f)\ \omega(\overline{g} * g), \tag{7.2.3}$$

il s'ensuit que *l'idéal de Gel'fand*,

$$\mathcal{I}_\omega := \{ f \in \mathcal{A}' \mid \omega(\overline{f} * f) = 0 \} \tag{7.2.4}$$

est un idéal gauche de $\mathcal{A}$. L'espace quotient

$$\mathcal{H}_\omega := \mathcal{A}'/\mathcal{I}_\omega \tag{7.2.5}$$

(pour lequel on note $f \mapsto \psi_f$ la projection canonique) est un $\mathcal{A}$-module gauche de façon naturelle

$$\rho_\omega(f)\psi_g := \psi_{f*g}. \tag{7.2.6}$$

En outre $\mathcal{H}_\omega$ est muni d'un produit scalaire $\langle\ ,\ \rangle$ à valeurs dans C défini par

$$\langle \psi_f, \psi_g \rangle := \omega(\overline{f} * g) \tag{7.2.7}$$

avec les propriétés de sequilinéarité et positivité ($\langle \psi_f, \psi_f \rangle > 0$ quel que soit $\psi_f \neq 0$), et la représentation ρ_ω satisfait

$$\langle \rho_\omega(f)\psi_g, \psi_h \rangle = \langle \psi_g, \rho_\omega(\overline{f})\psi_h \rangle. \tag{7.2.8}$$

Alors $\mathcal{H}_\omega$ peut être regardé comme un espace préhilbertien sur l'anneau C et l'algèbre $\mathcal{A}$ se représente dans $\mathcal{H}_\omega$. La construction précédente (qui s'appelle la construction GNS dans le cadre des algèbres stellaires) est indépendante de la nature de $C = R \oplus iR$: les seules choses importantes sont les faits que R est un anneau commutatif unitaire ordonné, que $C = R \oplus iR$ avec $i^2 = -1$ et que $(\mathcal{A}, *)$ est une algèbre associative sur C munie d'un antihomomorphisme antilinéaire involutif $f \mapsto \overline{f}$ (c.-à-d.: $\forall \alpha, \beta \in C$ et $\forall f, g \in \mathcal{A}$ on a $\overline{\alpha f + \beta g} = \overline{\alpha}\overline{f} + \overline{\beta}\overline{g}$ et $\overline{f * g} = \overline{g} * \overline{f}$).

Un exemple simple est donné par la variété symplectique $(\mathbb{R}^{2n}, \sum_{k=1}^n dq_k \wedge dp_k)$ (où $j : Q = \mathbb{R}^n \to M : (q_1, \ldots, q_n) \mapsto (q_1, \ldots, q_n, 0, \ldots, 0)$ désigne l'espace des configurations), $\mathcal{A} := (\mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{C})[[\lambda]], *_w)$,

$$\mathcal{A}' := \{\sum_{r=0}^{\infty} \lambda^r f_r \mid f_r \in \mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{C}) \text{ et } supp(f_r) \cap Q \text{ est compact}\} \tag{7.2.9}$$

et la fonction linéaire ω

$$\omega : \mathcal{A}' \to \mathbb{C}[[\lambda]] : f \mapsto \int_Q d^n q \; f\big(j(q)\big). \tag{7.2.10}$$

On peut montrer que ω est positive et que $\mathcal{H}_\omega$ est isomorphe à l'espace $\mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{C})$ muni du produit scalaire L^2 standard, c.-à-d.

$$\langle \phi_1, \phi_2 \rangle = \int_Q d^n q \; \overline{\phi_1(q)} \phi_2(q)$$

et la représentation ρ_ω est égale à ρ_w (voir (3.2.8) dans le cas $n = 1$) ce qui est une espèce de représentation de Schrödinger sur les fonctions d'onde définies sur l'espace des configurations (voir [17] pour plus de détails).

Il y a beaucoup d'autres exemples de représentations de physique qui se formulent dans le cadre des représentations GNS formelles précédentes comme la représentation de Schrödinger pour un fibré cotangent T^*Q d'une variété différentiable arbitraire Q (voir le paragraphe prochain et [14] et [13] pour plus de détails où se trouve également le cas de la représentation semiclassique WKB) et la représentation dans un fibré en droites complexes holomorphe pour des monopôles magnétiques (voir [15]). Dans ce domaine, surtout Stefan Waldmann a continué la recherche, voir par exemple [78].

7.3 Fibrés cotangent

Soit Q une variété différentiable arbitraire de dimension n, $\tau_Q^* : T^*Q \to Q$ son fibré cotangent, $i : Q \to T^*Q$ la section nulle et ∇^Q une connection sans torsion dans le fibré tangent de Q. La généralisation directe de la représentation standard ρ_s dans Diffop($\mathbb{R}$)[[λ]] (voir eqn (3.1.5) est la suivante où $f \in \mathcal{C}^\infty(T^*Q, \mathbb{C})[[\lambda]]$ et $\phi \in \mathcal{C}^\infty(Q, \mathbb{C})[[\lambda]]$

$$\rho_s(f)\phi := \sum_{r=0}^{\infty} \frac{(\lambda/i)^r}{r!} \sum_{i_1,\ldots,i_r=1}^{n} i^*\big(\frac{\partial f}{\partial p_{i_1} \cdots \partial p_{i_r}}\big) \ (\nabla^Q)^{(r)}_{\big(\frac{\partial}{\partial q^{i_1}},\ldots,\frac{\partial}{\partial q^{i_r}}\big)}\phi, \quad (7.3.1)$$

où on a utilisé une carte (q, p) pour rendre l'expression moins encombrante et $(\nabla^Q)^{(r)}$ désigne la r^{me} dérivée covariante de ϕ, c.-à-d. quels que soient les champs de vecteurs $X_1, \ldots, X_{r+1} \in \Gamma(TQ)$:

$$\begin{aligned} (\nabla^Q)^{(1)}_{X_1}\phi &:= L_{X_1}\phi = X_1\phi \\ (\nabla^Q)^{(r+1)}_{(X_1,\ldots,X_{r+1})}\psi &:= (\nabla^Q)^{(1)}_{X_1}\big((\nabla^Q)^{(r)}_{(X_2,\ldots,X_{r+1})}\phi\big) \\ &\quad -\sum_{k=2}^{r+1}(\nabla^Q)^{(r)}_{(X_2,\ldots,X_{k-1},\nabla^Q_{X_1}X_k,X_{k+1},\ldots,X_{r+1})}\phi. \end{aligned}$$

On peut montrer à l'aide de la construction de Fedosov qu'il existe un star-produit $*_s$ sur la variété symplectique (T^*Q, ω_0) (voir paragraphe 1.2.1) –qui est d'une certaine manière défini par eqn (7.3.1), voir [14],[13]–, tel que ρ_s est une représentation de $\big(\mathcal{C}^\infty(T^*Q, \mathbb{C}), *_s)[[\lambda]], *_s\big)$ dans Diffop(Q)[[λ]].

Il y a aussi une possibilité (nonunique) de définir un analogue au star-produit Weyl-Moyal $*_w$ (voir (3.2.10)) à l'aide d'une série opérateurs différentiels, N, due à N.Neumaier, qui généralise l'application N (eqn (3.2.9)) du paragraphe 3.2:

Soit R^Q le tenseur de courbure de ∇^Q. On fixe un champ de densités positives μ sur Q. Alors il y a une 1-forme unique α sur Q définie par $\nabla^Q\mu =: \alpha\mu$, et l'on calcule $d\alpha = -\text{trace}R^Q$. On considère l'opérateur différentiel

$$\Delta \ := \ \sum_{k=1}^{n} \frac{\partial^2}{\partial q^k \partial p_k} + \sum_{j,k,l=1}^{n} (\tau_Q^*)^*\Gamma^j_{kl}p_j \frac{\partial^2}{\partial p_k \partial p_l} + \sum_{k,l=1}^{n} (\tau_Q^*)^*\Gamma^k_{kl} \frac{\partial}{\partial p_l} + \sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial p_k} \quad (7.3.2)$$

(où $\alpha = \sum_{k=1}^n \alpha_k dq^k$ et $\Gamma^j_{kl} := dq^j(\nabla^Q_{\frac{\partial}{\partial q^k}} \frac{\partial}{\partial q^k}$ désignent les symboles de Christoffel de la connection ∇^Q) qui ne dépend pas de la carte choisie (en fait, la somme des premiers trois termes constituent le Laplacien de la métrique semi-riemannienne sur T^*Q obtenue par l'accouplement naturel entre les champs de vecteurs horizontaux (définis par la connection ∇^Q) et les champs de vecteurs verticaux) et l'on pose

$$N := e^{\frac{\lambda}{2i}\Delta}. \quad (7.3.3)$$

Alors le star-produit

$$f *_w g := N^{-1}\big((Nf) *_s (Ng)\big) \tag{7.3.4}$$

est symétrique et est représenté par

$$\rho_w(f) := \rho_s(Nf). \tag{7.3.5}$$

Cette représentation est GNS: on définie la fonction linéaire positive ω_μ sur l'idéal bilatère $C_0^\infty(T^*Q,\mathbb{C})[[\lambda]]$ de $C^\infty(T^*Q,\mathbb{C})[[\lambda]]$:

$$\omega_\mu(f) := \int_Q \mu(i^*f) \quad \forall f \in C_0^\infty(T^*Q,\mathbb{C})[[\lambda]] \quad . \tag{7.3.6}$$

On a montré [14, 13] que ω_μ est positive, que l'idéal de Gel'fand est donné par l'image par N^{-1} du sous-espace de toutes les fonctions appartenant à $C_0^\infty(T^*Q,\mathbb{C})[[\lambda]]$ qui s'annulent sur Q, et que l'espace préhilbertien $\mathcal{H}_{\omega_\mu}$ de la construction GNS est isométrique en tant que $C^\infty(T^*Q)[[\lambda]]$-module à $C_0^\infty(Q)[[\lambda]]$ (muni du produit scalaire L^2 moyennant μ) via $\psi_f \mapsto i^*(Nf)$ quelle que soit $f \in C_0^\infty(T^*Q,\mathbb{C})[[\lambda]]$. La représentation ρ_w mentionnée ci-dessus coïncide avec la représentation GNS.

