Basic prerequisites in differential geometry and operator theory in view of applications to quantum field theory

Sylvie Paycha

May 27, 2009

Acknowledgements

These lecture notes provide a summary of some prerequisites in differential geometry and functional analysis which can be of use to both mathematicians and physicists. They correspond to lectures held at two summer schools in Villa de Leyva, Colombia in 1999 and 2001 on *Geometric and topological methods for quantum field theory* aswell as to parts of some postgraduate lectures delivered at the Université Blaise Pascal, Clermont-Ferrand in 2003, and later during a CIMPA school *Index theory and interactions with physics* at the University of Ouagadougou, Burkina Faso in May 2009. I am grateful to all the participants of these schools, whose enthusiasm encouraged me to write down the lectures and whose many questions helped me improve this text. Thanks to all of them!

Introduction

These lectures are intended for graduate students in mathematics or physics who need some basic concepts in differential geometry, global analysis, operator algebras and pseudodifferential operators in view of understanding how these are used in quantum field theory.

Far from being complete, these notes offer a first guide for the layperson, suggesting further references for the interested reader. The list of references suggested at the beginning of each section is also far from complete and is just meant to give the reader a first hint of the (often huge) literature on the subject. I have mostly chosen to refer to text books and survey type articles, in order to limit the number of references.
Because of lack of space, I unfortunately have had to leave out numerous examples that illustrate the sometimes rather abstract concepts presented here. The last chapter somewhat compensates for this lack of example by illustrating in Yang-Mills, Seiberg-Witten and string theory how the various concepts introduced in the previous chapter can come into play to investigate the structure of the configuration and moduli spaces.

For the sake of simplicity, I chose to introduce the concepts of manifold and vector bundle in the simplest infinite dimensional setting, namely the Hilbert setting, leaving aside the more subtle concepts needed in the Fréchet setting. A more general infinite dimensional setting is described in [26]. The Hilbert setting offers various simplifications; we have the very useful implicit function theorem at hand and any closed subspace of a Hilbert space splits the Hilbert space as a direct sum of this subspace with its orthogonal complement. Also, a partition of unity can be defined on a Hilbert manifold, which is not always the case on Banach manifolds.
In fact the most appropriate setting for our needs is the I.L.H. setting, namely the inverse limit of

Hilbert spaces [41] which we shall only briefly mention in the applications at the end of these notes.

These notes start at a leisurly pace but the material gets denser as one goes along, relying on the fact that the reader who has read the beginning chapters has got acquainted with the geometric concepts enough to be able to use them rather loosely in the last chapters.
The present lecture notes are organized in five chapters; the first three are dedicated to prerequisites in differential geometry (including infinite dimensional Banach structures), chapter 4 to operators and operator algebras of different types, including pseudodifferential operators and a brief incursion into index theory. Chapter 5 is dedicated to the geometry of configuration and moduli spaces one comes across in Yang-Mills, Seiberg-Witten and string theory.

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1 Manifolds, Lie algebras and Lie groups

1.1 Banach vector spaces

Useful references are [5], [9], [30], [47], [54].

Recall that a Banach vector space (we shall say Banach space for short) is a vector space equipped with a norm for which it is closed.

Definition: Let E and F be two Banach spaces and U an open subset of E. A map $f : E \to F$ is *differentiable* at a point x_0 of U provided there exists a continuous linear map $L : E \to F$ and a map $\phi : U \subset E \to F$ defined on a neighborhood U of $0 \in E$ such that

$$f(x_0 + y) = f(x_0) + L(y) + \phi(y)$$

with $\lim_{y\to 0} \frac{\|\phi(y)\|_F}{\|y\|_E} = 0$, where $\|\cdot\|_E$ is the norm on E and $\|\cdot\|_F$ the norm on F. Then L is a uniquely defined map, called the *differential* at point x_0 and denoted by $D_{x_0} f$.

The space $\mathcal{L}(E, F)$ of continuous linear maps from E to F is also a Banach space when equipped with the norm $|||L||| := \sup_{u\neq 0} \frac{\|Lu\|_F}{\|u\|_E}$.

Definition: Let E and F be two Banach spaces and U an open subset of E. The map $f : U \subset E \to F$ is of class C^1 on U provided f is differentiable at any point of U and the map:

$$\begin{array}{rcl} Df : U & \to & \mathcal{L}(E, F) \\ x & \mapsto & D_x f \end{array}$$

is continuous.

Indentifying $\mathcal{L}\left(\mathcal{L}(E,\mathcal{L}\left(E,\cdots(E,F)\cdots,F\right)F\right)$ –where E arises k times– with the Banach space $\mathcal{L}^k(E,F)$ of k-linear maps from E^k to F, we can define the notion of C^k differentiability.

Definition: A differentiable map $f: U\subset E\to F$ is of class C^k provided Df is of class C^{k-1}.

Definition: Let E and F be two Banach spaces and U, V two open subsets of E and F, respectively. A differentiable map $f: U\to V$ is a *diffeomorphism* whenever it is one to one and onto and its inverse map is differentiable. It is a C^k diffeomorphism whenever it is a diffeomorphism and both f and its inverse f^{-1} are of class C^k.

The following well-known results in Banach spaces will be used later in these notes.

Local inverse function theorem: Let E and F be Banach spaces, U an open subset of E and $f: U\to F$ a C^k map for some $k\geq 1$. If for some point $x_0\in U$ the map $D_{x_0}f: E\to F$ is an isomorphism then there exists a neighborhood W of x_0 such that the restriction $f_{|_W}: W\to f(W)$ of f to W is a C^k diffeomorphism.

Hahn-Banach theorem: Let $F\subset E$ be a linear subspace of E s.t. $\bar{F}\neq E$. Then there is a continuous linear form L on E such that $L(u)\neq 0 \quad \forall u\in F$.

Applying this result to the vector space $F=\langle u_0\rangle$ generated by some $u_0\in E$ yields the following:

Corollary: Let $u_0\neq 0\in E$, where E is a Banach space. Then there is a continuous linear form L on E such that $L(u_0)\neq 0$.

Another fundamental result for the following is the

Open mapping theorem: Let E and F be two Banach spaces. A continuous linear map $L: E\to F$ which is onto takes an open subset to an open subset. If it is continuous and one to one, it is a homeomorphism.

Corollary 1: Let F and G be two closed linear subspaces in E such that $F\oplus G=E$. Then the map:

$$\begin{array}{rcl} F\times G & \to & E \\ (u,v) & \mapsto & u+v \end{array}$$

is an isomorphism of Banach spaces.

Restricting oneself to the Hilbert setting is convenient because of the existence of orthogonal complements for closed subspaces. This property can be formulated as follows.

Definition: A linear subspace F of a Banach vector space E *splits* the space E if it is closed and if there exists a closed linear subspace G of E such that $E=F\oplus G$.

In the finite dimensional setting, any subspace splits a vector space. In the Hilbert setting, any closed subspace of a Hilbert space splits the space; the orthogonal complement does the job and the above Corollary takes the following form.

Corollary 2: Let E be a Hilbert vector space and F a closed linear subspace of E then the map:

$$\begin{array}{rcl} F \times F^{\perp} & \rightarrow & E \\ (u,v) & \mapsto & u+v \end{array}$$

is an isomorphism of Hilbert spaces.

1.2 Manifolds, submanifolds

Useful references are [22], [27],[30], [33], [55].

Definition: A manifold M of class C^k, $k \geq 0$ (or C^k-manifold) modelled on a Banach space E (called the *model space*) is a topological space equipped with a C^k-*atlas* i.e. a set of *local charts* $\{(U_i, \phi_i), i \in I\}$ satisfying the following requirements:

i) for any $i \in I$ the subset U_i is open and $M = \bigcup_{i \in I} U_i$,

ii) for any $i \in I$, the map $\phi_i : U_i \rightarrow \phi_i(U_i)$ is a homomeorphism onto an open subset of E,

iii) for any $i, j \in I$ the maps

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$$

are diffeomorphisms of class C^k called *transition maps.*

It is of class C^∞ if it is of class C^k for all $k \geq 0$.

An atlas is not unique and any C^k (resp. C^∞) atlas could do; one usually picks out the *maximal* atlas, i.e. one that contains all the others.

A (real) finite dimensional manifold of dimension n is modelled on $E = I\!\!R^n$ and local charts provide *local coordinates*:

$$\begin{array}{rcl} \phi_i : U_i & \rightarrow & \phi_i(U_i) \\ x & \mapsto & (x_1, \cdots, x_n) \in I\!\!R^n. \end{array}$$

Transition maps are elements of $GL(n, I\!\!R)$.
Examples.

- The unit sphere in R^{n+1} defined as:

$$S^n = \{(x_0, \cdots, x_n) \in I\!\!R^{n+1}, \sum_{i=0}^n |x_i|^2 = 1\}$$

is a smooth manifold of dimension n. Let $U_1 = S^n - \{N\}$ and $U_2 = S^n - \{S\}$, with $N = (0, \cdots, 0, 1)$ the north pole and $S = (0, \cdots, 0, -1)$ the south pole. Given a point $M(x_0, \cdots, x_{n-1}, x_n) \in U_1$, we have $\overrightarrow{NM} = (x_n - 1)\,(\frac{x_0}{x_n - 1}, \cdots, \frac{x_{n-1}}{x_n - 1}, 1)$ and for a point a point $M(x_0, \cdots, x_{n-1}, x_n) \in U_1$ we have $\overrightarrow{SM} = (x_n + 1)\,(\frac{x_0}{x_n+1}, \cdots, \frac{x_{n-1}}{x_n+1}, 1)$. Local charts are (U_1, ϕ_1) and (U_2, ϕ_2) where $\phi_1(x_0, \cdots, x_n) = \left(\frac{x_0}{1-x_n}, \cdots, \frac{x_{n-1}}{1-x_n}\right)$,$\phi_2(x_0, \cdots, x_n) = \left(\frac{x_0}{1+x_n}, \cdots, \frac{x_{n-1}}{1+x_n}\right)$.

- The n-th dimensional torus $T^n = R^n / \sim$ where

$$z_1 \sim z_2 \Leftrightarrow \exists\, n_i \in \mathbb{Z}, i = 1 \cdots, n, \quad z_1 - z_2 = \sum_{i=1}^{n} n_i e_i$$

  where we have set $e_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{R}^n$ with 1 at the i-th place.

- The projective plane $P_n(R) = R^{n+1} - \{0\}/ \sim = S^n / \simeq$ where $z_1 \sim z_2 \Leftrightarrow \exists \lambda \in \mathbb{R}, \quad z_1 = \lambda z_2$ and $z_1 \simeq z_2 \Leftrightarrow \exists \lambda \in \{-1, 1\}, \quad z_1 = \lambda z_2$. Local charts are given by $U_i = \{(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}, x_i \neq 0\}$ and $\phi_i(x) = \left(\frac{x_0}{x_i}, \cdots, \frac{\hat{x_i}}{x_i}, \cdots, \frac{x_n}{x_i}\right)$ where the $\hat{}$ means we have deleted the variable.

- The Grassmann manifold $G_k^n(\mathbb{R})$ of k-dimensional submanifolds of $\mathbb{R}^n$. Given $V \subset \mathbb{R}^n$, we can identify $\mathbb{R}^n = V \times V^\perp$. A neighborhood of $V \in G_k^n$ is mapped homeomorphically onto an open set in the vector space of linear maps $V \to V^\perp$. This makes G_k^n a manifold of dimension $k(n-k)$. The case $k = 1$ yields back the projective space $P_n(\mathbb{R})$.

Definition: Let F be a linear subspace of a Banach vector space E that splits E. Given a C^k manifold M modelled on E, a subset N of M is a *submanifold* of M modelled on F provided there is a C^k-atlas $\{(U_i, \phi_i), i \in I\}$ on M that induces an atlas on N, i.e. for any $i \in I$ there are open subsets V_i, W_i of E, F such that

$$\phi_i(U_i) = V_i \oplus W_i$$

and

$$\phi_i(U_i \cap N) = V_i \oplus \{0\}.$$

In the case of an n-dimensional real manifold, the model space is R^n and a subspace F of dimension $k \leq n$ of $\mathbb{R}^n$ can be equipped with a basis which we complete to a basis of $\mathbb{R}^n$. In this basis local coordinates on N will be of the form:

$$\begin{aligned} (\phi_i)_{|_N} : U_i \cap N &\to \phi_i(U_i) \\ x &\mapsto (x_1, \cdots, x_k, 0, \cdots, 0) \in \mathbb{R}^n. \end{aligned}$$

In what follows, using local charts, we carry out to manifolds the notion of differentiability, C^k-regularity, and the notion of diffeomorphism, to maps between manifolds. Although the construction uses local charts, the concept thereby defined is shown to be independent of the choice of local chart. All the manifolds involved are Banach manifolds.

Definition: Let M, N be two Banach manifolds of class C^k, C^l, $k, l \geq 1$ respectively and modelled on E, F respectively. A map $f : M \to N$ is *differentiable* at a point $x_0 \in M$ provided there is a local chart (U, ϕ) of M containing x_0, a local chart (V, ψ) containing $f(x_0)$ such that the map

$$\psi \circ f \circ \phi^{-1} : \phi(U) \subset E \to \psi(V) \subset F$$

is differentiable at point $\phi(x_0)$.

A tangent vector at a point x of a C^k-Banach manifold M ($k \geq 1$) modelled on E is an equivalence class ξ of triples (U, ϕ, v) where (U, ϕ) is a local chart on M containing x and v a vector in the Banach space E, the equivalence relation being defined by:

$$(U, \phi, v) \sim (V, \psi, w) \Leftrightarrow w = D_{\phi(x)} \left(\psi \circ \phi^{-1}\right)(v).$$

In other words, v is the tangent vector ξ *read in the local chart* (U, ϕ) whereas w is the tangent vector ξ read in the local chart (V, ψ).

In the finite dimensional setting, say in dimension n, given a local system of coordinates $(x_1, \cdots, x_n)$, a tangent vector reads $v := \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$.