8 Géométrie de Poisson II

Pour préparer la discussion de la déformation (ou quantification) des morphismes de Poisson, je rappelle quelques propriétés des variétés de Poisson et des applications intéressantes entre elles (voir [76], [61], [80], [11], [24], [37], [43], [50], [54], [60], [79]).

8.1 Applications de Poisson

Definition 8.1 *Soient (M,P) et (M',P') deux variétés de Poisson. Une application $\Phi : M \to M'$ de classe $\mathcal{C}^\infty$ s'appelle* application de Poisson *ssi P et P' sont Φ-liées, c.-à-d.*

$$T_m\Phi \otimes T_m\Phi\big(P_m\big) = P'_{\Phi(m)} \text{ quel que soit } m \in M.$$

La proposition suivante est une conséquence directe de la définition:

Proposition 8.1 *Soit $\Phi : (M,P) \to (M',P')$ une application de Poisson entre deux variétés de Poisson.*
Alors l'application $\Phi^* : \mathcal{C}^\infty(M',\mathbb{C}) \to \mathcal{C}^\infty(M,\mathbb{C}) : g \mapsto g \circ \Phi$ est un homomorphisme d'algèbres de Poisson, c.-à-d.:

$$\begin{aligned} \Phi^*(g_1g_2) &= (\Phi^*g_1)(\Phi^*g_2) \\ \Phi^*\{g_1,g_2\}' &= \{\Phi^*g_1,\Phi^*g_2\} \end{aligned}$$

quelles que soient $g_1, g_2 \in \mathcal{C}^\infty(M', \mathbb{C})$.

8.1.1 Applications moment

L'outil principal pour la description des symétries dans le cadre des variétés de Poisson et celui des application moment:
Soit $(\mathfrak{g}, [\ ,\ ])$ une algèbre de Lie réelle de dimension finie et $\mathfrak{g}^*$ son espace dual.

Definition 8.2 *Soit* (M, P) *une variété de Poisson. Une application* $J : M \to \mathfrak{g}^*$ *s'appelle* application moment (pour l'algèbre de Lie $\mathfrak{g}$) *ssi*

$$\{\langle J, x\rangle, \langle J, y\rangle\} = \langle J, [x, y]\rangle \text{ quels que soient } x, y \in \mathfrak{g}$$

Cette définition entraîne évidemment la proposition suivante:

Proposition 8.2 *Toute application moment est une application de Poisson* $(M, P) \to (\mathfrak{g}^*, P_{\mathfrak{g}})$.

La définition classique d'une application moment par J.-M. Souriau commence par une action gauche d'une groupe de Lie G sur M, $G \times M \to M : (g, m) \mapsto gm =: \Phi_g(m)$ telle que 1. l'algèbre de Lie de G soit égale à $\mathfrak{g}$, 2. l'action préserve la structure de Poisson, c.-à.-d. toutes les Φ_g sont des application de Poisson, 3. il existe une application moment $J : M \to \mathfrak{g}^*$ comme dans définition 8.2, 4. les champs hamiltoniens $X_{\langle J,x\rangle}$ coïncident avec les générateurs infinitésimaux $x_M(m) := d/dt\big(exp(tx)m\big)|_{t=0}$ et 5. J est G-équivariante: $J(gm) = Ad^*(g)\big(m\big)\ \ \forall\ g \in G$. Ici, propriétés 4. et 5. impliquent propriété 3.

8.1.2 Systèmes intégrables

Definition 8.3 *Soit* (M, ω, H) *un système hamiltonien sur une variété symplectique* (M, ω) *de dimension* $2n$. *Il est dit* complètement intégrable (dans le sens de Liouville) *ss'il existe n fonctions de classe* $\mathcal{C}^\infty$ $F_1, \ldots, F_n : M \to \mathbb{R}$ *telles que*

1. *Les fonctions* $F_1, \ldots, F_n$ *sont des intégrales premières, c.-à-d.* $\{H, F_i\} = 0$ quel que soit $1 \leq i \leq n$,
2. *les fonctions* $F_1, \ldots, F_n$ *sont en involution, c.-à-d.* $\{F_i, F_j\} = 0$ *quels que soient* $1 \leq i, j \leq n$,
3. $F_1, \ldots, F_n$ *sont indépendantes, c.-à-d. la mesure (par rapport à la forme de volume* $\omega^{\wedge n}$*) de l'ensemble singulier* $S := \{m \in M \mid dF_1(m) \wedge \cdots \wedge dF_n(m) = 0\}$ *s'annule, et*

4. *il existe une fonction* $h : \mathbb{R}^n \to \mathbb{R}$ *de classe* $\mathcal{C}^\infty$ *telle que* $H(m) = h\big(F_1(m), \ldots, F_n(m)\big)$ *quel que soit* $m \in M$.

Le nom 'intégrabilité' provient du fait qu'il y a une procédure algébrique due à Liouville de trouver des coordonnées locales $(Q_1, \ldots, Q_n)$ autour de tout point régulier de l'application $F := (F_1, \ldots, F_n)$ telles que les coordonnées $(Q_1, \ldots, Q_n, F_1, \ldots, F_n)$ forment une carte de Darboux: les solutions des équations d'Hamilton se simplifient drastiquement:

$$\begin{aligned} \frac{dQ_k}{dt} &= \frac{\partial h}{\partial F_k}(F) =: \alpha_k(F) \\ \frac{dF_k}{dt} &= 0 \end{aligned}$$

alors

$$\big(Q(t), F(t)\big) = \big(Q(0) + t\alpha(F(0)), F(0)\big).$$

Les sous-variétés $F^{-1}(\mu)$ pour des valeurs régulières $\mu \in \mathbb{R}^n$ sont invariantes par le flot de H, et au cas où elles sont compactes et connexes elles sont difféomorphes au tore $S^1 \times \cdots \times S^1$ (Théorème de Liouville-Arnol'd, voir [4], p.271-285, ou [1], p.392-400).

On vérifie rapidement l'intégrabilité des systèmes hamiltoniens (M, ω, H) importants suivants:

1. *La particule libre dans* $\mathbb{R}^n$: $(M, \omega) = (\mathbb{R}^{2n}, \sum_{k=1}^n dq_k \wedge dp_k)$ et

$$H(q,p) := \frac{1}{2}\sum_{k_1}^{n} p_k^2 \tag{8.1.1}$$

$$F_k(q,p) := p_k \quad \forall 1 \leq k \leq n. \tag{8.1.2}$$

2. *L'oscillateur harmonique dans* $\mathbb{R}^n$: $(M, \omega) = (\mathbb{R}^{2n}, \sum_{k=1}^n dq_k \wedge dp_k)$ et

$$H(q,p) := \frac{1}{2}\sum_{k_1}^{n} \big(p_k^2 + q_k^2\big) \tag{8.1.3}$$

$$F_k(q,p) := \frac{1}{2}\big(p_k^2 + q_k^2\big) \quad \forall 1 \leq k \leq n. \tag{8.1.4}$$

3. *Le flot géodésique sur la sphère* S^n (voir section 6.1): on commence par $(M,\omega) = (\mathbb{R}^{n+1} \setminus \{0\} \times \mathbb{R}^{n+1}, \sum_{k=1}^{n+1} dq_k \wedge dp_k)$ et

$$\begin{aligned} H(q,p) &:= \frac{1}{2}\sum_{l,l'=1}^{n+1} \big(q_l^2 p_{l'}^2 - q_l p_l q_{l'} p_{l'}\big) & (8.1.5)\\ F_1(q,p) &:= \sum_{l=1}^{n+1} q_l p_l & (8.1.6)\\ F_k(q,p) &:= \frac{1}{2}\sum_{l,l'=1}^{k} \big(q_l^2 p_{l'}^2 - q_l p_l q_{l'} p_{l'}\big) \quad \forall 2 \le k \le n+1. & (8.1.7) \end{aligned}$$

Etant invariantes par le groupe G de dimension 2 (voir (6.1.6)) ces fonctions se restreignent bien sur la sous-variété symplectique T^*S^n (donnée par $F_1(q,p) = 0$ et $\sum_{l=1}^{n+1} q_l^2 = 1$) où $F_2, \ldots, F_{n+1} = H$ définissent un système intégrable qui représente le flot géodésique sur S^n dont les solutions sont des grands cercles paramétrés avec leurs vitesses.