The space $T_x M$ of tangent vectors at a point $x \in M$ can be equipped with a vector space structure induced from that of the model space E. Since transition maps are diffeomorphisms, the maps $D_{\phi(x)}\left(\psi \circ \phi^{-1}\right)$ are isomorphisms of Banach spaces and $T_x M$ can be equipped with a Banach structure induced from that on E. Thus $T_x M \simeq E$ is a Banach space.

Definition-Proposition: Let M and N be two Banach manifolds of class C^k, C^l respectively, with $k, l \geq 1$. Let $f : M \to N$ be a differentiable map, then $D_x f : T_x M \to T_{f(x)} N$ is a linear map, called the *differential of* f *at point* x defined by:

$$D_x f(\xi) = \eta \Leftrightarrow v = D_{\phi(x)}\left(\psi \circ f \circ \phi^{-1}\right) u$$

whenever u corresponds to ξ read in a local chart (U, ϕ) containing x and v corresponds to η read in a local chart (V, ψ) containing $f(x)$.

This definition is independent of the choice of local chart.

Given a manifold M of class C^k, $k \geq 1$, the set $TM := \bigcup_{x \in M} T_x M$ can be equipped with a C^{k-1}-manifold structure, a local chart at (x, ξ), $\xi \in T_x M$ being of the type $(U, \phi, D\phi)$ where (U, ϕ) is a local chart on M at point x.

Definition: Given manifolds M, N of class C^k, C^l, $k, l \geq 1$, a map $f : M \to N$ is *of class* C^j, with $1 \leq j \leq inf\{k, l\}$ provided it is differentiable and $Df : TM \to TN$ is of class C^{j-1}.

Submanifolds can be obtained via embeddings, a particular class of immersion. As we saw in the first section, a subspace of a Banach space does not automatically split the space E, so that we need to encode a "splitting" condition in the definition of immersion:

Definition: A differentiable map $f : M \to N$ from a C^k-manifold M, to a C^l-manifold N with $k, l \geq 1$ is *an immersion* (resp. *submersion*) provided $D_x f$ is injective (resp. onto) and the range $R(D_x f)$ (resp. the kernel $\mathrm{Ker}(D_x f)$ splits $T_{f(x)} N$ (resp. $T_x M$) for any $x \in M$.

Here again, the Hilbert setting offers a simplification:

A differentiable map $f : M \to N$ from a Hilbert manifold M to a Hilbert manifold N is an immersion (resp. submersion) provided $D_x f$ is injective (resp. onto) and $R(D_x f)$ is closed for any $x \in M$. (Note that the kernel is always closed when the operator is closed).

An injective immersion is called an *embedding*. The following result which is a manifold version of the global inverse map theorem will be very useful in the slice theorem.

Global inverse map theorem: An embedding $f : M \to N$ that is a homeomorphism onto its range yields a submanifold $f(M)$ of N and $f(M) \simeq M$, namely M is diffeomorphic to its range.

1.3 Partitions of unity

Useful references are [26], [30].

Partitions of unity provided means of gluing together local objects in order to build a globally defined one. It is therefore important to define conditions under which partitions of unity with a certain degree of regularity exist.

Definition: A *partition of unity* of class C^k of a C^k-manifold M is given by a locally finite covering $(u_i)_{i\in I}$ of M and a family $\{\psi_i, i \in I\}$ of maps of class C^k:

$$\psi_i : E \to I\!R$$

such that:

1. $0 \leq \psi_i, \quad \forall i \in I$
2. The support of ψ_i is contained in U_i
3. $\sum_{i\in I} \psi_i(p) = 1$.

Such a partition of unity is said to be subordinated to an atlas $(W_i, \phi_i)_{i\in I}$ on the manifold if $\bar{U}_i \subset W_i$. A partition of unity is smooth whenever it is of class C^k for any $k \in I\!N$.

Let us recall that a manifold is *paracompact* if from any cover of the manifold, one can extract a locally finite sub-cover, i.e. a subcover such that every point of the manifold admits a neighborhood which only intersects a finite number of the open subsets of this covering.
The following topological lemma will be useful in the sequel:

Lemma:

1) Any paracompact manifold is *normal*, i.e. two disjoint closed subsets have dijoint neighborhoods.
2) (Urysohn's Lemma) Given two closed disjoints subsets A and B of a topological normal vector space E, there is a continuous map $f : E \to [0,1]$ which vanishes on A and is equal to 1 on B.
3) Given a locally finite covering (U_i) of a paracompact topological vector space, there is a locally finite subcovering (V_i) such that $\bar{V}_i \subset U_i$.

Proposition: Any paracompact topological manifold has a *continuous* partition of unity.

Proof: Let U be an open set on a manifold modelled on some sparable space E and let $x \in U$. Let (U_i, ϕ_i) be an atlas and (U_{i_0}, ϕ_{i_0}) be a local chart at point x_0. ϕ_{i_0} can be composed with a map that sends an open ball of E onto E in such a way that the resulting map (also denoted by ϕ_0) satisfies $\phi_{i_0}(U_{i_0}) = E$. Since the manifold is Banach, it is metrisable, since it is moreover separable, it is paracompact ([30], chap. II, par. 3, Lemma 1). Thus one can extract from the above subcovering a locally finite one. The third part of the above lemma then yields locally finite subcoverings (V_i) and (W_i) such that $\bar{W}_i \subset V_i \subset \bar{V}_i \subset U_i$. Since every $\bar{V}_i$ is closed, given the way the ϕ_i were chosen, so are $\phi_i(\bar{V}_i)$ and $\phi_i(\bar{W}_i)$ closed subsets in E. E being Banach and separable, it is paracompact and hence normal. The second part of the above lemma yields a continuous map $\psi_i : E \to [0,1]$ which is 1 on $\bar{W}_i$ and 0 outside V_i. Composing it with ϕ_i yields continuous maps $\Psi_i : M \to [0,1]$ which are 1 on $\bar{W}_i$ and that vanish outside V_i. Setting $\xi_i := \frac{\Psi_i}{\sum \Psi_i}$ yields a partition of unity. ■

However, smooth partitions of unity do not always exist on smooth Banach manifolds. They do on smooth Hilbert manifolds as a consequence of the following result which provides a smooth version of the Urysohn Lemma. It essentially relies on the smoothness of the squared norm on a Hilbert space.

Lemma ([30] Th.3.7): Given two disjoint closed subets A and B of a separable Hilbert space, there is a smooth map $\phi : H \to [0,1]$ which is 1 on A and which vanishes on B.

Idea of the proof: First of all, using the smoothness of the squared norm $\|\cdot\|^2$ on the Hilbert space H, given any open ball $B(x,R)$ in H centered at point x with radius R, one can build a smooth function $\phi : H \to [0,1]$ which is positive on $B(x,R)$ and vanishes elsewhere. For this, one picks any smooth function $\eta : I\!R \to [0,1]$ which is positive for $t < R$ and vanishes beyond R, and composes it with the squared norm to build $\phi := \eta \circ \|\cdot\|^2$. Using the separability and metrisability of H, one can build a countable set of open balls $\{U_i = B(x_i, r_i)\}$ (with $x_i \neq x_j$) which cover A and do not meet B. One can then inductively construct a locally finite refinement $\{V_i \subset \bigcup_i U_i\}$, and correspondingly find smooth functions η_i built as above, which are positive on V_i and vanish outside V_i. The sum $\eta := \sum_i \eta_i$, which is finite at each point of W, defines a smooth function that is positive on A and vanishes on B. Letting σ be a smooth function positive on the complement W^c of W and that vanishes on W, the map $\phi := \frac{\eta}{\eta+\sigma}$ fulfills the requirements of the Lemma. ■

Proposition ([30] Corollary 3.8): A paracompact manifold of class C^p modelled on a separable Hilbert space admits a partition of unity *of class* C^p.

Idea of the Proof: The proof goes as in the construction of a continuous partition of unity. One transports an open covering (which by paracompactness is locally finite) of the manifold by local charts to the model space and applies the above lemma to the closure of the open subsets obtained this way. Carrying back the smooth functions thus obtained to the manifold via the local charts yields a smooth partition of unity on the manifold. ■

1.4 Vector fields

Useful references are [22], [27], [30], [33], [55].

Let M be a C^k-manifold and let $j \leq k$. A *C^j-vector field* is a C^j-map $\xi : M \to TM$ such that $\xi(x) \in T_xM$ for all $x \in M$. If M is smooth, a smooth vector field is one that is of class C^j for any $j \geq 0$.
Let us denote by $\Xi(M)$ the vector space of smooth vector fields on M. If M is n-dimensional, given a local system of coordinates $(x_1, \cdots, x_n)$ around a point x, a vector field $\xi \in \Xi(M)$ reads $\xi(x) := \sum_{i=1}^n \xi_i(x) \frac{d}{dx^i}$.

Definition: The *integral curve* of a vector field ξ on a manifold M is a curve $c : I \to M$ (I an open interval in $I\!R$) with tangent vector $\xi(c(t))$ at point $c(t)$ i.e. such that:

$$\frac{d}{dt}c(t) = \xi(c(t)) \quad \forall t \in I.$$

From the theory of classical differential equations in Banach spaces we know, that given some initial condition $c(0) = x$ (we assume $0 \in I$), for some $x \in M$, there exists a neighborhood I of 0 and an integral curve uniquely defined on I. The union of the domains of all integral curves with a given initial condition x is an open interval which we denote by I_x with end points $t_x^- \leq t_x^+$ (which could be $+\infty$ or $-\infty$).
These integral curves are smooth w.r. to initial conditions namely, given an integral curve c_x starting at point x, there is an open neighborhood U_x of x and a neighborhood J_x of 0 such that for for $y \in U$,

the integral curve c_y starting at point y is defined on J_x. Furthermore the map:

$$\begin{array}{rcl} J_x \times U_x & \to & M \\ (t, y) & \mapsto & c_y(t) \end{array}$$

is smooth.

For some given vector field ξ, let $\mathcal{D}(\xi)$ denote the subset in $I\!R \times M$ consisting of all points (t, x) such that $t_x^- < t < t_x^+$. The *flow of* ξ is a map:

$$\phi : \mathcal{D}(\xi) \to M$$

such that the map $\phi_x(t) := \phi(t, x)$ defined on the interval I_x is a morphism and an integral curve for ξ with initial condition x. In particular it satisfies the differential equation:

$$\frac{d\phi_x}{dt} = \xi \circ \phi_x.$$

The flow ϕ_x is *complete* if it can be extended to $I_x = I\!R$. Fixing the starting point $x \in M$ and setting $\phi_t := \phi_x(t)$ for any $t \in I\!R$ yields a *one parameter semi-group*:

$$\phi_t \circ \phi_s = \phi_{t+s} \quad \forall t, s \in I\!R^+.$$

The above property actually extends to $t, s \in I\!R$ which implies:

$$\phi_t^{-1} = \phi_{-t}$$

so that $\{\phi_t, t \in I\!R\}$ defines a *one parameter group of diffeomorphisms*.

Given a differentiable map ϕ on M and a vector field ξ, we call the vector field defined by:

$$\phi_* \xi(\phi(x)) = D_x \phi(\xi(x)),$$

the *push forward* of ξ and if ϕ is a diffeomorphism:

$$\phi^* \xi := \left(\phi^{-1}\right)_* \xi$$

the *pull-back* of ξ. For example, we can consider the pull-back $\phi_t^* \tilde{\xi}$ of $\tilde{\xi}$ along the integral curve ϕ_t of ξ.

Definition: The *Lie bracket* of two vector fields $\xi, \tilde{\xi}$ on a smooth manifold M is given by the variation of $\tilde{\xi}$ along an integral curve ϕ_t of ξ:

$$[\xi, \tilde{\xi}] := \frac{d}{dt}_{|t=0} (\phi_t^* \tilde{\xi}) = \frac{d}{dt}_{|t=0} \left((\phi_{-t})_* \tilde{\xi} \right).$$

On an n-dimensional manifold, in local coordinates around a point x the Lie bracket reads of two vector fields $\xi = \xi^i \frac{\partial}{\partial x_i}$ and $\eta = \eta^i \frac{\partial}{\partial x_i}$ reads:

$$[\xi, \eta](x) = \left(\xi^i \partial_{x_i} \eta^j - \eta^i \partial_{x_i} \xi^j \right) \frac{\partial}{\partial x_j}$$

with the usual summation convention over repeated indices which run from 1 to n. Given a smooth map ϕ from a manifold N to a manifold M and $\xi, \tilde{\xi} \in \Xi(M)$, we have:

$$[\phi_* \xi, \phi_* \tilde{\xi}] = \phi_* [\xi, \tilde{\xi}].$$

If ξ_1, ξ_2, ξ_3 are three vector fileds on a smooth manifold M and ϕ_t is a one parameter group of local diffeomorphisms generated by ξ_3, then differentiating the following relation:

$$\phi_{t*}[\xi_1, \xi_2] = [\phi_{t*}\xi_1, \phi_{t*}\xi_2]$$

w.r.to t at $t = 0$ yields the *Jacobi identity*:

$$[[\xi_1, \xi_2], \xi_3] + [[\xi_2, \xi_3], \xi_1] + [[\xi_3, \xi_1], \xi_2] = 0.$$

Vector fields can be identified with derivations on M and Lie brackets with operator brackets of the derivations. By a *derivation* on M we mean a linear map $L : C^\infty(M, I\!R) \to C^\infty(M, I\!R)$ which obeys the Leibniz rule

$$L(fg) = L(f)g + fL(g) \quad \forall f, g \in C^\infty(M, I\!R).$$

The set $Der(M)$ of derivations on M is a vector space over $I\!R$. To a given vector field ξ on M we associate the map:

$$\begin{array}{rcl} L_\xi : C^\infty(M, I\!R) & \to & C^\infty(M, I\!R) \\ f & \mapsto & Df(\xi) \end{array}$$

which is clearly a derivation.

Let $\xi \neq 0$, then there is some $x \in M$ such that $\xi(x) \neq 0$. Let (U, ϕ) be a local chart around x and u a representative of $\xi(x)$ in this chart. Since $u \neq 0$, by the Hahn-Banach theorem there is some linear form L on the model space E such that $L(u) \neq 0$. Thus $L \circ D\phi = D(L \circ \phi)$ does not vanish on $D\phi^{-1}(u)$ which can be identified with $\xi(x)$. Patching up the locally defined maps $f := L \circ \phi : U \to I\!R$ using a smooth partition of unity on M (provided there is one) shows the existence of a function $f \in C^\infty(U, I\!R)$ such that $L_\xi f(x) = Df(\xi(x)) \neq 0$ so that $L_\xi \neq 0$. Thus, provided there is a smooth partition of unity on M, there is a one to one correspondence:

$$\begin{array}{rcl} \Xi(M) & \to & Der(M) \\ \xi & \mapsto & L_\xi : f \to Df(\xi). \end{array}$$

The following identification holds:

Proposition: Given two vector fields $\xi, \tilde{\xi}$ on a smooth manifold M and $f \in C^\infty(M, I\!R)$:

$$[\xi, \tilde{\xi}]f := [L_\xi, L_{\tilde{\xi}}]f$$

where the bracket on the r.h.s is an operator bracket.

This identification yields back the antisymmetry of the bracket together with the Jacobi identity which hold for the operator bracket.

1.5 Lie groups and Lie algebras

Useful references are [1], [8], [27].

Definition: A Banach (resp. Hilbert) *Lie group* modelled on a Banach (resp. Hilbert) space E

is a C^∞-manifold modelled on E, equipped with a group structure such that the multiplication and inverse maps

$$\begin{array}{rcl} G \times G & \to & G \\ (g,h) & \mapsto & gh \end{array}$$

and

$$\begin{array}{rcl} G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$

are smooth.
In fact this second property follows from the former using the global inverse map theorem (see section 1.2).

A finite dimensional Lie group is one that has a finite dimensional manifold structure. Examples of finite dimensional Lie groups are the group $GL(n,\mathbb{R})$ of invertible $n \times n$ real matrices, the subgroup $O(n)$ of orthogonal matrices both of which arise as structure groups of frame bundles, the unitary groups $U(n) = \{A \in GL(n,\mathbb{C}), AA^* = 1\}$ and the special unitary groups $SU(n) = \{A \in U(n), \det A = 1\}$, that play an important role in gauge field theory.

Definition: Let H, G be two Lie groups and $f : H \to G$ an embedding which is homomorphism of Lie groups and a homeomorphism onto $f(H)$. Then $f(H)$ is called a *Lie subgroup* of H.

Remark: Notice that $f : H \to G$ being an immersion, $D_e f(T_e H)$ splits $T_e G$.

Definition: A *Lie algebra* A is a vector space equipped with an antisymmetric bilinear map:

$$\begin{array}{rcl} A \times A & \to & A \\ (a,b) & \mapsto & [a,b] \end{array}$$

that satisfies the Jacobi identity:

$$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0 \quad \forall a,b,c \in A.$$

A *Banach, resp. Hilbert Lie algebra* is a Banach, resp. Hilbert vector space equipped with a continuous antisymmetric bilinear map which satisfies the Jacobi identity.

On the grounds of the above remark, we call *Lie subalgebra* of a Banach Lie algebra A any closed linear subspace of A that splits A and that is stable under the Lie bracket, i.e.

$$a, b \in B \Rightarrow [a,b] \in B.$$

In particular, a Lie subalgebra of a Hilbert Lie algebra A is a closed linear subspace of A stable under the Lie bracket of A.

Definition: A left- (resp. right-) *action* of a Lie group G on a smooth manifold M is a map:

$$\begin{array}{rcl} \Theta : G \times M & \to & M \\ (g,x) & \mapsto & \Theta(g,x) \end{array}$$

such that:

$$\Theta(e,x) = x \quad \forall x \in M$$

and
$$\Theta(gh, x) = \Theta(g, \Theta(h, x)) \quad \forall g, h \in G$$
(resp.
$$\Theta(gh, x) = \Theta(h, \Theta(g, x)) \quad \forall g, h \in G.)$$
Such an action is smooth if the map Θ is smooth. It is convenient to denote a right action by $\Theta(g, x) := x \cdot g$ and a left action by $\Theta(g, x) := g \cdot x$.

A Lie group acts on itself by a right and a left action via the multiplication maps:
$$R_g(h) := hg, \quad L_g(h) := gh \quad \forall h, g \in G.$$
It also acts on itself via the *adjoint action*:
$$\begin{array}{rcl} G \times G & \to & G \\ ((g, h) & \mapsto & Ad_g(h) := L_g R_{g^{-1}} h = R_{g^{-1}} L_g h. \end{array}$$
A *right invariant field* on G is a vector field ξ such that:
$$\xi(hg) = D_g R(\xi(h)) \quad \text{i.e.} \quad R_{g*}\xi = \xi \quad \forall h, g \in G$$
and a *left invariant field* on G is a vector field ξ such that:
$$\xi(gh) = D_g L(\xi(h)) \quad \text{i.e.} \quad L_{g*}\xi = \xi \quad \forall h, g \in G.$$

Let $\Xi_L(G)$, resp. $\Xi_R(G)$ denotes the space of left, resp. right invariant vector fields on G. For any $u \in T_eG$ the vector field $g \to \xi_u^L(g) := D_e L_g(u)$ is left invariant, $g \mapsto \xi_u^R(g) := D_e R_g(u)$ is right invariant, and we can build two maps:
$$\begin{array}{rcl} \Xi_L : T_eG & \to & \Xi_L(G) \\ u & \mapsto & D_e L_g(u) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Xi_R : T_eG & \to & \Xi_R(G) \\ u & \mapsto & D_e R_g(u), \end{array}$$
which are one to one and onto.

If the manifold M is a Lie group, given two left invariant vector fields ξ^L and $\tilde{\xi}^L$ on G, their Lie bracket is also left invariant for we have:
$$[L_{g*}\xi, L_{g*}\tilde{\xi}] = L_{g*}[\xi^L, \tilde{\xi}^L] \quad \forall g \in G$$
and a similar property holds for right invariant vector fields on G. Thus the Lie bracket on vector fields induces two brackets on T_eG:
$$[u, v]_L := [\xi_u^L, \xi_v^L], \quad [u, v]_R := [\xi_u^R, \xi_v^R] \quad \forall u, v \in T_eG.$$
The map:
$$\begin{array}{rcl} J : G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$
satisfies $gJ(g) = e$, i.e. $R_{J(g)}(g) = e$ (or equivalently $J(g)g = e$ i.e. $L_g(J(g)) = e$ for any $g \in G$). Differentiating this relation at point $h \in G$ yields $D_h R_{g^{-1}} u + (D_{hg^{-1}} L_g) D_h J u = 0$, and gives the expression for its differential map at $h \in G$:
$$\begin{array}{rcl} D_h J : T_hG & \to & T_hG \\ v & \mapsto & -(D_{hg^{-1}} L_g)^{-1} D_h R_{g^{-1}} v = -D_{hg^{-1}} R_{g^{-1}} (D_h L_g)^{-1} v \end{array}$$

since DL_g commutes with DR_g. Hence DJ takes a left invariant vector field $\xi_u^L(g) = D_eL_g(u) \in T_gG$ to a right invariant vector field $D_gJ\xi_u^L(g) = -D_eR_{g^{-1}}\left(D_gL_g\right)^{-1}\xi_u^L(g) = -\xi_u^R(g^{-1})$, so that:

$$D_gJ(\xi_u^L) = -\xi_u^R \circ J \quad \text{i.e} \quad J_*\xi_u^R = -\xi_u^L.$$

Since J is a diffeomorphism, it follows that:

$$[\xi_u^L, \xi_v^L] = \left[J_*\xi_u^R, J_*\xi_v^R\right] = J_*\left[\xi_u^R, \xi_v^R\right]$$

and hence

$$[u,v]_L = [u,v]_R = [\xi_u^R, \xi_v^R](e) = [\xi_u^L, \xi_v^L](e).$$

The tangent space T_eG equipped with this Lie bracket becomes a Lie algebra denoted henceforth by $Lie(G)$.

Every automorphism ϕ of the Lie group G induces an automorphism ϕ_* of its Lie algebra $Lie(G)$ for if ξ is a left invariant vector field, then so is $\phi_*\xi$ and

$$[\phi_*\xi, \phi_*\tilde{\xi}] = \phi_*[\xi, \tilde{\xi}].$$

In particular, for any $g \in G$, the automorphism

$$\begin{array}{rcl} Ad(g) : G & \to & G \\ h & \mapsto & ghg^{-1} \end{array}$$

induces an automorphism of $Lie(G)$ also denoted by Ad_g.

To the left invariant and right invariant vector fields ξ_u^L and ξ_u^R built from an element $u \in Lie(G)$, we can associate two local flows ϕ_u^L and ϕ_u^R defined by

$$\frac{d\phi_u^L(t)}{dt} = \xi_u^L(\phi_u^L(t)), \qquad \frac{d\phi_u^R(t)}{dt} = \xi_u^R(\phi_u^R(t)).$$

Let us assume such a flow ϕ_u is defined up to time t_1. For simplicity we drop the superscript L and set $g_1 = \phi_u(t_1)$. Then ξ_u^L being left invariant $\psi_u(t) := g_1\phi_u(t)$ verifies:

$$\begin{array}{rcl} \frac{d}{dt}\psi_u(t) & = & D_{\phi_u(t)}L_{g_1}\frac{d}{dt}D_{\phi_u(t)} = D_{\phi_u(t)}L_{g_1}\xi_u^L(\phi_u(t)) \\ & = & \xi_u^L(\phi_u(t))(g_1\phi_u(t)) = \xi_u^L(u)(\psi_u(t)) \end{array}$$

and $\psi_u(0) = g_1$. As before, ψ_u can be defined on an interval $[0, t_1[$ thus extending the flow ϕ_u defined on $[0, t_1[$ to a flow on, $[0, 2t_1[$. Iterating this procedure shows that the flow can be extended to all $I\!R$. The same holds for the flow ϕ_u^R.

The *left invariant and right invariant integral flows* ϕ_u^L and ϕ_u^R of some vector u in the Lie algebra of a Lie group are therefore *complete.*
Let us compare these two flows:

$$\begin{array}{rcl} \frac{d}{dt}(J \circ \phi_u^L)(t) & = & D_{\phi_u^L(t)}J\left(\xi_u^L(\phi_u^L(t))\right) \\ & = & -\xi_u^R\left(J \circ \phi_u^L(t)\right) \end{array}$$

and hence

$$\frac{d}{dt}\left(\phi_u^L(t)\right) = \frac{d}{dt}\left(J \circ \phi_u^L(-t)\right) = \xi_u^R\left(J \circ \phi_u^L\right)(-t)$$

so that $\phi_u^L(t)$ satisfies the same differential system as $\phi_u^R(t)$ with the same initial conditions.

From the uniqueness of such a solution, we conclude that

$$\phi_u^L(t) = \phi_u^R(t) := \phi_u(t).$$

Definition: The map:

$$\begin{array}{rcl} \exp : Lie(G) & \to & G \\ \xi & \mapsto & \phi_u(1) \end{array}$$

is called the *exponential map* on the Lie group G.

Since $D_e \exp = Id$, by the local inverse theorem recalled in section 1.1, it induces a local diffeomorphism:

$$\exp : U \subset Lie(G) \to V \subset G$$

from an open neighborhood of 0 to an open neighborhood of $e \in G$.
On $GL(n, I\!R)$ it coincides with the exponential map of matrices $\exp A = \sum_0^\infty \frac{1}{k!} A^k$.

Given $a \in Lie(G)$, we can define a one parameter family $g_t := \exp(ta), \quad t \in I \subset I\!R$ where I is a (small enough open) interval containing 0, and define the adjoint action of $Lie(G)$ on itself by differentiating that of G on $Lie(G)$:

$$\begin{array}{rcl} ad_a : Lie(G) & \to & Lie(G) \\ b & \mapsto & \frac{d}{dt}_{|t=0} Ad_{g_t}(b) = [a, b] \end{array}$$

thus recovering the Lie bracket of $Lie(G)$.

The exponential map is not a morphism from the vector space $(Lie(G), +)$ to the group $(G, \cdot)$ as can be seen from the Lie Campbell-Hausdorff formula:

$$\exp a \cdot \exp b = \exp \left(\sum_{k=0}^\infty C^{(k)}(a, b) \right)$$

using the Banach topology on $Lie(G)$ and where $C^{(1)}(a, b) = a + b$ and for $k > 1$, $C^{(k)}(a, b)$ is a linear combination of Lie monomials of degree k in a and b given by:

$$C^{(k)}(a, b) = \sum_{j=1}^\infty \frac{(-1)^{j+1}}{(j+1)} \sum \frac{(ad\, a)^{\alpha_1} (ad\, b)^{\beta_1} \cdots (ad\, a)^{\alpha_j} (ad\, b)^{\beta_j} b}{(1 + \sum_{l=1}^j \beta_l) \alpha_1! \cdots \alpha_j! \beta_1! \cdots \beta_j!}.$$

Here we have set $ad\, a(c) := [a, c]$ and the inner sum is over all j-tuples of pairs of nonnegative integers (α_l, β_l) with $\alpha_l + \beta_l > 0$ and $\alpha_1 + \cdots \alpha_j + \beta_1 + \cdots \beta_j + 1 = k$ (terms with $\beta_j \neq 0$ vanish).

2 Vector bundles and tensor fields

2.1 Definition and first properties

Useful references are [23], [?], [27], [39], [48], [52], [55].

Definition: A *fibre bundle of class* C^k with typical fibre a C^k Banach manifold V (we shall also say *modelled on* V and call V the *model space*) is a triple (F, B, π) often denoted by $\pi : F \to B$ where

- F and B are differentiable manifolds of class C^k, called *total space* and *base space* respectively,
- $\pi : F \to B$ is a map of class C^k, called *canonical projection* , such that there is a set of local charts $(U_i, \phi_i)_{i\in I}$ covering B and C^k diffeomorphisms

$$\tau_i : \pi^{-1}(U_i) \to \phi_i(U_i) \times V$$

satisfying the following requirements:

  i) the fibre $F_b = \pi^{-1}(b)$ is a Banach manifold and

  ii) $\tau_i(b) := \tau_{i|_{F_b}}$ is a diffeomorphism from F_b to F.

A triple (U_i, ϕ_i, τ_i) is called a *local trivialisation* of the bundle.