Il est clair que chaque système intégrable est un cas particulier d'une application moment

$$J : M \to \mathbb{R}^{n*} : m \mapsto \big(F_1(m), \ldots, F_n(m)\big) \qquad (8.1.8)$$

où $\mathbb{R}^n$ est considérée comme une algèbre de Lie abélienne (c.-à-d. où tous les crochets s'annulent).

8.2 Sous-variétés et applications coïsotropes

Soit M une variété différentiable. Soit E un sous-espace vectoriel de l'espace tangent T_mM au point $m \in M$. On note

$$E^{\mathrm{ann}} := \{\alpha \in T_mM^* \mid \alpha(v) = 0\ \forall v \in E\} \qquad (8.2.1)$$

l'espace annihilateur de E.

Definition 8.4 *Soient* (M,P) *une variété de Poisson et* C *une variété quelconque et* $\Phi : C \to M$ *une application de classe* $\mathcal{C}^\infty$.

1. Φ *s'appelle* coïsotrope *ssi*

$$P_{\Phi(c)}(\alpha,\beta) = 0 \text{ quels que soient } c \in C; \alpha,\beta \in (T_c\Phi\ T_cC)^{\mathrm{ann}}.$$

2. *En particulier, si* Φ *est l'injection canonique d'une sous-variété fermée* C *de* M*, alors* C *s'appelle* sous-variété coïsotrope *quand* Φ *est coïsotrope.*

Il est immédiat que l'application identique d'une variété de Poisson est une application coïsotrope.

Pour une variété symplectique (M, ω) et un sous-espace E d'un espace tangent $T_m M$ il y a la notion du sous-espace ω-orthogonal

$$E^\omega := \{w \in T_m M \mid \omega_m(v, w) = 0 \ \forall v \in E\}, \tag{8.2.2}$$

et l'on en déduit aisément la

Proposition 8.3 *Soit (M, ω) une variété symplectique et C une sous-variété fermée de M. Alors C est coïsotrope ssi*

$$T_c C^\omega \subset T_c C \text{ quel que soit } c \in C.$$

Remarque: Si une sous-variéte coïsotrope C d'une variété symplectique (M, ω) est telle que

$$T_c C^\omega = T_c C \text{ quel que soit } c \in C. \tag{8.2.3}$$

elle est appelée *sous-variété lagrangienne*. Les sous-variétés lagrangiennes jouent un rôle principal dans la théorie du *développement semiclassique* des opérateurs différentiels, voir par exemple [6].

Proposition 8.4 *Soient (M, P) et (M', P') des variétés de Poisson, $\Phi : M \to M'$ une application de Poisson et C' une sous-variété coïsotrope de M' qui soit transverse à Φ, c.-à-d. $T_m\Phi(T_m M) + T_{\Phi(m)}C' = T_{\Phi(m)}M'$ quel que soit $m \in M$.*
Alors l'image réciproque $C := \Phi^{-1}(C')$ est une sous-variété coïsotrope de M.

Proof: Grâce à la transversalité de Φ et C' il s'ensuit que C est une sous-variété de M qui a la même codimension que C'. Soit $c \in C$ et $\alpha', \beta' \in T_{\Phi(c)}C'^{\mathrm{ann}}$. Alors, puisque $T_c\Phi\ v \in T_{\Phi(c)}C'$ quel que soit $v \in T_c C$ il s'ensuit que $\alpha := \alpha' \circ T_c\Phi$ et $\beta := \beta' \circ T_c\Phi$ sont des éléments de $T_c C^{\mathrm{ann}}$. Si $\alpha = 0$ alors α' s'annule sur $T_c\Phi(T_c M)$ et sur $T_{\Phi(c)}C'$, donc $\alpha' = 0$ grâce au fait que Φ et C' sont transverses. Alors les élements de $T_c C^{\mathrm{ann}}$ sont tous de la forme $\alpha = \alpha' \circ T_c\Phi$. On calcule

$$P_c(\alpha, \beta) = P_c(\alpha' \circ T_c\Phi, \alpha' \circ T_c\Phi) = (T_c\Phi \otimes T_c\Phi)(P_c)(\alpha', \beta') = P'_{\Phi(c)}(\alpha', \beta') = 0,$$

et C est coïsotrope. □

Proposition 8.5 *Soient (M, P) et (M', P') deux variétés de Poisson et C une variété quelconque. Soit $\Psi : C \to (M, P)$ une application coïsotrope et $\Phi : (M, P) \to (M', P')$ une application de Poisson.*
Alors la composée $\Phi \circ \Psi : C \to M'$ est une application coïsotrope.
En particulier, le cas $C = M$ et $\Psi = 1_M$ montre que toute application de Poisson est une application coïsotrope.

Proof: Soient $c \in C$ et $\alpha', \beta' \in T_{\Phi(\Psi(c))}{M'}^*$ telles que $\alpha' \circ T_c(\Phi \circ \Psi) = 0$ et $\beta' \circ T_c(\Phi \circ \Psi) = 0$. Alors $\alpha \circ T_c\Psi := \big(\alpha' \circ T_{\Psi(c)}\Phi\big) \circ T_c\Psi = 0$ et $\beta \circ T_c\Psi := \big(\beta' \circ T_{\Psi(c)}\Phi\big) \circ T_c\Psi = 0$. Puisque Ψ est coïsotrope il s'ensuit:

$$\begin{aligned} 0 &= P_{\Psi(c)}(\alpha, \beta) = P_{\Psi(c)}(\alpha' \circ T_{\Psi(c)}\Phi, \beta' \circ T_{\Psi(c)}\Phi) \\ &= \big(T_{\Psi(c)}\Phi \otimes T_{\Psi(c)}\Phi\big)(P_{\Psi(c)})(\alpha', \beta') = P'_{\Phi(\Psi(c))}(\alpha', \beta') \end{aligned}$$

ce qui prouve que $\Phi \circ \Psi$ est coïsotrope. □

Soit (M, P) une variété de Poisson. On a les exemples des sous-variétes coïsotropes suivants:

1. La variété M elle-même.

2. Si $m_0 \in M$ tel que $P_{m_0} = 0$, alors $C := \{m_0\}$ est coïsotrope.

3. Soit $(\mathfrak{g}, [\ ,\ ])$ une algèbre de Lie réelle de dimension finie et $i : \mathfrak{h} \to \mathfrak{g}$ une sous-algèbre. Alors la restriction $i^* : (\mathfrak{g}^*, P_{\mathfrak{g}}) \to (\mathfrak{h}^*, P_{\mathfrak{h}})$ est une application de Poisson et une submersion surjective, $\{0\} \subset \mathfrak{h}^*$ est une sous-variété coisotrope de $(\mathfrak{h}^*, P_{\mathfrak{h}})$, alors

$$\mathfrak{h}^{\text{ann}} := \{\xi \in \mathfrak{g}^* \mid \xi(y) = 0\ \forall y \in \mathfrak{h}\} \tag{8.2.4}$$

est une sous-variété coïsotrope de $(\mathfrak{g}^*, P_{\mathfrak{g}})$ selon proposition 8.4.

4. Soit $J : (M, P) \to (\mathfrak{g}^*, P_{\mathfrak{g}})$ une application moment dont $0 \in \mathfrak{g}^*$ est une valeur régulière. Alors $C := J^{-1}(0)$ est une sous-variété coïsotrope de (M, P) d'après proposition 8.4.

5. Dans la variété de Poisson $(M \times M, P_{(1)} - P_{(2)})$ (voir proposition 1.5) la diagonale $\Delta(M) := \{(m, m) \mid m \in M\}$ est une sous-variété coïsotrope.

6. Soit $\Phi : (M, P) \to (M', P')$ une application de Poisson. Alors son graphe

$$C := \{\big(\Phi(m), m\big) \in M' \times M \mid m \in M\} \tag{8.2.5}$$

est une sous-variété coïsotrope de la variété de Poisson $(M' \times M, P'_{(1)} - P_{(2)})$: en fait, $id_{M'} \times \Phi : (M' \times M, P'_{(1)} - P_{(2)}) \to (M' \times M', P'_{(1)} - P'_{(2)}) : (m', m) \mapsto \big(m', \Phi(m)\big)$ est une application de Poisson, et $C = (id_{M'} \times \Phi)^{-1}\big(\Delta(M')\big)$ (A.Weinstein [80]).

On note que toute structure de Poisson définit un homomorphisme de fibrés vectoriels

$$P^\sharp : T^*M \to TM : \alpha_p \mapsto P^\sharp_p(\alpha_p) := P_p(\alpha_p,\ \ ) \tag{8.2.6}$$

avec l'identification naturelle $T_pM^{**} = T_pM$.