Two local trivialisations (U_i, ϕ_i, τ_i) and (U_j, ϕ_j, τ_j) give rise to maps $\tau_{ij} := \tau_i \circ \tau_j^{-1}$ called *transition maps* of the form:

$$\begin{array}{rcl} \tau_{ij} : \phi_i(U_i \cap U_j) \times V & \to & \phi_j(U_i \cap U_j) \times V \\ (b, v) & \mapsto & (b, \tau_{ij}(b)(v)) \end{array}$$

where the $\tau_{ij}(b)$ are diffeomorphisms of class C^k of V.

Transition maps satisfy the following properties:

$$\tau_{ii}(b) = id_V \quad \forall b \in U_i$$
$$\tau_{ij}(b) \circ \tau_{ji}(b) = id_V \quad \forall b \in U_i \cap U_j$$
$$\tau_{ij}(b) \circ \tau_{jk}(b) \circ \tau_{ki}(b) = id_V \quad \forall b \in U_i \cap U_j \cap U_k.$$

The family $\{\tau_{ij}\}$ is called a *cocycle* associated to the trivialization $\{U_i, \tau_i, i \in I\}$, and the last relation mentioned above a *cocycle* relation. From a covering of a manifold B together with a set of transition maps satisfying these relations one can reconstruct the fibre bundle on B.

A C^k *section* of a fibre bundle $\pi : F \to B$ is a map $s : B \to F$ of class C^k such that $\pi \circ s = Id_B$. It is smooth when it is of class C^k for all $k \in I\!N$.
The space of C^k sections of F is denoted by $C^k(F)$.
In the following we mainly consider smooth manifolds and smooth bundles as well as smooth sections.

Definition: A morphism of C^k fibre bundles $\pi : F \to B$ and $\pi' : F' \to B'$ is a couple (f_0, f) of C^k morphisms $f_0 : B \to B'$ and $f : F \to F'$ such that $\pi' \circ f = f_0 \circ \pi$ and the induced map on the fibres $f_x : \pi^{-1}(x) \to (\pi')^{-1}(x)$ is a morphism of the fibres.
In what follows we shall often take $B = B'$ and $f_0 = Id$.

Two fibre bundles are *isomorphic* if there is a diffeomorphism from one to the other.
A *trivial fibre bundle* is a fibre bundle isomorphic to the bundle $\pi : F = B \times V \to V$.

Definition: Let $B' \to B$ be a C^k morphism of Banach manifolds, and let $F \to B$ be a C^k-fibre bundle on B. The *pull-back* $\phi^* F$ of F by ϕ is a fibre bundle $\phi^*\pi : \phi^* F \to B'$ with total space:

$$\phi^* F := \{(b', v(\phi(b'))) \in B' \times F_{\phi(b'))}\}$$

where V is the model space of F and with projection $\phi^*\pi(b) = \pi(\phi(b))$.

Definition: A (real or complex) *vector bundle of class* C^k is a fibre bundle of class C^k with typical fibre a (real or complex) vector space V, and such that there is a local trivialization inducing automorphisms $\tau_{ij}(x)$ of the Banach vector space V, i.e. $\tau_{ij} \in GL(V)$.

When $V = \mathbb{R}^d$ (resp. $\mathbb{C}^d$), the vector bundle has *rank* d. If $d = 1$ it is called a *line bundle.*

Example. The *Grassmann bundle* γ_k^n over the Grassmann manifold $G_k^n(\mathbb{R})$ is the vector bundle with fibre above the vector space $V \subset \mathbb{R}^n$ given by the pairs (V, x) such that $x \in V$. It is a vector bundle of rank k.

A real finite rank vector bundle is *orientable* provided it has a trivialization with transition maps $\tau_{ij}(b)$ with positive determinant. A manifold is orientable whenever its tangent bundle is orientable.

Given two C^k-vector bundles $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ over a C^k- manifold B modelled repsectively on the linear spaces V_1 and V_2, we can build their Whitney sum $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \to B$ which is the C^k-vector bundle over B modelled on $V_1 \oplus V_2$ whose fibre above $b \in B$ is given by $\pi_1^{-1}(b) \oplus \pi_2^{-1}(b)$ and whose transition maps are given by the Whitney sum of the transition maps of E_1 and $\frac{E}{2}$.
Serre-Swan's theorem states that for any a finite rank C^k-vector bundle E over B, there is a finite rank C^k-vector bundle F over B such that $E \oplus F$ is C^k-isomorphic to a trivial bundle over B.

The set $C^0(E)$ (resp. $C^\infty(E)$) of continuous sections (resp. smooth sections) of a vector bundle E forms a linear space which is a $C^0(B)$- (resp. $\mathbb{C}^\infty(B)$-module. An equivalent formulation of Serre-Swan's theorem is the existence for the $C^0(B)-$ (resp. $C\infty(B)$-) module $C^0(E)$ (resp. $C^\infty(E)$ of a $C^0(B)-$ (resp. $C^\infty(B)-$) module $\mathcal{M}$ such that $C^0(E) \oplus \mathcal{M} \simeq (\mathbb{C}^\infty(B))^N$ (resp. $C^\infty(E) \oplus \mathcal{M} \simeq (\mathbb{C}^0(B))^N$ for some positive integer N.

Given a manifold M of class C^{k+1} (resp. of class C^∞) modelled on a Banach space V, the tangent bundle TM is a C^k (resp. C^∞)-vector bundle with fibres modelled on that same space V; given a local trivialization (U_i, ϕ_i) on M, a local trivialization (U_i, ϕ_i, τ_i) on TM is given by $(U_i, \phi_i, D\phi_i)$ and $D\phi_i \circ D\phi_j^{-1}$ is of class C^{k-1}.

Vector fields on a smooth manifold M are smooth sections of the tangent vector bundle so that the space $\Xi(M)$ is now viewed as the vector space of smooth sections $C^\infty(TM)$ of the tangent bundle TM.
When ϕ is a diffeomorphism, the pull-back $\phi^*\xi$ of a vector field is a section of the pull-back ϕ^*TM of the tangent bundle to M since $\phi^*\xi(\phi(m)) = D\phi(\xi(m))$.

Definition: A morphism of C^k vector bundles $\pi : E \to B$ and $\pi' : E' \to B'$ is a couple (f_0, f) of C^k morphisms $f_0 : B \to B'$ and $f : E \to E'$ such that $\pi' \circ f = f_0 \circ \pi$ and the induced map on the fibres $f_x : \pi^{-1}(x) \to (\pi')^{-1}(x)$ is a linear map. It is an isomorphism if it is invertible and if its inverse is a morphism of vector bundles
Here again, we often take $B = B'$ and $f_0 = Id$.
Let $k = 0$, in which case the vector bundles are topological vector bundles. An isomorphism preserves direct sums so that the Whitney sum induces a sum

$$[E]\dot{\oplus}[F] = [E \oplus F]$$

on the set

$$\mathrm{Vect}(B) := \{[E], \quad E \quad \text{vector} \quad \text{bundle} \quad \text{over} \quad B\}$$

of equivalence classes of vector bundles $\pi : E \to B$ modulo isomorphisms, turning it into an abelian semi-group. Let $[k]$ denote the equivalence class of a trivial bundle of rank k over B. Serre-Swan's

theorem implies the existence for any topological vector bundle $\pi_1 : E_1 \to B$ of a vector bundle $\pi_2 : E_2 \to B$ and a positive integer k such that $[E_1] \oplus [E_2] = [k]$.
The set:

$$K^0(B) := \{([E_1],[E_2])\}/_\sim; \quad \text{where} \quad ([E_1],[E_2]) \sim ([F_1],[F_2]) \Longleftrightarrow [E_1]\dot\oplus[F_2] = [F_1]\dot\oplus[E_2],$$

called the K^0-group of B inherits a group structure for the Whitney sum $\dot\oplus$. Morally, $([E_1],[E_2])$ stands for a difference $[E_1]\dot\ominus[E_2]$ and two topological vector bundles E and F define the same element in $K^0(B)$ if there is a vector bundle G such that $F \oplus G \simeq E \oplus G$.

2.2 Tensor, dual and morphism bundles

We refer the reader to the same references as the previous section.
The (topological) tensor product of two Banach spaces is built from their algebraic tensor product as follows.

Definition: Given two Banach vector spaces V_1 and V_2, the tensor product $V_1\hat\otimes V_2$ is the unique Banach vector space V such that the following map:

$$\begin{array}{rcl} \mathcal{L}(V,W) & \to & \mathcal{B}(V_1 \times V_2, W) \\ f & \mapsto & ((u_1,u_2) \mapsto f(u_1 \otimes u_2)) \end{array}$$

is continuous for any Banach space W. Here $\mathcal{B}(V_1 \times V_2, W)$ denotes the set of continuous bilinear forms on $V_1 \times V_2$ with values in W.

If $\|\cdot\|_i$ denotes the norm on V_i for $i = 1,2$ then $V_1\hat\otimes V_2$ coincides with the closure of the tensor product for the norm on V_i defined by:

$$\|v_1\hat\otimes v_2\| = \|v_1\|_1 \cdot \|v_2\|_2.$$

If both V_1 and V_2 are finite dimensional, then the tensor product $\hat\otimes$ coincides with the ordinary tensor product $\otimes$. In what follows we shall drop the explicit mention of the completion $\hat{}$.

Note that $K^0(B)$ for some manifold B, can be equipped with the induced tensor product on isomorphism classes which turns it into a ring.

Definition: Let $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ be two vector bundles of class C^k with fibres modelled on V_1 and V_2 respectively. The tensor product $\pi_1 \otimes \pi_2 : E_1 \otimes E_2 \to B$ is a vector bundle of class C^k modelled on $V_1 \otimes V_2$ with fibre $\pi_1^{-1}(b) \otimes \pi_2^{-1}(b)$ above $b \in B$ and the local trivializations of which are built from the tensor product of local trivializations (U_i, ϕ_i, τ_i^1), (U_i, ϕ_i, τ_i^2) and $(U_i, \phi_i, \tau_i^1 \otimes \tau_i^2)$.

Transition functions are given by tensor products $\tau_{ij}^1 \otimes \tau_{ij}^2$ where $\tau_{ij}^k, k = 1,2$ are transition maps for the bundles $E_k, k = 1,2$.
Whenever E_1 and E_2 have ranks d_1 and d_2, their tensor product has rank $d_1 d_2$.

Given a topological vector space V, the dual space V^* is the space of continuous linear forms on V.

Definition: Let $\pi : E \to B$ be a C^k vector bundle with fibres modelled on a Banach space V. The dual bundle $\pi^* : E^* \to B$ is a vector bundle of class C^k modelled on V^* with fibre $\left(\pi^{-1}(b)\right)^*$ above $b \in B$ and local trivializations $\left(U_i, \phi_i, \left(\tau_i^{-1}\right)^*\right)$ induced by some local trivialization (U_i, ϕ_i, τ_i) of E.

The transition maps are given by $\left(\tau_{ij}^{-1}\right)^*$, where the τ_{ij} are transition maps for E.

Combining duals and tensor products yields different types of bundles which are useful for geometric purposes. The homomorphism bundle is one of them:

Definition: Given two vector bundles $E \to B$ and $F \to B$, we can build the bundle $Hom(E, F) := E^* \otimes F$ of linear morphisms from E to F.

Also we shall use the notion of symmetrized and antisymmetrized tensor products of vector bundles:

Definition: Given vector bundles $E_1, \cdots, E_k$ based on some manifold B, we can build symmetric sections of their tensor product from sections $\sigma_1, \cdots, \sigma_k$ of $E_1, \cdots, E_k$:

$$\sigma_1 \otimes_s \sigma_2 \otimes_s \cdots \otimes_s \sigma_k := \frac{1}{k!} \sum_{\alpha \in \Sigma_k} \sigma_{\alpha(1)} \otimes \sigma_{\alpha(2)} \otimes \cdots \otimes \sigma_{\alpha(k)},$$

and similarly antisymmetric sections:

$$\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k := \frac{1}{k!} \sum_{\alpha \in \Sigma_k} (-1)^{sign(\alpha)} \sigma_{\alpha(1)} \otimes \sigma_{\alpha(2)} \otimes \cdots \otimes \sigma_{\alpha(k)}$$

where $sign(\alpha)$ is the signature of the permutation.

Another useful class of bundles is that of tensor bundles on a manifold:

Definition: Given a Banach manifold X of class C^k modelled on a Banach vector space E then:

- The dual bundle T^*X to the tangent bundle TX is a vector bundle called the *cotangent bundle*. It is a vector bundle of class C^{k-1} based on B and with fibres modelled on E^*. Its sections are called *cotangent vector fields*.

- The tensor bundle $TX^q := \otimes^q TX$, $q \in I\!N^*$ is a vector bundle of class C^{k-1} based on B and with fibres modelled on $\otimes^q E$. Its sections are called *contravariant q-tensor* fields.

- The tensor bundle $(T^*X)^p := \otimes^p T^*X$, $p \in I\!N^*$ is a vector bundle of class C^{k-1} based on B and with fibres modelled on $\otimes^p E^*$. Its sections are called *covariant p-tensor* fields.

- A (p, q) tensor field is a section of the bundle $(\otimes^q TX) \otimes (\otimes^p T^*X)$.

In finite dimensions, one often writes a (p, q) tensor T in local coordinates as $T_{i_1 \cdots i_p}^{j_1 \cdots j_q}$.
The p-th antisymetric power of the cotangent bundle denoted by

$$\Lambda^p TM := T^*M \wedge \cdots \wedge T^*M \quad (\mathrm{p-times})$$

is a vector bundle over M whose sections correspond to p-forms. If M is smooth they form the space $\Omega^p(M)$ of smooth p-forms on M.
Here we have set

$$V_1 \wedge \cdots \wedge V_p := \{v_1 \wedge \cdots \wedge v_p = \frac{1}{p!} \sum_{\tau \in \Sigma_p} (-1)^{|\tau|} v_{\tau(1)} \otimes \cdots v_{\tau(p)}, \quad v_i \in V_i,$$

Pull-backs can be extended to covariant tensor fields.