Proposition 8.6 *Soit C une sous-variété coïsotrope d'une variété de Poisson (M,P).*
Alors la distribution $E := \cup_{c\in C} P_c^\sharp\big(T_cC^{ann}\big)$ est lisse et involutive, c.-à-d. si deux champs de vecteurs X,Y sur C prennent leur valeurs dans E, alors il en est de même pour leur crochet $[X,Y]$.

On obtient la caractérisation algébrique des sous-variétés coïsotropes suivante:

Proposition 8.7 *Soit (M,P) une variété de Poisson et C une sous-variété fermée de M. Soit I_C* l'idéal annulateur de C*, c.-à-d.*

$$I_C := \{f \in \mathcal{C}^\infty(M,\mathbb{C}) \mid f(c) = 0 \ \forall c \in C\}.$$

Alors C est une sous-variété coïsotrope si et seulement si I_C est une sous-algèbre de Poisson, c.-à-d.:

$$\text{Si } f,g \in I_C \text{ alors } \{f,g\} \in C.$$

Voir [76], p.99, Prop.7.6 pour une démonstration.

8.3 Réduction symplectique

Dans ce sous-paragraphe on ne traite que les variétés symplectiques (pour des généralisations voir [61]):

Soit $i : C \to M$ une sous-variété coïsotrope d'une variété symplectique (M,ω). Ici la distribution E de proposition 8.6 est égal au sous-fibré $TC^\omega := \cup_{c\in C} T_cC^\omega$ (voir proposition 8.3. Le feuilletage $\mathcal{F}$ correspondant au fibré intégrable E s'obtient à l'aide du théorème classique de Frobenius, voir [59], p. 28, Thm.3.25. Supposons que l'espace des feuilles $M_{red} := C/\mathcal{F}$ est muni d'une structure différentiables compatible avec la topologie quotient telle que la projection canonique $\pi : C \to M_{red}$ soit une submersion surjective. Alors on a le théorème classique suivant:

Theorem 8.1 *Avec les hypothèses mentionnées ci-dessus, l'espace quotient M_{red} est muni d'une structure symplectique canonique, ω_{red}, définie par*

$$i^*\omega =: \pi^*\omega_{red}.$$

La variété symplectique (M_{red},ω_{red}) s'appelle la variété symplectique réduite.

Voir [1], p. 416, Thm. 5.3.23, pour une démonstration.

Un cas particulier important s'obtient par une application moment $J : M \to \mathfrak{g}^*$ pour laquelle 0 est une valeur régulière dont l'image réciproque

$C := J^{-1}(0)$ n'est pas vide. Dans ce cas-là, C est une sous-variété coïsotrope, et la variété réduite (au cas où elle existe) s'obtient en tant qu'espace quotient du groupe de Lie G (à algèbre de Lie $\mathfrak{g}$) agissant de façon libre et propre sur C. Cette construction importante et extrêmement utile est appelée la *réduction de Marsden-Weinstein* [62]. Par exemple l'espace projectif complexe s'obtient en tant que variété symplectique réduite de $M = \mathbb{R}^{2n+2}, \omega = \sum_{k=1}^{n+1} dq^k \wedge dp_k$ à l'aide de l'application moment $J(q,p) := \frac{\sum_{k=1}^{n+1}(q_k^2+p_k^2)}{2} - \frac{1}{2}$ pour l'action du groupe $U(1)$ sur $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$.

9 Quantification des applications de Poisson et des plongements coïsotropes?

Dans ce paragraphe je voudrais bien discuter quelques questions –à ma connaissance ouvertes– et quelques-uns de mes résultats au sujet de la quantification des applications de Poisson et des sous-variétés coïsotropes des variétés de Poisson.

9.1 Homomorphismes de star-produits

Definition 9.1 *Soient (M,P) et (M',P') deux variétés de Poisson munies des star-produits $*$ et $*'$, respectivement.*
Une application $\mathbb{C}[[\lambda]]$-linéaire $\Phi : \mathcal{C}^\infty(M,\mathbb{C})[[\lambda]] \to \mathcal{C}^\infty(M',\mathbb{C})[[\lambda]]$ est appelée homomorphisme de star-produits *ssi Φ est un homorphisme d'algèbres associatives unitaires sur $\mathbb{C}[[\lambda]]$:*

$$\Phi(F * G) = \big(\Phi(F)\big) *' \big(\Phi(G)\big)$$

quels que soient $F, G \in \mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$.

Le lien avec les applications de Poisson est contenu dans le lemme suivant:

Lemma 9.1 *Soit $\Phi = \sum_{r=0}^{\infty} \lambda^r \Phi_r : \mathcal{C}^\infty(M,\mathbb{C})[[\lambda]] \to \mathcal{C}^\infty(M',\mathbb{C})[[\lambda]]$ un homomorphisme de star-produits.*
Alors il existe une application de Poisson $\phi : (M',P') \to (M,P)$ telle que $\Phi_0(f) = \phi^* f := f \circ \phi$ quelle que soit $f \in \mathcal{C}^\infty(M,\mathbb{C})$.

Proof: Soient $f, g \in \mathcal{C}^\infty(M,\mathbb{C})$. La propriété d'homomorphisme de Φ s'écrit à l'ordre 0 de λ:

$$\Phi_0(fg) = \big(\Phi_0(f)\big)\big(\Phi_0(g)$$

Alors Φ_0 est un homomorphisme d'algèbres commutatives associatives unitaires $\mathcal{C}^\infty(M,\mathbb{C}) \to \mathcal{C}^\infty(M',\mathbb{C})$. D'après l'exercice de Milnor (voir [59], p. 301, Cor.

35.9) il existe une application de classe $\mathcal{C}^\infty$ $\phi : M' \to M$ telle que $\Phi_0(f) = \phi^* f$. Ensuite, le commutateur de la propriété d'homomorphismes à l'ordre 1 de λ s'écrit

$$\Phi_0\{f, g\} = \{\Phi_0(f), \Phi_0(g)\}$$

d'où le fait que ϕ est une application de Poisson. □

La question réciproque de savoir quand une application de Poisson donnée se déforme dans un homomorphisme de star-produits est sans doute intéressante:

Problem 9.1 *Quelles sont les conditions sur une application de Poisson $\phi : (M', P') \to (M, P)$ pour qu'il existent des star-produits $*'$ et $*$ sur les variétés de Poisson (M', P') et (M, P), respectivement, et des applications linéaires $\Phi_1, \Phi_2, \ldots : \mathcal{C}^\infty(M, \mathbb{C}) \to \mathcal{C}^\infty(M', \mathbb{C})$ tels que*

$$\Phi := \phi^* + \sum_{r=1}^{\infty} \lambda^r \Phi_r$$

soit un homomorphisme de star-produits?

9.1.1 Applications moment quantiques et systèmes intégrables quantiques

Un cas particulier très important est donné par les applications moments $J : M \to \mathfrak{g}^*$ (voir paragraphe 8.1.1). Puisque J est une application de Poisson on peut spécifier problème 9.1 de façon suivante:

Problem 9.2 *Quelles sont les conditions sur une application moment $J : M \to \mathfrak{g}^*$ pour qu'il existent un star-produit $*$ sur la variété de Poisson (M, P) et des applications linéaires $\Phi_1, \Phi_2, \ldots : \mathcal{C}^\infty(\mathfrak{g}^*, \mathbb{C}) \to \mathcal{C}^\infty(M, \mathbb{C})$ tels que*

$$\Phi := J^* + \sum_{r=1}^{\infty} \lambda^r \Phi_r$$

soit un homomorphisme de star-produits si la variété de Poisson $(\mathfrak{g}^*, P_{\mathfrak{g}})$ (voir eqn (1.3.6)) est munie du star-produit BCH (théorème 6.3)?

Si l'on définit les applications $\mathbb{J}_r : M \to \mathfrak{g}^*$, $r \in \mathbb{N}$ par ($\xi \in \mathfrak{g}$):

$$\mathbb{J}_0 := J \quad \text{et} \quad \langle \mathbb{J}_r, \xi \rangle := \Phi_r(\tilde{\xi}) \tag{9.1.1}$$

où $\tilde{\xi} : \mathfrak{g}^* \to \mathbb{C} : \alpha \mapsto \langle \alpha, \xi \rangle$, il résulte de la propriété d'homomorphismes de Φ et du fait que $\tilde{\xi} * \tilde{\eta} - \tilde{\eta} * \tilde{\xi} = i\lambda\widetilde{[\xi, \eta]}$ (voir thm 6.3):

$$\langle \mathbb{J}, \xi \rangle * \langle \mathbb{J}, \eta \rangle - \langle \mathbb{J}, \eta \rangle * \langle \mathbb{J}, \xi \rangle = i\lambda \langle \mathbb{J}, [\xi, \eta] \rangle. \tag{9.1.2}$$