Given a morphism $\phi : X \to Y$ between two C^k manifolds X and Y, the pull-back by ϕ of a covariant p-tensor field T on Y is given by:

$$(\phi^* T)_x(U_1, \cdots, U_p) := T_{\phi(x)}(D_x\phi(U_1), \cdots, D_x\phi(U_p)) \quad \forall U_1, \cdots, U_p \in T_x X.$$

In particular, the pull-back of a p-form is also a p-form. It is easy to check that

$$\phi^*(T_1 \otimes T_2) = \phi^* T_1 \otimes \phi^* T_2$$

and that given two morphisms ϕ, ψ we have:

$$(\phi \otimes \psi)^* = \psi^* \otimes \phi^*.$$

If ϕ is a diffeomorphism, the pull-back can be extended to contravariant vector fields:

$$\phi^*(\xi_1 \otimes \cdots \otimes \xi_q) := (\phi^{-1})_* \xi_1 \otimes \cdots \otimes (\phi^{-1})_* \xi_q.$$

2.3 Examples of tensors: metrics and almost complex structures

Important examples of covariant tensor fields are the Riemannian (resp. Hermitian) metrics.

Definition: A weak (resp. strong) *Riemannian metric* on a smooth real vector bundle with fibres modelled on a Banach space and based on a manifold B, is a smooth section g of $E^* \otimes E^*$ such that for any $b \in B$, g_b induces a symmetric positive definite form on each fibre E_b, producing a weaker topology than the Banach topology on the fibre (respectively the same topology as the Banach topology on the fibre).

Definition: A weak (resp. strong) *Hermitian metric* on a smooth complex vector bundle with fibres modelled on a Banach space and based on a manifold B, is a smooth section h of $E^* \otimes E^*$ such that for any $b \in B$, h_b induces a Hermitian positive definite form on each fibre E_b, producing a weaker topology than the Banach topology on the fibre (respectively the same topology as the Banach topology on the fibre).

In the following when there is no other explicit mention, we shall be thinking of strong metrics.

A weak (resp. strong) Riemannian (Hermitian) metric on a Banach manifold is a weak (resp. strong) Riemannian (Hermitian) metric on the tangent bundle TB.

If M is finite dimensional, then weak and strong topologies coincide and one only requires that g_b (resp. h_b) be a positive definite symmetric (resp. Hermitian) form on the fibres. Given a metric on M of dimension n, a local orthonormal system of coordinates $(x_1, \cdots, x_n)$ around a point x is such that setting $e_i := \frac{\partial}{\partial x_i}$ we have $g_{ij}(x) := g_x(e_i, e_j) = \delta_{ij}$, i.e. the matrix representing g_x in this coordinate system is the identity matrix.

Given a smooth map $\phi : N \to M$ between two manifolds and a Riemannian (resp. Hermitian) metric g (resp. h) on a vector bundle based on M, the pull-back $\phi^* g$ (resp. $\phi^* h$) yields a Riemannian (resp. Hermitian) metric on the pull-back vector bundle $\phi^* E$ based on N.
In particular, if ξ is a vector field on a Riemannian manifold (M, g) the local one parameter group of diffeomorphisms ϕ_t generated by ξ acts on the metric by pull-back $\phi_t^* g$. A *Killing vector field* also

called an infinitesimal isometry, is a vector field ξ such that the Lie derivative of the metric in the direction ξ vanishes, i.e.:

$$L_\xi g := \left(\frac{d}{dt}\right)_{t=0} \phi_t^* g = 0.$$

If $\xi, \tilde{\xi}$ are two Killing vector fields, then so is their bracket $[\xi, \tilde{\xi}]$.

A vector bundle equipped with a (strong) Riemannian (resp. Hermitian) metric is called a *Riemannian* (resp. *Hermitian*) vector bundle. A manifold M such that TM is equipped with a (strong) Riemannian (resp. Hermitian) metric is called a Riemannian (resp. Hermitian) manifold.

Notice that a Banach vector bundle equipped with a strong Riemannian metric becomes a Hilbert bundle since the fibres become Hilbert spaces when equipped with the inner product induced by the metric. This is of course not the case anymore if the Riemannian structure is weak.

Metrics do not always exist on a manifold; however, provided there is a smooth partition of unity on the manifold, one can always build a Riemannian metric patching up locally defined positive definite forms. Also, if M is a Riemannian manifold, tensor bundles over M can be equipped with a metric structure induced from that of M.

The existence of a Riemannian metric on a manifold M provides explicit isomorphisms between the tangent and cotangent vector fields called *musical isomorphisms:*

$$\begin{array}{rcl} T_xM & \to & T_x^*M \\ V & \mapsto & V^\flat \end{array}$$

defined by

$$V^\flat(W) = \langle V, W\rangle_x, \quad \forall W \in T_xM$$

where $\langle\cdot,\cdot\rangle_x$ is the scalar product on the fibre T_xM of the tangent bundle above $x \in M$ induced by the Riemannian metric. Similarly, using the Riesz theorem, one defines:

$$\begin{array}{rcl} T_x^*M & \to & T_xM \\ \alpha & \mapsto & \alpha^\sharp \end{array}$$

by

$$\alpha(W) = \langle \alpha^\sharp, W\rangle_x, \quad \forall W \in T_xM.$$

Using these musical isomorphisms, we can equip tensor bundles with Riemannian structures:

$$\langle T, T'\rangle_x = g^{i_1 j_1} \cdots g^{i_p j_p}\, g_{\mu_1\nu_1} \cdots g_{\mu_q\nu_q}\, T^{\mu_1\cdots\mu_q}_{i_1\ldots i_p} T'^{\nu_1\cdots\nu_q}_{j_1\ldots j_p} \quad \forall T \in (TM)^{\otimes q} \otimes (T^*M)^{\otimes p}.$$

Definition: An *almost complex structure* on an oriented Banach vector bundle $\pi : E \to B$ is a smooth section J of $E^* \otimes E$ which induces a morphism, also denoted by J, preserving orientation and such that $J^2 = -Id$. An *almost complex* structure on an oriented manifold M is one on the tangent bundle TM, i.e. it is a $(1,1)$ tensor J inducing a morphism J on TM which preserves orientation and satisfies $J^2 = -Id$.

An almost complex structure J on a real vector bundle E extends linearly to its complexification $J^{\mathbb{C}} : E^{\mathbb{C}} \to E^{\mathbb{C}}$. This complexified vector bundle splits $E^{\mathbb{C}} = E^{1,0} \oplus E^{0,1}$ where $E^{1,0}$ is the vector bundle over B with fibre $Ker(J(b) - i) := \{u_b \in E_b^{\mathbb{C}}, J(b)(u_b) = iu_b\}$ above $b \in B$, resp. $E^{0,1}$ the vector bundle over B with fibre $Ker(J(b)+i) := \{v_b \in E_b^{\mathbb{C}}, J(b)(v_b) = -iv_b\}$ above $b \in B$.

Definition: Let M be a manifold equipped with an almost complex structure J which induces a splitting $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$. If $T^{1,0}M$ is stable under brackets of vector fields, then J is said to be *integrable*.

Proposition: An almost complex structure J on M is integrable if and only if the *Nuijenhuis tensor* field $N : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ defined by

$$N(U,V) = [U,V] + J[JU,V] + J[U,JV] - [JU,JV]$$

vanishes for any $U \in C^\infty(TM), V \in C^\infty(TM)$.

*Proof:*Extending the Nuijenhuis tensor to complex vector fields, $W = U + iV, Z = X + iY$, we can write:

$$N(W,Z) = N(U,X) - N(V,Y) + i\,(N(V,X) + N(U,Y))\,.$$

Thus if N vanishes on real vector fields, it also vanishes on complex vector fields. Assume that $W, Z \in C^\infty(T^{1,0}M)$. Then $JW = iW$ and $JZ = iZ$ so that $N(W,Z) = 2\,([W,Z] + iJ[W,Z])$. Hence $N(W,Z) = 0 \Rightarrow J[W,Z] = -i[W,Z]$, i.e. $[W,Z] \in C^\infty(T^{1,0}M)$. It follows that J is integrable.

Conversely, let us write $W = W^+ + W^-$ and $Z = Z^+ + Z^-$ according to the splitting $C^\infty(T^{\mathbb{C}}M) = C^\infty(T^{1,0}M) \oplus C^\infty(T^{0,1}M)$. Then

$$N(W,Z) = N(W^+,Z^+) - N(W^-,Z^-) + i\left(N(Z^-,W^+) + N(W^+,Z^-)\right).$$

Since $JW^+ = iW^+$, $JW^- = -iW^-$, $JZ^+ = iZ^+$, $JZ^- = -iZ^-$, it follows that $N(W^+,Z^-) = N(W^-,Z^+) = 0$ and $N(W^+,Z^+) = N(W^-,Z^-) = 0$ so that N finally vanishes on all complex tangent fields. ∎

Definition: A *complex manifold* is a manifold M equipped with a complex structure, i.ewith an atlas (U_i, ϕ_i) with transition maps given by holomorphic maps. A complex structure on a manifold induces an almost complex structure from the local charts.

Conversely we have:

Theorem (Newlander and Nirenberg) Let M be an (even dimensional) real manifold equipped with an almost complex structure J. If J is integrable, it yields a *complex structure* on the manifold with associated almost complex structure J.

If M and N are two complex manifolds, a map $f : M \to N$ is called *holomorphic* if it is holomorphic in any local chart, this requirement being independent of the choice of local chart since the transition maps are holomorphic.

Let M be a real even dimensional manifold M equipped with a complex structure c. Given another real even dimensional manifold N, a smooth map $f : N \to M$ induces a complex structure $f^*c := \{f^{-1}(U_i), \phi_i \circ f\}$ on N called the *pull-back* of c by f. If $N = M$ and if f is a diffeomorphism of M, f^*c is a priori different from the initial complex structure c in the sense that the charts are not only different from the initial ones but also incompatible with them. Yet (M,c) and (M, f^*c) are holomorphically equivalent in the sense that $f : (M,c) \to (M, f^*c)$ is a holomorphic map and so is its inverse.

Let us comment on the finite dimensional case. Letting $\{z_k := x_k + iy_k, k = 1, \cdots, n\}$ be a system of local coordinates on the complex manifold M, we can set

$$J\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k}, \quad J\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k}.$$

This defines a $(1,1)$ tensor on M independently of the choice of local coordinates. Indeed, given another system of local coordinates $\{z'_k := x'_k + iy'_k, k = 1, \cdots, n\}$, the Cauchy-Riemann equations

$$\frac{\partial x_i}{\partial x'_j} = \frac{\partial y_i}{\partial y'_j}, \quad \frac{\partial x_i}{\partial y'_j} = \frac{\partial y_i}{\partial x'_j}$$

lead to a similar expression

$$J\left(\frac{\partial}{\partial x'_k}\right) = \frac{\partial}{\partial y'_k}, \quad J\left(\frac{\partial}{\partial y'_k}\right) = -\frac{\partial}{\partial x'_k}$$

so that we obtain an almost complex structure J on M.

A Riemannian metric g combined with an almost complex structure yields a Hermitian metric:

$$h(\sigma, \rho) := g(\sigma, J\rho).$$

2.4 Differential forms

Useful references are [6], [38].

Given a smooth manifold M, we previously introduced the space $\Omega^p(M)$ of p-valued forms on M. This generalises to E-valued p-forms on a manifold B, with E a smooth fibre bundle over a smooth manifold B.

Definition: Let $\pi : E \to B$ be a smooth fibre bundle. An E-valued *p-form* α on B is a smooth section of the tensor product $\otimes^p T^*B \otimes E$ such that:

$$\alpha\left(U_{\sigma(1)}, \cdots, U_{\sigma(p)}\right) = (-1)^{\varepsilon(\sigma)} \alpha(U_1, \cdots, U_p) \quad \forall U_1, \cdots, U_p \in T_bB \quad \forall \sigma \in \Sigma_p$$

where $\varepsilon(\sigma)$ is the signature of σ.
In particular, such an expression vanishes whenever two vectors U_i and U_j coincide so that if the manifold B is n-dimensional, using the multilinearity property, one can show that a p-form with $p > n$ vanishes identically.

We denote by $\Omega^p(E)$ the space of smooth E-valued p-forms and $\Omega(E) := \oplus_{p=0}\Omega^p(E)$, which becomes a finite sum when B is finite dimensional. For $p = 0$ we get back the space of smooth sections of E. The degree of a p-form α is the integer p also denoted by $|\alpha|$.

If E is the trivial vector bundle $E = B \times \mathbb{R}$ (or $B \times \mathbb{C}$) we get back the space $\Omega^p(B)$ of p-forms on B, and set $\Omega(B) := \oplus_{p \in \mathbb{N}} \Omega^p(B)$ which is a finite sum as soon as B is finite dimensional. Given a local system of coordinates $(x_1, \cdots, x_n)$ around a point x of an n-dimensional manifold, a one form $\alpha(x)$ reads $\alpha(x) := \sum_{i=1}^n \alpha_i(x) dx^i$.

Whenever $\mathcal{A}$ is a fibration of algebras, the space $\Omega(\mathcal{A})$ can be equipped with the *exterior product* or *wedge product* which sends $\alpha \in \Omega^p(\mathcal{A})$ and $\beta \in \Omega^q(\mathcal{A})$ to $\alpha \wedge \beta \in \Omega^{p+q}(\mathcal{A})$:

$$\begin{aligned}
&(\alpha \wedge \beta)(U_1, \cdots, U_p, U_{p+1}, \cdots, U_{p+q}) \\
&:= \frac{1}{p!\, q!} \sum_{\sigma \in \Sigma_{p+q}} (-1)^{sign(\sigma)} \alpha(U_{\sigma(1)}, \cdots, U_{\sigma(p)}) \cdot \beta(U_{\sigma(p+1)}, \cdots, U_{\sigma(p+q)}).
\end{aligned}$$

In particular, for two forms α and β and two vector fields U, V we have $\alpha \wedge \beta(U, V) = \alpha(U) \cdot \beta(V) - \alpha(V) \cdot \beta(U)$. Here the dot denotes the product of sections of $\mathcal{A}$. Thus $\Omega(\mathcal{A})$ becomes a graded algebra with the grading given by the degree of forms. here are two important examples:

- Starting from the bundle $E = B \times K$ where K is a field, yields a graded algebra structure on $\Omega(B, K)$ using the product on K.
- Starting from a vector bundle E based on B, the bundle $\mathcal{A} = Hom(E)$ yields a fibration of algebras on B and $\Omega(Hom(E))$ can be equipped with a graded algebra structure using the composition of homomorphisms.