Une série formelle de fonctions $\mathbb{J} \in \mathfrak{g}^* \otimes \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]]$ satisfaisant eqn (9.1.2) s'appelle une *application moment quantique* d'après Xu [82]. Si $\mathbb{J} = J = \mathbb{J}_0$ le star-produit $*$ s'appelle $\mathfrak{g}$-*covariant* d'après Arnal, Cortet, Molin et Pinczon [2]. Plus particulièrement, un star-produit $*$ s'appelle *fortement* $\mathfrak{g}$-*invariant* d'après ces auteurs si

$$\langle \mathbb{J}, \xi \rangle * f - f * \langle \mathbb{J}, \xi \rangle = i\lambda \{ \langle \mathbb{J}, \xi \rangle, f \} \tag{9.1.3}$$

quelle que soit la fonction $f \in \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]$. On a le critère suffisant suivant pour l'existence de ces star-produits:

Theorem 9.1 (Fedosov,1996) *Soit* (M, ω) *une variété symplectique,* $J : M \to \mathfrak{g}^*$ *une application moment et* ∇ *une connection dans le fibré tangent telle que*

$$0 = \big(L_{X_{\langle J,\xi\rangle}} \nabla\big)_X Y := [X_{\langle J,\xi\rangle}, \nabla_X Y] - \nabla_{[X_{\langle J,\xi\rangle}, X]} Y - \nabla_X [X_{\langle J,\xi\rangle}, Y].$$

quel que soit ξ *dans l'algèbre de Lie* $\mathfrak{g}$*. Alors il existe un star-produit fortement* $\mathfrak{g}$*-invariant* $*$*.*

Voir [42] pour la démonstration. Par exemple, si les flots des champs de vecteurs $X_{\langle J,\xi\rangle}$ définissent l'action d'un groupe de Lie compacte ou plus généralement une action propre d'un groupe de Lie, il résulte d'un théorème classique de R.Palais que ces champs de vecteurs préservent une métrique riemannienne sur M, alors sa connection Levi-Civita, et le théorème de Fedosov est applicable.

Puisque les systèmes hamiltoniens intégrables constituent une sous-classe des applications moment (voir paragraphe 8.1.2 et eqn (8.1.8)) on peut appeler un système hamiltonien (M, ω, H) un *système intégrable quantique* ss'il ya une application $\mathbb{F} = \sum_{r=0}^\infty \lambda^r \mathbb{F}_r \in \mathbb{R}^n \otimes \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]]$ telle que $\big(M, \omega, H, \mathbb{F}_0 =: F =: (F_1, \ldots, F_n)\big)$ soit un système intégrable classique et

$$\mathbb{F}_k * \mathbb{F}_l - \mathbb{F}_l * \mathbb{F}_k = 0 \tag{9.1.4}$$

quels que soient $1 \leq k, l \leq n$. La plupart des systèmes hamiltoniens intégrables connus sont aussi intégrables quantiques, par exemple tous les exemples mentionnés en paragraphe 8.1.2 si l'on choisit $* = *_w$ et $\mathbb{F} = F$ ou les exemples de [12].

9.2 Représentations de star-produits II

On rappelle la définition d'une représentation de star-produit du paragraphe 7 dans Définition 7.1: ceci était un homomorphisme d'algèbres associatives entre l'algèbre $(\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]],*)$ ((M,P) étant une variété de Poisson munie d'un star-produit $*$) et l'algèbre d'opérateurs différentiels sur une variété différentiable C.

Le lien entre les représentations de star-produits et les sous-variétés coïsotropes est contenu dans la proposition suivante:

Proposition 9.1 *Soit (M,P) une variété de Poisson munie d'un star-produit $*$, C une variété différentiable et*

$$\rho = \sum_{r=0}^{\infty} \lambda^r \rho_r : \mathcal{C}^\infty(M,\mathbb{C})[[\lambda]] \to \mathrm{Diffop}(C)[[\lambda]]$$

une représentation de star-produits.
Alors il existe une application de classe $\mathcal{C}^\infty$, $i : C \to M$ telle que $\rho_0(f)(\psi) = (i^*f)\psi := (f \circ i)\psi$ quelles que soient $f \in \mathcal{C}^\infty(M,\mathbb{C})$ et $\psi \in \mathcal{C}^\infty(C,\mathbb{C})$.
Ensuite, au cas où i est un plongement sur une sous-variété fermée $i(C)$ de M, alors $i(C)$ est une sous-variété coïsotrope.

Proof: La propriété de représentation s'écrit à l'ordre 0: $\rho_0(f)\rho_0(g) = \rho_0(fg)$. Donc ρ_0 est un homomorphisme de l'algèbre associative commutative $\mathcal{C}^\infty(M,\mathbb{C})$ dans l'algèbre associative $\mathrm{Diffop}(C)$. Puisque $\rho_0(1) =$ l'application identique, alors ρ_0 envoie des fonctions qui ne s'annulent nulle part sur des opérateurs différentiels inversibles. Si l'on regarde le symbole standard (voir paragraphe 4.5) d'un opérateur différentiel inversible dans des coordonnées locales, on voit qu'il ne contient aucune puissance strictement positive d'une dérivée partielle. Alors, un tel opérateur différentiel prend la forme $\psi \mapsto \chi\psi$ où $\chi \in \mathcal{C}^\infty(C,\mathbb{C})$. Soit $f \in \mathcal{C}^\infty(M,\mathbb{R})$. Alors la fonction $1+f^2$ est un élément inversible dans l'algèbre $\mathcal{C}^\infty(M,\mathbb{C})$. Par conséquent, il existe une fonction $\chi \in \mathcal{C}^\infty(C,\mathbb{R})$ telle que

$$\psi + \rho_0(f)^2(\psi) = \rho_0(1+f^2)(\psi) = \chi\psi$$

quelle que soit $\psi \in \mathcal{C}^\infty(C,\mathbb{C})$. Il s'ensuit qu'il existe une fonction $\chi' \in \mathcal{C}^\infty(C,\mathbb{C})$ telle que $\rho(f)(\psi) = \chi'\psi$. Pour une fonction à valeurs complexes on arrive à la même conclusion tout en séparant en partie réelle et partie imaginaire. Alors il existe un homomorphisme d'algèbres associatives commutatives $\tilde{\rho}_0 : \mathcal{C}^\infty(M,\mathbb{C}) \to \mathcal{C}^\infty(C,\mathbb{C})$ tel que $\rho_0(f)\psi = \tilde{\rho}_0(f)\,\psi$. D'après l'exercice de Milnor (voir [59], p. 301, Corollary 35.10) il existe une application de classe $\mathcal{C}^\infty$ $i : C \to M$ telle que $\tilde{\rho}_0(f) = f \circ i$.
Soit i maintenant un plongement. Tout en identifiant C et son image $i(C)$ nous considérons deux fonctions $f,g \in \mathcal{C}^\infty(M,C)$ qui s'annulent sur C. Le commutateur de l'identité de représentation à l'ordre 1 s'écrit

$$[\rho_1(f),\rho_0(g)] - [\rho_1(g),\rho_0(f)] = \rho_0(\{f,g\})$$

Puisque $\tilde{\rho}_0(f) = i^*f = 0 = i^*g = \tilde{\rho}_0(g)$ il s'ensuit que $\{f,g\} \circ i = 0$, alors C est coïsotrope selon proposition 8.7. □

Encore une fois, la question réciproque de savoir quand l'injection canonique i d'une sous-variété coïsotrope fermée C d'une variété de Poisson donnée se déforme dans une représentation de star-produits me semble aussi intéressante:

Problem 9.3 *Quelles sont les conditions sur l'injection canonique* $i : C \to (M, P)$ *d'une sous-variété coïsotrope fermée* C *d'une variété de Poisson* (M, P) *pour qu'il existent un star-produit* $*$ *sur la variété de Poisson* (M, P) *et des applications linéaires* $\rho_1, \rho_2, \ldots : \mathcal{C}^\infty(M, \mathbb{C}) \to \mathrm{Diffop}(C)$ *tels que*

$$\rho := i^* + \sum_{r=1}^{\infty} \lambda^r \rho_r$$

soit une représentation de star-produits?

Ici j'ai utilisé la notation simplifiée $\rho_0 = i^*$ pour $\rho_0(f)(\psi) = (i^* f)\psi$.