We introduce two operators on forms which are useful to construct a Clifford multiplication on forms later in these notes.

- Given a Riemannian manifold M, the exterior multiplication $\varepsilon(V) : \Omega^*(M) \to \Omega^{*+1}(M)$ is the operator defined by:
$$\varepsilon(V)\alpha = V^\sharp \wedge \alpha$$
where $V^\sharp$ is the 1-form associated to the vector field V by the musical isomorphisms.
- Given a fibration of algebras $\mathcal{A}$ and a vector field V on B, the *contraction* operator $i(V)$: $\Omega^*(\mathcal{A}) \to \Omega^{*-1}(\mathcal{A})$ is the unique operator such that:
$$i(V)\alpha = \alpha(V) \quad \forall \alpha \in \Omega^1(\mathcal{A})$$
and
$$i(V)(\alpha \wedge \beta) = i(V)\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge i(V)\beta \quad \forall \beta \in \Omega(\mathcal{A}).$$

On a smooth oriented closed n-dimensional manifold M, any smooth n-form ω can be integrated to give a complex (or real) number $\int_M \omega$. Given a smooth map $f : N \to M$ between two closed oriented n-dimensional smooth manifolds N and M, the pull-back $f^*\omega$ by f of this form can be integrated on N and we have:

$$\int_N f^*\omega = \deg(f) \cdot \int_M \omega$$

where $\deg(f)$ is an integer called the *degree* of the map f.
The bilinear map:

$$\begin{array}{rcl}
\Omega^p(M) \times \Omega^{n-p}(M) & \to & \mathbb{C} \\
(\alpha, \beta) & \mapsto & \int_M \alpha \wedge \beta
\end{array}$$

will later yield a dual pairing between p-th and $n-p$-th cohomology groups. Notice that when $n = 2k$, for two k-forms α and β we have $\int_M \alpha \wedge \beta = (-1)^{\frac{k}{2}} \int \beta \wedge \alpha$, so that this bilinear map yields a symmetric bilinear form on $\Omega^k(M)$whenever k is even.

A Riemannian structure on a finite n-dimensional oriented manifold M yields a particular n-form, the *volume form* given in a local system of coordinates $(x_1, \cdots, x_n)$ at a point x by:

$$dvol(x) = \sqrt{\det g_x}\, dx_1 \wedge \cdots \wedge dx_n = e_1^* \wedge \cdots \wedge e_n^*$$

where $\det g_x$ is the positive determinant of the matrices representing the metric locally at point x and where $\{e_1^*(x), \cdots, e_n^*(x)\}$ is an orthonormal basis of T_x^*M equipped with the inner product induced by the Riemannian metric.
Given an n-dimensional Riemannian manifold (M, g), the *Hodge star* operator is defined pointwise as the linear operator

$$* : \Lambda^p T_x^* M \to \Lambda^{n-p} T_x^* M$$

on a positively oriented orthonormal local basis $\{e_1^*, \cdots, e_n^*\}$ of T_x^*M by:

$$e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \wedge \star(e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*) = dvol(x)$$

for any $i_1 < \cdots < i_p$. This definition is independent of the choice of oriented orthonormal basis and one can check that $\star^2 = (-1)^{p(n-p)}$ on $\Lambda^p T_x^* M$.
The Hodge $\star$ operator induces a duality on forms $\Omega^p(M) \simeq \Omega^{n-p}(M)$ called *Hodge duality.* When M is closed, the above bilinear form on differential forms yields the following bilinear form on $\Omega^p(M)$:

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha, \beta \rangle_x dvol(x) = \int_M \alpha(x) \wedge \star\beta(x).$$

When M is a *complex manifold,* just as $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$, the complexified space of forms $\Omega^r(M) \otimes \mathbb{C}$ splits:

$$\Omega^r(M) \otimes \mathbb{C} = \sum_{p+q}^{r} \Omega^{p,q}(M),$$

where $\Omega^{p,q}(M)$ is the space of smooth antisymmetric sections of the tensor bundle $\left((T^{1,0}M)^*\right)^{\otimes p} \otimes \left((T^{0,1}M)^*\right)^{\otimes q}$.

A Hermitian metric h on M is a $(1,1)$ covariant two tensor which, if M is n-dimensional reads in local coordinates:

$$h(z) = \sum_{1 \leq i,j \leq n} h_{jk} dz_j \otimes d\bar{z}_k.$$

There is a 2-form associated to it called the *fundamental* $(1,1)$ *form*:

$$\omega(z) := -\mathrm{Im} h = \frac{i}{2} \sum_{1 \leq i,j \leq n} h_{jk} dz_j \wedge d\bar{z}_k$$

which plays an important role for Kähler structures we shall come across later in these notes.

2.5 De Rham and Dolbeault cohomology

Useful references are [6], [38].

Exterior differentiation on forms on a given smooth manifold M is defined as follows:

Definition: The derivation $f \to Df$ defined on the space of smooth functions on a manifold M extends to a unique linear map $d : \Omega(M) \to \Omega(M)$ such that

i) d sends $\Omega^p(M)$ to $\Omega^{p+1}(M)$,

ii) d is a graded derivation i.e. it satisfies the (graded) *Leibniz rule*:

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta \quad \forall \alpha, \beta \in \Omega(M),$$

iii) $(d \circ d)\, f = 0 \quad \forall f \in C^\infty(M)$.

We set $df = Df$ for $f \in C^\infty(M) = \Omega^0(M)$.

For smooth vector fields $U_0, \cdots, U_p$ on M, the exterior differentiation reads:

$$\begin{aligned} d\alpha(U_0, \cdots, U_p) &= \sum_{k=0}^{p} (-1)^k U_i \left(\alpha(U_0, \cdots, \hat{U}_k, \cdots, U_p) \right) \\ &+ \sum_{1 \le k, l \le p} (-1)^{k+l} \alpha \left([U_k, U_l], U_0, \cdots, \hat{U}_k, \cdots, \hat{U}_l, \cdots, U_p \right) \end{aligned}$$

where the "hat" above the vector fields means we have deleted them.
In local coordinates around a point x of an n-dimensional manifold M, the exterior differentiation on a p form $\alpha(x) = \alpha_{i_1 \cdots i_p}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ reads:

$$d\,(\alpha)\,(x) = \sum_{k=1}^{n} \partial_{x_k} \alpha_{i_1 \cdots i_p}(x)\, dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Remark: As can be seen from the above explicit descriptions of d, the requirement that $d \circ d$ vanishes on functions actually implies (using the Leibniz rule and the fact that d coincides on functions with D) that it vanishes on all forms i.e.

$$d \circ d(\alpha) = 0 \quad \forall \quad \alpha \in \Omega(M).$$

On a compact finite dimensional oriented Riemannian manifold, one can define the adjoint d^* of d setting:

$$\langle d\alpha, \beta \rangle = \langle \alpha, d^* \beta \rangle \quad \forall \alpha \in \Omega^p(M), \forall \beta \in \Omega^{p+1}(M),$$

so that by Stokes' formula we have:

$$\begin{aligned} \langle \alpha, d^* \beta \rangle &= \int_M \alpha \wedge d^* \beta \\ &= \int_M d\alpha \wedge \star \beta \\ &= \int_M d(\alpha \wedge \star \beta) - (-1)^p \int_M \alpha \wedge d * \beta \\ &= (-1)^{p+1} \int_M \alpha \wedge d \star \beta \\ & \quad \left(\text{since} \int_M d\gamma = 0 \right) \\ &= (-1)^{p+1} (-1)^{(n-p)(n-(n-p))} \int_M \alpha \wedge \star \star d * \beta \\ &= (-1)^{np+1} \int_M \alpha \wedge \star (\star d * \beta) \\ &= (-1)^{np+1} \langle \alpha, \star d \star \beta \rangle \end{aligned}$$

Thus on $\Omega^p(M)$ we get:

$$d^* = (-1)^{np+1} \star d \star.$$

Combining d and d^* yields a *Laplacian on forms* given by

$$\Delta = d^* \circ d + d \circ d^*$$

which restricts to operators $\Delta_p = d_p^* \circ d_p + d_{p-1} \circ d_{p-1}^*$ on each $\Omega^p(M)$ where $d_p : \Omega^p(M) \to \Omega^{p+1}(M)$.

A form α is *closed* whenever $d\alpha = 0$, and *exact* whenever there is a form β such that $\alpha = d\beta$. Since $d \circ d = 0$, exact forms are closed but closed forms are not expected to be exact, they are only locally exact by the Poincaré lemma. The obstruction to their global exactness is measured by the de Rham cohomology groups:

$$H^p(M) := \mathrm{Ker}(d|_{\Omega^p(M)})/\mathrm{R}(d|_{\Omega^{p-1}(M)})$$

where $\mathrm{R}(d|_{\Omega^{p-1}(M)})$ denotes the range of the map $d|_{\Omega^{p-1}(M)}$. The theory of elliptic operators on closed manifolds which we describe later in these notes shows that these cohomology groups are finite dimensional. The dimension of $H^p(M)$ is called the *Betti number* of M.
If M has dimension n, the sequence

$$0 \to \Omega^0(M) \to^d \cdots \Omega^k(M) \to^d \Omega^{k+1}(M) \to^d \cdots \to^d \Omega^n(M) \to^d 0$$

defines a differentiable complex and the cohomology measures in how far this complex is not exact. When M is a complex manifold, the exterior differentiation splits $d = \partial + \bar\partial$ where $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\bar\partial : \Omega^{p,q} \to \Omega^{p,q+1}$ and it follows from the relation $d \circ d = 0$ that $\delta \circ \bar\delta + \bar\delta \circ \partial = 0$, $\partial^2 = 0$, $\bar\partial^2 = 0$. Since $\bar\partial^2 = 0$,a $\bar\partial$-exact form α (i.e.$\alpha = \bar\partial\beta$) is $\bar\partial$-closed (i.e. $\bar\partial\alpha = 0$) and there is an associated complex

$$0 \to \Omega^{0,0}(M) \to \Omega^{0,1}(M) \to \Omega^{0,2}(M) \to \cdots,$$

the *Dolbeault complex*. A $\bar\partial$-closed form is however generally not $\bar\delta$-exact and the obstruction to the extacness of closed forms is measured by the *Dolbeaut cohomology* groups:

$$H^{p,q}(M) := \mathrm{Ker}(\bar\partial|_{\Omega^{p,q}(M)})/\mathrm{R}(\bar\partial|_{\Omega^{p,q-1}(M)})$$

where $\mathrm{R}(d|_{\Omega^{p,q-1}(M)})$ denotes the range of the map $\bar\partial|_{\Omega^{p,q-1}(M)}$. The *Hodge decomposition theorem* gives a relation between the de Rham and the Dolbeault cohomology groups:

$$H^k(M) = \oplus_{p+q=k} H^{p,q}(M).$$

Using the theory of elliptic operators on closed manifolds one can show that these cohomology groups are finite dimensional; their dimensions are called the *Hodge numbers* $h^{p,q} := \mathrm{dim} H^{p,q}(M)$.

From the Hodge decomposition theorem it follows that the Betti numbers relate to the Hodge numbers as follows:

$$b_k = \sum_{p+q=k} h^{p,q}.$$

Moreover, since $h^{p,q} = h^{q,p}$ (by Hodge symmetry), this relation shows that the Betti numbers on a closed Kähler manifold are always even. There are many more interesting properties of these Hodge numbers which we cannot discuss here and refer the reader to the litterature.

A *Kählerian* manifold is a complex manifold M which can be equipped with a positive definite $(1,1)$ form ω called a *Kählerian metric* which is closed, i.e. such that $d\omega = 0$.
Example: The projective space $P_n(\mathbb{C})$ has a natural Kählerian metric called the *Fubini Study* metric defined by:

$$\pi^*\omega = \frac{1}{2\pi}\partial\bar\partial \log\left(|\eta_0|^2 + |\eta_1|^2 + \cdots + |\eta_n|^2\right)$$

where the $\eta_i, i = 1, \cdots, n$ are the coordinates on $\mathbb{C}^{n+1}$ and where $\pi : \mathbb{C}^{n+1}/\{0\} \to P_n(\mathbb{C})$ is the canonical projection. Let $z = (\frac{\zeta_1}{\zeta_0}, \cdots, \frac{\zeta_n}{\zeta_0})$ be the homogeneous coordinates of the chart $\mathbb{C}^n \subset P_n(\mathbb{C})$ then:

$$\omega = \frac{1}{2pi} \partial\bar{\partial} \log(1 + |z|^2).$$

Using the Hodge decomposition theorem, on a closed kählerian manifold M, one can relate the de Rham cohomology groups to the Dolbeault cohomology groups by:

$$H^k(M) = \oplus_{p+q=k} H^{p,q}(M).$$

2.6 Covariant derivatives and geodesics

Useful references are [16], [24], [25], [27], [30].

Covariant derivatives extend the exterior differentiation to sections of vector bundles.

Definition: Given a vector bundle $\pi : E \to B$ based on a manifold B, a covariant derivative (also abusively called connection) on E is a differential operator:

$$\nabla : C^\infty(E) \to C^\infty(T^*B \otimes E)$$

which satisfies the *Leibniz rule*:

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma.$$

It extends in a unique way to the space $\Omega(B, E)$ of E-valued forms on B:

$$\nabla(\alpha \wedge \theta) := d\alpha \wedge \theta + (-1)^{|\alpha|}\alpha \wedge \nabla\theta \quad \forall \alpha \in \Omega(B), \beta \in \Omega(B, E).$$

Notice that

$$\nabla_{fU}\sigma = f\nabla_U\sigma$$

and

$$\nabla_{U+V}\sigma = \nabla_U\sigma + \nabla_V\sigma \quad \forall \sigma \in C^\infty(E), f \in \Omega^0(B), U, V \in TB.$$

A covariant derivation ∇ on a vector bundle E induces a *dual connection* ∇^* on the dual bundle E^*, given by the Leibniz rule using the duality product:

$$d\langle\sigma, \rho\rangle = \langle\nabla\sigma, \rho\rangle + \langle\sigma, \nabla^*\rho\rangle, \quad \forall \sigma, \rho \in C^\infty(E),$$

and a connection ∇^{Hom} on the bundle $Hom(E) \simeq E^* \otimes E$ defined by:

$$\nabla^{Hom} := \nabla^* \otimes 1 + 1 \otimes \nabla.$$

For a trivial vector bundle $E \to B$, any connection is given by an $Hom(E)$-valued one form θ via the formula $\nabla = d + \theta$. As a consequence, a connection on a general vector bundle can locally be described by $\nabla = d + \theta_U$ where now θ_U is a $Hom(E)$ valued one form on an open subset $U \in B$ over which we have trivialised the bundle. Another consequence is that two connections on E differ by a (globally defined) $Hom(E)$-valued one form on B. An easy computation yields that if $\nabla = d + \theta_U$ locally, then $\nabla^* = d - \theta_U$ and $\nabla^{Hom} = d + [\theta_U, \cdot]$.

A similar formula to that of the differentiation on ordinary forms holds for a covariant derivative on E-valued forms $\alpha \in \Omega^p(B, E)$:

$$\begin{aligned} \nabla\alpha(U_0, \cdots, U_p) &= \sum_{k=0}^{p}(-1)^k \nabla_{U_i}\left(\alpha(U_0, \cdots, \hat{U}_i, \cdots, U_k)\right) \\ &+ \sum_{0 \leq k < l \leq p} (-1)^{k+l}\alpha([U_k, U_l], U_0, \cdots, \hat{U}_k, \cdots, \hat{U}_l, \cdots, U_p) \end{aligned}$$

where $\hat{U}_i$ means that we have left out the vector field U_i. In particular, for $p=1$ and $\alpha \in \Omega^1(B)$ we have:

$$\nabla\alpha(U,V) = \nabla_U V - \nabla_V U - \nabla_{[U,V]} \quad \forall U,V \in C^\infty(B,TB).$$

A connection ∇ on a Riemannian (resp. Hermitian) bundle $\pi : E \to B$ based on a manifold B is *Riemannian* provided it is compatible with the Riemannian (resp. Hermitian) metric in the following sense:

$$d\langle\sigma,\rho\rangle_b = \langle\nabla\sigma,\rho\rangle_b + \langle\sigma,\nabla\rho\rangle_b \quad \forall\sigma,\rho \in C^\infty(E) \quad \forall b \in B$$

where $\langle\cdot,\cdot\rangle_b$ is the inner product on the fibre above b.

Given a connection ∇ on a finite n-dimensional manifold M, and given a local system of coordinates $(x_1,\cdots,x_n)$ at a point $x \in M$, we define the *Christoffel symbols:*

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

These extend to the Banach setting as follows. Let $\pi : E \to B$ be a vector bundle with base B modelled on a linear Banach space V and fibre modelled on a linear Banach space V_1. Let (U,ϕ,Φ) be a local trivialization of the bundle E over an open subset U of B. A Christoffel coefficient corresponding to this trivialization is given by a map:

$$\Gamma_\Phi : \Phi(\pi^{-1}(U)) \to \mathcal{L}(V \times V_1, V_1)$$

with the following property. If (W,ψ,Ψ) is another trivialization then

$$D(\Psi\circ\Phi^{-1})\Gamma_\Phi(\phi_* X, \tau\sigma) = D^2((\Psi\circ\Phi^{-1})(\phi_* X, \Phi\sigma) + \Gamma_\psi \circ (D(\psi\circ\phi^{-1})X, D(\Psi\circ\Phi^{-1})$$

where X is a vector at a point of $U \cap W$ and σ a section of E. Under this assumption, it makes sense to define a connection $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ in a local trivialization (U,ϕ,Φ) using the Christoffel symbol Γ_Φ by:

$$\Phi(\nabla_X\sigma) = D(\Phi\sigma).\phi_* X + \Gamma_\Phi(\phi_* X, \Phi\sigma)$$

since the latter definition is independent of the choice of local trivialization.

Definition: The *torsion* of a connection on the tangent bundle TM to a manifold M is given by:

$$T(U,V) := \nabla_U V - \nabla_V U - [U,V] \quad \forall U,V \in C^\infty(M,TM).$$

In a system of local coordinates $(x_1,\cdots,x_n)$ around a point x of a finite n-dimensional manifold M, setting $e_i = \frac{\partial}{\partial x_i}$ we have $T(e_i,e_j) = \nabla_{e_i}e_j - \nabla_{e_j}e_i$ so that if the torsion vanishes then $\nabla_{e_i}e_j = \nabla_{e_j}e_i$, i.e. $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Notice that whern $E = TM$ is the tangent bundle, in the absence of torsion, i.e. when $T=0$ the covariant derivative on forms $\alpha \in \Omega^p(M)$ reads:

$$\begin{aligned}\nabla\alpha(U_0,\cdots,U_p) &= \sum_{k=0}^p (-1)^k \nabla_{U_i}\left(\alpha(U_0,\cdots,\hat{U}_i,\cdots,U_k)\right) \\ &+ \sum_{0\leq k<l\leq p}(-1)^{k+l}\alpha(\nabla_{U_k}U_l - \nabla_{U_l}U_k, U_0,\cdots,\hat{U}_k,\cdots,\hat{U}_l,\cdots,U_p).\end{aligned}$$

Proposition-Definition: There is a unique connection on a Riemannian manifold which has vanishing torsion and is compatible with the (strong) Riemannian metric; it is called the *Levi-Civita connection.*
Idea of proof: We first write

$$U\langle V,W\rangle = d\langle V,W\rangle(U) = \langle\nabla_U V,W\rangle + \langle V,\nabla_U W\rangle$$

as well as circular combinations of this expression. Using the fact that the torsion vanishes yields the following expression of $\langle\nabla_U V,W\rangle$:

$$\begin{aligned} 2\langle\nabla_U V,W\rangle &= \langle[U,V],W\rangle - \langle[V,W],U\rangle + \langle[W,U],V\rangle \\ &+ U\langle V,W\rangle + V\langle W,U\rangle - W\langle U,V\rangle. \end{aligned}$$

in terms of differentials of the inner product $\langle U,V\rangle$, $\langle V,W\rangle$ and $\langle U,W\rangle$. The existence and uniqueness of $\nabla_U V$ then follows from Riesz's theorem. ■
A torsion free connection relates to the exterior differentation:

Proposition: If ∇ is a torsion free connection on M then the exterior differential coincides with $\varepsilon\circ\nabla$ where ε is the exterior multiplication. In particular, if ∇ is the Levi-Civita connection on a Riemannian manifold M, then $d = \varepsilon\circ\nabla$.

Idea of the proof: Setting $\tilde{d} = \varepsilon\circ\nabla$ one proves that $\tilde{d}^2 f = -\langle T, df\rangle$ for any smooth function f on M where T is the torsion. Since the torsion vanishes by assumption, this will prove that $\tilde{d}^2 f = 0$. One is then left to check the Leibniz property for $\tilde{d}$ and the fact that it coincides with the ordinary differentiation on smooth functions in order to conclude that it coincides withg $\tilde{d}$ on all differential forms. ■

A Hermitian complex manifold (M,h) can be equipped with a Riemannian metric $g(\cdot,\cdot) := h(\cdot, J\cdot)$ where J is the almost complex structure on M induced by the complex structure.
Proposition: A Hermitian complex manifold M is *Kählerian* provided the bundles $T^{1,0}M$ and $T^{0,1}M$ are preserved by the Levi-Civita connection ∇, or equivalently provided the Levi-Civita connection ∇ is compatible with the complex structure J i.e. $[\nabla, J] = 0$.

Idea or the Proof: Recall that if M is a complex manifold with a Hermitian metric h, the real part of h restricted to the tangent bundle TM is a Riemannian metric g on M, while the imaginary part ω restricted to TM is a two form on M. For any two vector fields on M, we have $g(U,V) = \omega(JU,V)$ where J is the almost complex structure on M. Letting $\langle\cdot,\cdot\rangle$ denote the Riemannian scalar product, we have:

$$\begin{aligned} \langle(\nabla_U J)V,W\rangle &= \langle\nabla_U(JV),W\rangle + \omega(\nabla_U V,W\rangle \\ &= -U\omega(V,W) + \omega(\nabla_U V,W) + \omega(V,\nabla_V W) \\ &= -(\nabla_U\omega)(V,W) \end{aligned}$$

Since ∇ is torsion free, $d = \varepsilon\circ\nabla$ where ε is the exterior product and $d\omega(U,V,W) = (\nabla_U)\omega(V,W) - (\nabla_V)(W,U) + (\nabla_W)(U,V)$, which vanishes as a consequence of the condition $\nabla J = 0$. Hence $\nabla J \Rightarrow d\omega = 0$. ■
On the other hand, the formula for the Levi-Civita connection applied to the holomorphic coordinate system z_i yields:

$$\begin{aligned} 2\langle\nabla_{\partial_{z^j}}\partial_{z^k},\partial_{z^l}\rangle &= 0 \\ 2\langle\nabla_{\partial_{\bar{z}^j}}\partial_{z^k},\partial_{z^l}\rangle &= i\,d\omega(\partial_{z^k},\partial_{z^l},\partial_{\bar{z}^j}). \end{aligned}$$

from which it follows that if $d\omega = 0$, then the Levi-Civita connection preserves $T^{1,0}M$.

Definition: A *geodesic* on a Banach Riemannian manifold (M, g) is a smooth curve $c : I \to M$ on M solution of the second order differential equation:

$$\nabla_{\dot{c}(t)}\dot{c}(t) = 0$$

where I is some open interval in $\mathbb{R}$.

Such a solution exists locally by the theory of differential equations on Banach spaces and there is a unique solution $c_{x,u}$ determined by the initial conditions $c(0) = x, \quad \dot{c}(0) = u \in T_xM$ provided $0 \in I$.
Geodesics correspond to critical points of the energy functional on curves

$$E(c) = \int_I \|\dot{c}(t)\|^2 \, dt = \int_I g\left(\dot{c}(t), \dot{c}(t)\right) \, dt$$

and correspond to curves with minimal length. On $\mathbb{R}^d$ equipped with the Euclidean metric, they correspond to straight lines.

Choosing u in a small enough neighborhood of 0 ensures the existence of the geodesic up to time 1 and we define the *exponential map:*

$$\begin{array}{rcl} \exp : U \subset T_xM & \to & M \\ u & \mapsto & c_{x,u}(1) \end{array}$$

which yields, by the local inverse map theorem (see section 1.1), a local diffeomorphism from U onto its range. The *injectivity radius* at a point in M is the largest radius ρ for which the exponential is a diffeomorphism onto the ball of radius ρ centered at that point.

The Riemannian manifold is *complete* provided all geodesics are defined on $\mathbb{R}$, in which case the exponential map is defined on the whole tangent bundle. A compact Riemannian manifold is complete.

The exponential map defined on Lie groups can in some cases be described as an exponential map built from geodesics, choosing an adapted left invariant metric on the group, e.g. on $GL(n, \mathbb{R})$ the one given by the inner product $\langle A, B\rangle := \mathrm{tr}(A^tB)$ on $gl(n, \mathbb{R})$.

2.7 The curvature and characteristic classes

Useful references are [6], [31], [37], [39], [48].

Definition: The *curvature* of a covariant derivation is given by the $Hom(E)$-valued two form

$$\Omega^E = \left(\nabla^E\right)^2 \in \Omega^2(B, Hom(E)).$$

Applying the above formula for connexions extended to forms, to the 1-form $\nabla^E\sigma$ with $\sigma \in C^\infty(B, E)$ yields Equivalently,

$$\left(\Omega^E(U, V)\sigma\right)(U, V) = [\nabla^E_U, \nabla^E_V]\sigma - \nabla^E_{[U,V]}\sigma \quad \forall U, V \in C^\infty(B, TB).$$

An easy computation shows that the curvature is a tensor, meaning by this that $\Omega^E(U, V)f = f\Omega^E(U, V)$, although one could expect apriori from the above formula that f might get differentiated.

It is clear from the definition of the curvature that the *Bianchi identity*

$$[\nabla^E, \Omega^E] = 0$$

holds.
Writing the connection on a vector bundle in a trivialization over an open subset U of the base manifold $\nabla^E = d + \theta^E_U$, the curvature reads

$$\Omega^E = d\theta^E_U + \theta^E_U \wedge \theta^E_U.$$

Lemma: Let E be a Riemannian vector bundle equipped with a connection ∇^E which is compatible with the metric. Its curvature Ω^E is an $so(E)$-valued 2-form on M where $so(E)$ is the subbundle of $Hom(E)$ of antisymmetric morphisms of E.

Proof Let U, V be two vector fields on the base manifold:

$$\begin{aligned}
0 &= (UV - VU - [U,V])\langle\sigma,\rho\rangle \\
&= U\langle\nabla^E_V\sigma,\rho\rangle + U\langle\sigma,\nabla^E_V\rho\rangle \\
&- V\langle\nabla^E_U\sigma,\rho\rangle - V\langle\sigma,\nabla^E_U\rho\rangle \\
&- \langle\nabla^E_{[U,V]}\sigma,\rho\rangle - \langle\sigma,\nabla^E_{[U,V]}\rho\rangle \\
&= \langle\nabla^E_U\nabla^E_V\sigma,\rho\rangle + \langle\nabla^E_V\sigma,\nabla^E_U\rho\rangle \\
&+ \langle\nabla^E_U\sigma,\nabla^E_V\rho\rangle + \langle\sigma,\nabla^E_U\nabla^E_V\rho\rangle \\
&- \langle\nabla^E_V\nabla^E_U\sigma,\nabla^E_V\rho\rangle - \langle\sigma,\nabla^E_U\nabla^E_V\rho\rangle \\
&- \langle\nabla^E_V\sigma,\nabla^E_V\rho\rangle - \langle\sigma,\nabla^E_V\nabla^E_U\rho\rangle \\
&- \langle\nabla^E_{[U,V]}\sigma,\rho\rangle - \langle\sigma,\nabla^E_{[U,V]}\rho\rangle \\
&= \langle\Omega^E(U,V)\sigma,\rho\rangle + \langle\sigma,\Omega^E(U,V)\rho\rangle
\end{aligned}$$

so that $\langle\Omega^E(U,V)\sigma,\rho\rangle = -\langle\sigma,\Omega^E U,V\rho\rangle$ which shows that $\Omega^E(U,V)$ is antisymmetric. ■
Let now $E = TM$ where M is a Riemannian manifold. We drop the upper index E in the notation.