9.2.1 Lien entre homomorphismes et représentations de star-produits

Soient (M, P) et (M', P') deux variétés de Poisson munies des star-produits $* = \sum_{r=0}^{\infty} \lambda^r C_r$ et $*' = \sum_{r=0}^{\infty} \lambda^r C'_r$, respectivement. Pour deux entiers positifs s, t on définit dans une carte $(U \times U', x^1, \ldots, x^n, y^1, \ldots, y^{n'})$ l'opérateur bidifférentiel suivant

$$\begin{aligned}\big(C_s \otimes C'_t\big)(F, G) := \\ \sum_{a,b=0}^{N_s} \sum_{c,d=0}^{N_t} \sum_{\substack{1 \le i_1, \ldots, i_a \le n \\ 1 \le j_1, \ldots, j_b \le n}} \sum_{\substack{1 \le i'_1, \ldots, i'_c \le n' \\ 1 \le j'_1, \ldots, j'_d \le n'}} C_s^{(a,b), i_1 \cdots i_a, j_1 \cdots j_b} \; C'^{(c,d), i'_1 \cdots i'_c, j'_1 \cdots j'_d}_t \\ \frac{\partial^{a+c} F}{\partial x^{i_1} \cdots \partial x^{i_a} \partial y^{i'_1} \cdots \partial y^{i'_c}} \; \frac{\partial^{b+d} G}{\partial x^{j_1} \cdots \partial x^{j_b} \partial y^{j'_1} \cdots \partial y^{j'_d}}.\end{aligned}$$

quelles que soient $F, G \in \mathcal{C}^\infty(M \times M', \mathbb{C})$. La définition ne dépend pas des cartes choisies. On pose

$$* \otimes *' := \sum_{r=0}^{\infty} \lambda^r \sum_{s+t=r} C_s \otimes C'_t$$

ce qui définit évidemment un star-produit sur $(M \times M', P_{(1)} + P'_{(2)})$ (voir Proposition 1.5). En outre, on définit la multiplication $*^{\mathrm{opp}}$

$$f *^{\mathrm{opp}} g := g * f \quad \text{quelles que soient } f, g \in \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]]$$

qui est évidemment un star-produit pour la variété de Poisson $(M, -P)$. Le lien entre la quantification des application de Poisson et la quantification des sous-variétés coïsotropes se trouve dans la suivante

Proposition 9.2 (M.B. 2000) *Soit $\phi : (M', P') \to (M, P)$ une application de Poisson entre deux variétés de Poisson. Soit $i : C := M' \to M \times M'$ le plongement canonique dans le graphe de ϕ, $\{(\phi(p'), p') := i(p') \in M \times M' \mid p' \in M'\}$ (ce qui est une sous-variété coïsotrope de $(M \times M', P_{(1)} - P'_{(2)}$ d'après le théorème de Weinstein, voir exemple 6 dans paragraphe 8.2). On suppose qu'il y ait une représentation de star-produits ρ de $\mathcal{C}^\infty(M \times M', \mathbb{C})[[\lambda]]$ munie du star-produit $*\otimes*'^{\mathrm{opp}}$ dans $\mathrm{Diffop}(C)[[\lambda]] = \mathrm{Diffop}(M')[[\lambda]]$ telle que $\rho_0 = i^*$. Soit $r : \mathcal{C}^\infty(M', \mathbb{C})[[\lambda]] \to \mathcal{C}^\infty(M', \mathbb{C})[[\lambda]]$ l'application $r(g) := \rho(1 \otimes g)(1)$ et $l : \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]] \to \mathcal{C}^\infty(M', \mathbb{C})[[\lambda]]$ l'application $l(f) := \rho(f \otimes 1)(1)$.*
Alors r est inversible et

$$\Phi := r^{-1} \circ l : \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]] \to \mathcal{C}^\infty(M', \mathbb{C})[[\lambda]]$$

est un homomorphism de star-produits tel que $\Phi_0 = \phi^*$.

Proof: Soient $f, f_1, f_2 \in \mathcal{C}^\infty(M, \mathbb{C})$ et $g, g_1, g_2 \in \mathcal{C}^\infty(M', \mathbb{C})$. On a $r_0(g) = \rho_0(1 \otimes g)(1) = i^*(1 \otimes g) = g$, alors r_0 est l'application identique ce qui entraîne que r est inversible. Ensuite, $l_0(f) = \rho_0(f \otimes 1)(1) = i^*(f \otimes 1) = \phi^* f$, par conséquent $\Phi_0 = (r^{-1} \circ l)_0 = l_0 = \phi^*$. On a

$$\begin{aligned} \rho(1 \otimes g_1) r(g_2) = \rho(1 \otimes g_1)\rho(1 \otimes g_2)(1) &= \rho\big(1 \otimes (g_1 *'^{\mathrm{opp}} g_2)\big)(1) \\ &= r(g_1 *'^{\mathrm{opp}} g_2) = r(g_2 *' g_1). \end{aligned}$$

d'où

$$g_2 *' g_1 = \big(r^{-1} \circ \rho(1 \otimes g_1) \circ r\big)(g_2)$$

En outre,

$$\begin{aligned} \big(r^{-1} \circ \rho(f \otimes 1) \circ r\big)(g) &= r^{-1}\big(\rho(f \otimes 1)\rho(1 \otimes g)(1)\big) = r^{-1}\big(\rho(f \otimes g)(1)\big) \\ &= r^{-1}\big(\rho(1 \otimes g)\rho(f \otimes 1)(1)\big) = \big(r^{-1} \circ \rho(1 \otimes g) \circ r\big)\big(r^{-1}(l(f))\big) \\ &= \Phi(f) *' g \end{aligned}$$

d'après l'équation précédente. Par conséquent

$$\begin{aligned} \Phi(f_1 * f_2) *' g &= \big(r^{-1} \circ \rho((f_1 * f_2) \otimes 1) \circ r\big)(g) \\ &= \big(r^{-1} \circ \rho(f_1 \otimes 1) \circ r\big)\big(\big(r^{-1} \circ \rho(f_2 \otimes 1) \circ r\big)(g)\big) \\ &= \Phi(f_1) *' \big(\Phi(f_2) *' g\big) = \big(\Phi(f_1) *' \Phi(f_2)\big) *' g \end{aligned}$$

ce qui montre que Φ est un homomorphisme de star-produits. $\square$

9.2.2 Représentation de star-produits quand l'espace réduit existe

Voici un résultat positif simple:

Theorem 9.2 (M.B. 2001) *Soit (M,ω) une variété symplectique munie d'un star-produit $*$. On note $[*]$ sa classe de Deligne. Soit $i : C \to M$ une sous-variété coïsotrope fermée de M telle que la variété symplectique réduite $\pi : C \to (M_{\mathrm{red}}, \omega_{\mathrm{red}})$ (voir théorème 8.1 en paragraphe 8.3) existe. Supposons en outre qu'il existe une série formelle β à coefficients dans le deuxième groupe de cohomologie de de Rham de M_{red} telle que*

$$i^*[*] = \pi^*\beta.$$

Alors il existe une représentation de star-produits $\rho : (\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]], *) \to \mathrm{Diffop}(C)[[\lambda]]$. En outre, il est toujours possible de choisir ρ, un star-produit $*_{\mathrm{red}}$ sur $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ de classe de Deligne $[*_{\mathrm{red}}] = \beta$ et une anti-représentation $\rho_{\mathrm{red}} : \mathcal{C}^\infty(M_{\mathrm{red}},\mathbb{C})[[\lambda]], *_{\mathrm{red}}) \to \mathrm{Diffop}(C)[[\lambda]]$ ($\rho_{\mathrm{red}}(g_1 *_{\mathrm{red}} g_2) = \rho_{\mathrm{red}}(g_2)\rho_{\mathrm{red}}(g_1)$ quelles que soient $g_1, g_2 \in \mathcal{C}^\infty(M_{\mathrm{red}},\mathbb{C})[[\lambda]]$) de telle façon que

$$\rho(f)\rho_{\mathrm{red}}(g) = \rho_{\mathrm{red}}(g)\rho(f)$$

quelles que soient $f \in \mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$ et $g \in \mathcal{C}^\infty(M_{\mathrm{red}},\mathbb{C})[[\lambda]]$.

Proof: On considère la variété symplectique $(M \times M_{\mathrm{red}}, \omega_{(1)} - \omega_{\mathrm{red}(2)})$. Grâce au théorème de classification 5.5 des star-produits symplectiques il existe un star-produit $*_{\mathrm{red}}$ sur M_{red} tel que sa classe de Deligne $[*_{\mathrm{red}}]$ soit égale à β. On considère le star-produit $\hat{*} := * \otimes *_{\mathrm{red}}^{\mathrm{opp}}$, et sa classe de Deligne vaut

$$[\hat{*}] = pr_1^*[*] - pr_2^*[*_{\mathrm{red}}]$$

où $pr_1 : M \times M_{\mathrm{red}} \to M$ et $pr_2 : M \times M_{\mathrm{red}} \to M_{\mathrm{red}}$ désignent les projections canoniques. Grâce à l'équation $i^*\omega = \pi^*\omega_{\mathrm{red}}$ on voit que

$$j : C \to M \times M_{\mathrm{red}} : c \mapsto \big(i(c), \pi(c)\big)$$

est un plongement sur une variété lagrangienne $L := j(C)$ (voir eqn 8.2.3) de $M \times M_{\mathrm{red}}$. D'après un théorème de Weinstein (voir [79] ou [1], p.411, thm 5.3.18) il existe un voisinage ouvert $U \supset L$ dans $M \times M_{\mathrm{red}}$ un voisinage ouvert $L \subset V \subset T^*L$ de la section nulle du fibré cotangent de L et un difféomorphisme symplectique $\phi : U \to V$ dont la restriction à L donne l'identification usuelle de L avec la section nulle $L \to T^*L$. On peut choisir U de telle façon que $V \cap T_lL^*$ est contractile quel que soit $l \in L$. Soit $\tau : U \to C$ la submersion surjective induite par la projection du fibré $\tau_L^* : T^*L \to L$ moyennant ϕ (c.-à-d.: $\tau_L^* \circ \phi =: j \circ \tau$) et soient $i_L : L \to U$ et $i_U : U \to M$ les injections canoniques. U est une sous-variété ouverte, donc symplectique de $M \times M_{\mathrm{red}}$, et le star-produit $\hat{*}$ se restreint à U, $\hat{*}|_U$. L'application

$(j \circ \tau)^* : \Gamma(\Lambda T^*L) \to \Gamma(\Lambda T^*U)$ induit un isomorphisme des cohomologies de de Rham (dont l'application réciproque est induite par $i_L^* : \Gamma(\Lambda T^*U) \to \Gamma(\Lambda T^*L)$) puisque $j \circ \tau$ est une rétraction par déformation de U sur L. Par conséquent, la classe de Deligne de $\hat{*}|_U$ vaut

$$
\begin{aligned}
[\hat{*}|_U] = (j \circ \tau)^* i_L^* [\hat{*}|_U] &= (j \circ \tau)^* i_L^* i_U^* [\hat{*}] = \tau^* j^* i_L^* pr_1^* [*] - \tau^* j^* i_L^* pr_2^* [*_{\mathrm{red}}] \\
&= \tau^* \big(i^*[*] - \pi^*[*_{\mathrm{red}}] \big) = 0
\end{aligned}
$$

puisque $i_U \circ i_L = i_L$, $pr_1 \circ i_L \circ j = pr_1 \circ j = i$ et $pr_2 \circ i_L \circ j = pr_2 \circ j = \pi$. Dans [15] on a montré qu'un star-produit sur $T^*L \cong T^*C$ dont la classe de Deligne s'annule est toujours équivalent au star-produit $*_s$ (voir eqn 7.3.1). Par conséquent, en utilisant le symplectomorphisme ϕ on peut montrer qu'il existe une série d'opérateur différentiels $S = id + \sum_{r=1}^{\infty} \lambda^r S_r$ sur U telle que

$$
S(F \hat{*}|_U G) = (SF) *_s (SG) \quad \text{quelles que soient } F, G \in \mathcal{C}^\infty(U, \mathbb{C})[[\lambda]],
$$

et il suit directement que la representation $\rho_s : \mathcal{C}^\infty(U, \mathbb{C})[[\lambda]] \to \mathrm{Diffop}(C)[[\lambda]]$ relative à une connection sans torsion ∇ sur $C \cong L$ (voir eqn 7.3.1) définit une représentation $\hat{\rho}$ pour l'algèbre $\big(\mathcal{C}^\infty(M \times M_{\mathrm{red}}, \mathbb{C})[[\lambda]], \hat{*}\big)$ par

$$
\hat{\rho}(F) := \rho_s\big(S(F|_U)\big)
$$

quelle que soit $F \in \mathcal{C}^\infty(M \times M_{\mathrm{red}}, \mathbb{C})[[\lambda]]$. Evidemment, $\hat{\rho}_0 = j^*$. En particulier, quand on restreint la représentation $\hat{\rho}$ à la sous-algèbre $\big(\mathcal{C}^\infty(M, \mathbb{C})[[\lambda]], *\big)$ de $\big(\mathcal{C}^\infty(M \times M_{\mathrm{red}}, \mathbb{C})[[\lambda]], \hat{*}\big)$ (c.-à-d.: $\rho(f) := \hat{\rho}(pr_1^* f) = \hat{\rho}(f \otimes 1)$) on obtient la représentation ρ souhaitée. En plus, $\rho_0 = \hat{\rho}_0 \circ pr_1^* = j^* pr_1 * = i^*$. Quand on restreint la représentation $\hat{\rho}$ à la sous-algèbre $\big(\mathcal{C}^\infty(M_{\mathrm{red}}, \mathbb{C})[[\lambda]], *_{\mathrm{red}}^{\mathrm{opp}}\big)$ on obtient l'anti-représentation souhaitée par un raisonnement analogue. Puisque les deux sous-algèbres commutent, il en est de même pour ses représentations. □

References

[1] Abraham, R., Marsden, J. E.: *Foundations of Mechanics*, second edition. Addison Wesley Publishing Company, Inc., Reading Mass. 1985.

[2] Arnal, D., Cortet, J. C., Molin, P., Pinczon, G.:*Covariance and Geometrical Invariance in $*$-Quantization.* J. Math. Phys. **24**.2 (1983), 276–283.

[3] Arnal, D., Manchon, D., Masmoudi, M.: *Choix des signes pour la formalité de Kontsevitch.* Prépublication Universités de Metz et de Nancy, `math.QA/0003003`, 2000.

[4] Arnol'd, V. I.: *Mathematical Methods of Classical Mechanics.* Second Edition, Springer, New York, 1989.

[5] Batchelor, M.: *Two Approaches to Supermanifolds.* Trans. Am. Math. Soc. **258**(1) (1980), 257-270.

[6] Bates, S., Weinstein, A.: *Lectures on the Geometry on Quantization.* Berkeley Mathematics Lecture Notes, Volume 8, 1995.

[7] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: *Deformation Theory and Quantization.* Annals of Physics **111** (1978), part I: 61-110, part II: 111-151.

[8] Berezin, F.: *Quantization.* Izv. Mat. NAUK **38** (1974), 1109-1165.

[9] Bertelson, M., Cahen, M., Gutt, S.: *Equivalence of Star-Products.* Class.Quant.Grav. **14** (1997), A93-A107.

[10] Bordemann, M.: *The deformation quantization of certain super-Poisson brackets and BRST cohomology.* Dans: Dito, G., Sternheimer, D.: *Conférence Moshé Flato 1999. Volume II.* Kluwer, Dordrecht, 2000, 45-68.

[11] Bordemann,M., Herbig, H.-C., Waldmann, S.: *BRST cohomology and Phase Space Reduction in Deformation Quantisation*, Commun.Math.Phys. **210** (2000), 107-144.

[12] Bordemann, M., Walter, M.: *Quantum Integrable Toda-Like Systems in Deformation Quantization.* Lett. Math. Phys. **48** (1999), 123-133.

[13] Bordemann, M., Neumaier, N., Waldmann, S.: *Homogeneous Fedosov Star Products on Cotangent Bundles II: GNS Representations, the WKB Expansion, traces, and applications*, J. Geom. Phys. **29** (1999), 199-234.

[14] Bordemann, M., Neumaier, N., Waldmann, S.: *Homogeneous Fedosov Star Products on Cotangent Bundles I: Weyl and Standard Ordering with Differential Operator Representation*, Comm. Math. Phys. **198** (1998), 363-396.

[15] Bordemann, M., Neumaier, N., Pflaum, M., Waldmann, S.: *On representations of star product algebras over cotangent spaces on Hermitean line bundles.* Preprint `math.QA`/9811055v2, December 1998.

[16] M. Bordemann, H. Römer, S. Waldmann: *A Remark on Formal KMS States in Deformation Quantization*, Lett. Math. Phys. **45** (1998), 49-61.

[17] M. Bordemann, S. Waldmann: *Formal GNS Construction and States in Deformation Quantization*, Comm. Math. Phys. **195** (1998), 549-583.

[18] Bordemann, M., Waldmann, S.: *A Fedosov Star-Product of Wick Type for Kähler Manifolds*, Lett. Math. Phys. **41** (1997), 243-253.

[19] M. Bordemann, M. Brischle, C. Emmrich, S. Waldmann: *Subalgebras with Converging Star Products in Deformation Quantization: An Algebraic Construction for* $\mathbb{C}P^n$, J. Math. Phys. **37** (1996), 6311-6323.

[20] M. Bordemann, M. Brischle, C. Emmrich, S. Waldmann: *Phase Space Reduction for Star Products: An Explicit Construction for* $\mathbb{C}P^n$, Lett. Math. Phys. **36** (1996), 357-371.

[21] Bordemann, M.: *On the deformation quantization of super-Poisson brackets.* preprint Uni Freiburg, FR-THEP-96/8, `q-alg`/9605038, May 1996.

[22] Cahen, M., DeWilde, M., Gutt, S.: *Local cohomology of the algebra of* $\mathcal{C}^\infty$*-functions on a connected manifold.* Lett. Math. Phys. **4** (1980), p.157-167.

[23] Cahen, M., Gutt, S.: *Regular* $*$*-representations of Lie Algebras.* Lett.Math.Phys. **6** (1983), 395-404.

[24] Cannas da Silva, A., Hartshorn, K., Weinstein, A.: *Lectures on Geometric Models for Noncommutative Algebras.* University of Berkeley, 1998.

[25] Cattaneo, A., Felder, G.: *A path integral approach to the Kontsevitch quantization formula.* Commun.Math.Phys. **212** (2000), 591-611.

[26] Cattaneo, A., Felder, G., Tomassini, L.: *From local to global defromation quantization of Poisson manifolds.* Prépublication `math.QA/0012228`, 2000.

[27] Connes, A.: *Noncommutative Differential Geometry.* Academic Press, San Diego, 1994.

[28] Connes, A., Flato, M., Sternheimer, D.: *Closed Star Products and Cyclic Cohomology.* Lett.Math.Phys. **24** (1992), 1-12.

[29] Deligne, P.: *Déformations de l'algèbre des fonctions d'une variété symplectique: comparaison entre Fedosov et DeWilde, Lecomte.* Sel.Math., New series **1 (4)** (1995), 667-697.

[30] d'Espagnat, Bernard: *Le réel voilé.* Fayard, Paris, 1994.

[31] DeWilde, M., Lecomte, P.B.A.: *Star-products on cotangent bundles.* Lett. Math. Phys. **7** (1983), 235-241.

[32] DeWilde, M., Lecomte, P.B.A.: *Existence of star-products and of formal deformations of the Poisson Lie Algebra of arbitrary symplectic manifolds.* Lett. Math. Phys. **7** (1983), 487-49.

[33] DeWilde, M., Lecomte, P.B.A.: *Formal Deformations of the Poisson Lie Algebra of a Symplectic Manifold and Star Products. Existence, Equivalence, Derivations.* Dans: Hazewinkel, M., Gerstenhaber, M. (eds.) *Deformation Theory of Algebras and Structures and Applications.* Dordrecht, Kluwer, 1988.

[34] Dimakis, A., Müller-Hoissen, F.: *Stochastic differential calculus, the Moyal ∗-product, and noncommutative differential geometry.* Lett.Math.Phys. **28** (1993), p. 123-137.

[35] Dito, G., Sternheimer, D.: *Conférence Moshé Flato 1999. Volume I, II.* Kluwer, Dordrecht, 2000, 45-68.

[36] Drinfel'd, V.: On constant quasiclassical solutions of the Yang-Baxter Quantum Equation. Sov.Math.Doklady **28** (1983), 667-671.

[37] Dubois-Violette, M.: *Systèmes dynamiques contraints: l'approche homologique.* Ann. Inst. Fourier **37**, No.4, 45-57 (1987).

[38] Eckel, R.: *Eine geometrische Formulierung von Supermannigfaltigkeiten, deren Super-Poisson-Klammern und Sternprodukten.* Thèse de DEA de physique, en allemand, Fakultät für Physik, Université de Freiburg, R.F.A., avril 1996.

[39] Eckel, R.: *Quantisierung von Supermannigfaltigkeiten à la Fedosov.* Thèse de doctorat de physique, en allemand, Fakultät für Physik, Université de Freiburg, R.F.A., septembre 2000.

[40] Fedosov, B.: *Formal Quantization.* Some Topics of Modern Mathematics and Their Applications to Problems of Mathematical Physics, Moscow (1985), 129-136.

[41] Fedosov, B.: *A Simple Geometrical Construction of Deformation Quantization.* J. of Diff. Geom. **40** (1994), 213-238.

[42] Fedosov, B.: *Deformation Quantization and Index Theory.* Akademie Verlag, Berlin, 1996.

[43] Fish, J., Henneaux, M., Stasheff, J., Teitelboim, C.: *Existence, uniqueness and cohomology of the classical BRST charge with ghosts of ghosts.* Commun. Math. Phys. **120** (1989), 379–407.

[44] Gerstenhaber, M.: *The Cohomology Structure of an Associative Ring.* Ann. Math. **78** (1963), 267-288.

[45] Gerstenhaber, M.: *On the deformation of rings and algebras.* Ann.Maths **79** (1964), 59-103.

[46] Gerstenhaber, M.: *On the deformation of rings and algebras II.* Ann.Maths **84** (1966), 1-19.

[47] Gerstenhaber, M.: *On the deformation of rings and algebras III.* Ann.Maths **88** (1968), 1-34.

[48] Gerstenhaber, M.: *On the deformation of rings and algebras IV.* Ann.Maths **99** (1974), 257-276.

[49] El Gradechi, A.M., Nieto, L.M.: *Supercoherent states, super-Kähler geometry and geometric quantization.* Commun. Math. Phys. **175** (1996), 521-563.

[50] Gotay, M.: *On coisotropic imbeddings of presysmplectic manifolds.* Proc. Am. Math. Soc. **84** (1982), 111-114.

[51] Greub, W.: *Multilinear Algebra*. Springer, New York, 1978.

[52] Gutt, S.: *An explicit $*$-product on the cotangent bundle of a Lie group.* Lett.Math.Phys. **7** (1983), 249-258.

[53] Gutt, S., Rawnsley, J.: *Equivalence of star products on a symplectic manifold; an introduction to Deligne's Čech cohomology classes.* J.Geom.Phys. **29** (1999), 347-399.

[54] Henneaux, M., Teitelboim, C.: *Quantization of Gauge Systems.* Princeton University Press, New Jersey, 1992.

[55] Heß, H.: *Symplectic connections in geometric quantization and factor orderings*. Dissertation (Fachbereich Physik, Freie Universität Berlin, F.R.G., 1981).

[56] Hochschild, G., Kostant, B., Rosenberg, A.: *Differential forms on regular affine algebras.* Trans. Am. Math. Soc. **102** (1962), 383-408.

[57] Karabegov, A. V.: *Deformation quantization with separation of variables on a Kaehler manifold.* Commun. Math. Phys. **180** (1996), 745-755.

[58] Kobayashi S. & Nomizu K. Fondations of Differential Geometry, Vol I, Insterscience Publishers, Wiley and Sons, 1963.

[59] Kolář, I., Michor, P., Slovák, J.: *Natural Operations in Differential Geometry.* Springer, Berlin, 1993.

[60] Landsman, N.P., Pflaum, M., Schlichenmaier, M. (éditeurs): *Quantization of Singular Symplectic Quotients.* Birkhäuser, Basel, 2001.

[61] Marsden, J.E., Raţiu, T.: *Reduction of Poisson manifolds.* Lett.Math.Phys. **11** (1986), 161-169.

[62] Marsden, J.E., Weinstein, Alan: *Reduction of symplectic manifolds with symmetry.* Rep. on Math. Phys. **5** (1974), 121-130.

[63] Kontsevitch, M.: *Deformation Quantization of Poisson Manifolds. I.* Preprint `q-alg`/9709040, September 1997.

[64] Nadaud, F.: *On continuous and differential Hochschild cohomology.* Lett. Math. Phys. **47** (1999), 85-95.

[65] Neroslavski, O., Vlassov, A.T.: *Sur les déformations de l'algèbre des fonctions d'une variété symplectique.* C.R.Acad.Sc. Paris I **292** (1981), 71-73.

[66] Nest, R., Tsygan, B.: *Algebraic Index Theorem.* Commun.Math.Phys. **172** (1995), 223-262.

[67] Nest, R., Tsygan, B.: *Algebraic Index Theorem for Families.* Adv.Math. **113** (1995), 151-205.

[68] Neumaier, N.: *Local ν-Euler Derivations and Deligne's Characteristic Class for Fedosov Star-Products and Star-Products of Special Type.* Prépublication Université de Freiburg, R.F.A., FR-THEP-99/3, juin 1999, `math.QA/9905176v2`.

[69] Omori, H., Maeda, Y., Yoshioka, A.: *Weyl manifolds and Deformation Quantization.* Adv.Math. **85** (1991), 224-255.

[70] Omori, H., Maeda, Y., Yoshioka, A.: *Existence of a Closed Star-Product.* Lett.Math.Phys. **26** (1992), 285-294.

[71] Pflaum, M.: *The normal symbol of Riemannian manifolds.* New York J. Math. **4** (1998), 97-125.

[72] Pflaum, M.: *On Continuous Hochschild Homology and Cohomology Groups.* Lett. Math. Phys. **44** (1998), 43-51.

[73] Rothstein, Mitchell.: *The structure of supersymplectic supermanifolds.* In: Bartocci, C. et al., Eds. *Differential Geometric Methods in Mathematical Physics*, Proc. 19th Int. Conf., Rapallo/Italy 1990, Lect. Notes Phys. **375**, 331-343 (1991).

[74] Ruiz, J.M.: *The Basic Theory of Power Series.* Vieweg Verlag, Braunschweig, 1993.

[75] Sternheimer, D.: *Deformation Quantization: Twenty Years After.* Dans: Rembieliński, J. (éd.) *Particles, Fields, and Gravitation. (Łódź 1998).* AIP Press, New York, 1998, p. 107-145; voir aussi `math.QA/9809056`.

[76] Vaisman, I.: *Lectures on the Geometry of Poisson Manifolds.* Birkhäuser, Basel, 1994.

[77] Voronov, T.: *Quantization of Forms on the Cotangent Bundle.* Commun. Math. Phys. **205**, 315-336 (1999).

[78] Waldmann, S.: *Locality in GNS Representations of Deformation Quantization.* Commun.Math.Phys. **210** (2000), 467-495.

[79] Weinstein, A.: *Symplectic manifolds and their Lagrangian submanifolds.* Adv. Math. **6** (1971), 329-346.

[80] Weinstein, A.: *Coisotropic calculus and Poisson groupoids.* J.Math.Soc.Japan **40** (1988), 705-727.

[81] Weinstein, A.: *Deformation Quantization, Séminaire Bourbaki.* Vol. 1993/94. Astérisque **227** (1995), Exp. No. 789, 5, p. 389-409.

[82] Xu, P.: *Fedosov $*$-Products and Quantum Momentum Maps.* Commun. Math. Phys. **197** (1998), 167–197.