The *Ricci tensor* of a connection ∇ on a Riemannian manifold M is defined by

$$R(X,Y,W,Z) := \langle\Omega(X,Y)W,Z\rangle$$

where X, Y, W, Z are vector fields on M and $\langle\cdot,\cdot\rangle$ the inner product induced by the Riemannian structure. We have:

$$R(X,Y,W,Z) = -R(Y,X,W,Z) = -R(X,Y,Z,W)$$

and

$$R(X,Y,W,Z) = R(W,Z,X,Y).$$

When M is finite dimensional, the *Ricci curvature* is given by the trace of the operator $\Omega(X,\cdot)Y$, i.e. $\mathrm{Ricc}(X,Y) := \mathrm{tr}(\Omega(X,\cdot)Y)$. The *scalar curvature* is the trace of the Ricci curvature $s(x) = \sum_{i=1}^n \mathrm{Ricc}(e_i(x), e_i(x))$, where $(e_i(x))_{i\in\mathbb{N}}$ is any local orthonormal frame of T_xM.

A connection with vanishing curvature is called a *flat* connection. When the Ricci curvature vanishes, the manifold is called *Ricci flat*.

The ordinary differentiation on sections of a trivial bundle is flat since $d \circ d = 0$.

Characteristic classes:

- *Complex vector bundles:* Recall that the trace $\mathrm{tr} : gl_n(\mathbb{C}) \to \mathbb{C}$ on matrices has the following invariance property:
$$tr(C^{-1}AC) = \mathrm{tr}(A) \quad \forall C \in Gl_n(\mathbb{C}).$$
As a consequence it extends to a morphism of cvector bundles:
$$\mathrm{tr} : Hom(E) \to B \times \mathbb{C}$$
where E is a vector bundle based on B. It furthermore extends to $Hom(E)$-valued forms setting:
$$\mathrm{tr}(\alpha \otimes \sigma) := \alpha \mathrm{tr}(\sigma) \quad \forall \alpha \in \Omega(B), \sigma \in C^\infty(E).$$
Differentiating the above invariance property yields a *cyclicity* property:
$$\mathrm{tr}\left([A, B]\right) = 0 \quad \forall A, B \in gl_n(\mathbb{C}).$$
Combining the cyclicity of the trace with the Bianchi identity, provides closed forms $\mathrm{tr}\left(\left(\Omega^E\right)^k\right)$. Indeed:
$$d\,\mathrm{tr}(\Omega^k) = \mathrm{tr}(\Omega^{k-1} d\Omega) = \mathrm{tr}(\Omega^{k-1}[\nabla, \Omega]) = 0,$$
where we have used the local description $\nabla = d + [\theta_U, \cdot]$ of a connection on $Hom(E)$ induced by a connection on E, combined with the cyclicity of the trace $\mathrm{tr}([A, B]) = 0$ to establish the second identity. In the above formula, the product Ω^k uses both the exterior product and the composition in $Hom(E)$ since Ω is a $Hom(E)$-valued form. This formula extends replacing the k-th power by any analytic function f so that $\mathrm{tr}(f(\Omega))$ (which is in fact a polynomial expression in Ω of degree $\left[\frac{n}{2}\right]$, the integer part of half the dimension of the manifold M) is closed in the de Rham cohomology. Its cohomology class, called *Chern-Weil cohomology class*, is in fact independent of the choice of connection, as can be shown using similar arguments to those used to show that it is closed.

  Different Chern-Weil classes carry different names according to the choice of the function f. As an example, the *first Chern form* is obtained from $f(x) := x$,
$$r_1(\nabla) := \mathrm{tr}(\Omega),$$
the *Chern character* is obtained from $f(x) := e^{-x}$,
$$\mathrm{ch}(\nabla) := \mathrm{tr}(e^{-\Omega})$$
where we have set $\Omega = \nabla^2$ for the curvature. The exponential map involves wedge products as well as composition of morphisms since Ω is a $Hom(E)$-valued two-form. Notice that $r_1(\nabla) = -[\mathrm{ch}(\nabla)]_{[2]}$, namely minus the part of degree 2 of the form $\mathrm{ch}(\nabla)$.

- *Real vector bundles:* Since the trace vanishes on antisymmetric matrices, the trace is not very useful to define characteristic classes from real vector bundles for which the curvature is an antisymmetric tensor. We therefore use another tool to define characteristic classes form on real vector bundles, namely the Pfaffian, which in turn is related to another very useful tool, Berezin integration.

  Let $G = SO_n(\mathbb{R})$ and $\gamma = so_n(\mathbb{R})$, there is a one to one correspondance:
$$\begin{aligned} \Lambda^2 \mathbb{R}^n \quad &\leftrightarrow \quad so_n(\mathbb{R}) \\ a_{ij} e_i \wedge e_j \quad &\leftrightarrow \quad (a_{ij}) \end{aligned}$$

where $e_i, i = 1, \cdots, n$ is an orthonormal basis for the canonical scalar product on $\mathbb{R}^n$.

Definition: *Berezin integration* on $\Lambda\mathbb{R}^n$ is the linear map defined by:

$$\begin{array}{rcl} T : \Lambda\mathbb{R}^n & \longrightarrow & \mathbb{R} \\ \alpha & \mapsto & e_i^* \wedge \cdots e_n^*(\alpha) \end{array}$$

where $e_i^*, i = 1, \cdots, n$ is the dual basis $e_i^*(e_j) = \delta_{ij}$.

Notice that T vanishes on $\Lambda^p\mathbb{R}^n$ for any $p < n$ so that for any $v \in \mathbb{R}^n$ and any $\alpha \in \Lambda\mathbb{R}^n$, $T(i(v)\alpha) = 0$ where $i(v)$ is the interior product. The fact that T yields a linear map which vanishes on derivations justifies the terminology "integral" (analogy with Stoke's theorem).

Given a real metric vector bundle E of rank n based on a manifold M, Berezin integration generalises to a vector bundle morphism:

$$\begin{array}{rcl} T : \Lambda E & \longrightarrow & M \times \mathbb{R} \\ \alpha & \mapsto & e_i^* \wedge \cdots e_n^*(\alpha) \end{array}$$

where $e_i^*, i = 1, \cdots, n$ is now an orthonormal frame of E. T in turn induces a map on sections (denoted by the same symbol) $T : C^\infty(M, \Lambda E) \to C^\infty(M, \mathbb{R})$ in an obvious way.

Definition:

Under the above assumptions on E, the *Pfaffian* of $A = (a_{ij}) \in C^\infty(M, \Lambda^2 E \simeq so(E))$ is the real valued function on M defined by:

$$\mathrm{Pf}(A) := T\left(e^{\frac{1}{2}\sum_{i,j=1}^n a_{ij} e_i \wedge e_j}\right) = T\left(e^{\sum_{i<j;i,j=1}^n a_{ij} e_i \wedge e_j}\right).$$

In some cases, the Pfaffian is identified to the top form $\mathrm{Pf}(A) e_1 \wedge \cdots e_n$.

We state the following result without proof, leaving the proof as an exercise.

Lemma

Given $A = (a_{ij}) \in C^\infty(M, \Lambda^2 E \simeq so(E))$, if the rank n of E is even, setting $n = 2k$ we have:

$$\mathrm{Pf}(A) = \frac{(-1)^k}{2^k k!} \sum_{\sigma \in \Sigma_n} \varepsilon(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma_{2k-1}\sigma_{2k}}$$

and the Pfaffian vanishes if the rank of E is odd. Here $\varepsilon(\sigma)$ denotes the signature of σ.

Given a function with Taylor expansion at all orders at 0, namely $f(z) = \sum_{k=0}^K \frac{f^{(k)}}{k!}(0) z^k + o(z^K) \quad \forall K \in \mathbb{N}$ and an oriented metric real vector bundle (E, ∇^E) equipped with a connection compatible with the metric, similarly to the construction of characteristic classes via the trace, here again, using the Bianchi indentity and the properties of the Pfaffian, one can show that $P(\Omega) = \mathrm{Pf}\left(f(\Omega^E)\right)$ defines a closed form with cohomology class independent of the choice of connection.

Choosing $f(z) = -z$ yields the *Euler class*

$$e(\nabla^E) = \mathrm{Pf}(-\Omega^E) \in \Omega^N(M)$$

where N is the rank of E. The Euler class vanishes if N is odd as a consequence of the vanishing of the Pfaffian in odd dimensions. Moreover, as a consequence of the multiplicativity of the Pfaffian on tensor products, this characteristic class obeys the following property:

$$e(\nabla^{E\oplus F}) = e(\nabla^E) \wedge e(\nabla^F).$$

If M is an oriented Riemannian surface, and $(E = TM, \nabla^{TM})$ is the tangent bundle equipped with the Levi-Civita connection, then

$$e(\nabla^{TM}) = \kappa \quad d\mathrm{vol}$$

where κ is the Gaussian curvature.

Choosing $f(z) = \frac{\frac{z}{2}}{\mathrm{sh}\frac{z}{2}}$ yields the $\hat{A}$-*genus*

$$\hat{A}(\nabla^E) = \mathrm{Pf}\left(\frac{\frac{\Omega^E}{2}}{\mathrm{sh}\frac{\Omega^E}{2}}\right)$$

and $f(z) = \frac{\frac{z}{2}}{\mathrm{th}\frac{z}{2}}$ the L-*genus*

$$L(\nabla^E) = \mathrm{Pf}\left(\frac{\frac{\Omega^E}{2}}{\mathrm{th}\frac{\Omega^E}{2}}\right).$$

As a consequence of the multiplicativity of the Pfaffian on tensor products, these characteristic classes obey the following property:

$$\hat{A}(\nabla^{E\oplus F}) = \hat{A}(\nabla^E) \wedge \hat{A}(\nabla^F); \quad L(\nabla^{E\oplus F}) = L(\nabla^E) \wedge L(\nabla^F).$$

3 Principal bundles

3.1 Classification of principal bundles

Useful references are [34], [19], [48].

Definition: A (Banach) C^k-*principal* G-*bundle* based on a C^k-manifold B, where G is a Banach Lie group is a (Banach) C^k-fibre bundle P based on B with typical fibre G, such that if Φ and Ψ are two trivializations above some open subset $U \in B$, there exists a local map $\gamma : U \subset B \to G$ verifying:

$$\Phi_b \circ \Psi_b^{-1} = \gamma(b) \quad \forall b \in B.$$

G is called the *structure group* of P.

Notice that letting the group G acts on itself by left translation $L_g : h \to g \cdot h$ and letting (U, ϕ, Φ) and (W, ψ, Ψ) be two local trivialization with $b \in U \cap W$, we have:

$$\Phi_b(p_b) = g \cdot \Phi_b(q_b) \Rightarrow \Psi_b(p_b) = \gamma(b)^{-1} g \gamma(b) \Psi_b(q_b)$$

for any $g \in G$, $p_b, q_b \in E_b$ where E_b is the fibre over b. Thus, a change of trivialization induces an inner automorphism $g \mapsto \gamma(b)^{-1} g \gamma(b)$ of G.

Given a C^k-morphism $\phi : B' \to B$ between two C^k-manifolds, the pull-back $\phi^* P$ to B' of a C^k-principal G-bundle on B is a C^k-principal G-bundle on B'.

Let us now restrict ourselves to C^0-bundles. One can show that two homotopic maps $\phi : B' \to B$ and $\psi : B' \to B$ give rise to equivalent principal G-bundles $\phi^* P \simeq \psi^* P$.
One can therefore associate to the homotopy class $[\phi] \in [B', B]$ of a map $\phi : B' \to B$ the equivalence class of $\phi^* P$. This leads to the following definition.

Definition: A *classifying space* for a Lie group G is a connected topological space BG together with a principal G-bundle $PG \to BG$, such that for any compact Hausdorff space X, there is a one to one correspondence between the homotopy classes $[\phi]$ of maps $\phi : X \to BG$ and equivalence classes of principal G-bundles on X. A principal G-bundle PG on BG yields a pull-back bundle $\phi^* PG$ on X. The base space BG is defined up to homotopy type and the bundle $PG \to BG$ is called the *universal principal* G-bundle.

A principal G-bundle $P \to B$ with the property that the total space is contractible yields a classifying space B for G. An important example is the Grassmannian $G_n(\mathbb{C}^\infty) := \cup_{N=n}^{\infty} G_n(\mathbb{C}^N)$ which yields a classifying space for the unitary group $U(n)$ so that $BU(n) = G_n(\mathbb{C}^\infty)$.

Letting $\pi_n(G) := [S^n, G]$ denote the n-th homotopy group of G, the long exact sequence of homotopy groups:

$$\cdots \to \pi_n(P) \to \pi_n(B) \to \pi_{n-1}(G) \to \pi_{n-1}(P) \to \cdots$$

yields $\pi_n(B) \simeq \pi_{n-1}(G)$, using the fact that $\pi_n(P) = \{1\}$. *Singular cohomology* is needed for further information on the principal bundle (we refer the reader to any classical text on algebraic topology). A *universal characteristic class* for a principal G-bundle is a non zero class in the singular cohomology $H^*(BG, \Lambda)$ with coefficients in a ring Λ. Given a class $c \in H^k(BG, \Lambda)$, and any principal G-bundle $P \to B$, there is a map $\phi : B \to BG$ such that $P \simeq \phi^* PG$ and $c(P) := \phi^*(c) \in H^k(B, \Lambda)$ is the c-characteristic class of P. In particular the cohomology ring $H^*(BU(n), \mathbb{Z})$ is a $\mathbb{Z}$-polynomial ring with canonical generators $c_k \in H^{2k}(BU(n), \mathbb{Z})$, called the universal k-th Chern class. Thus to any $U(n)$-principal bundle $P \to B$, classified by a map $\phi : B \to BU(n)$, one can associate the k-th Chern class $c_k(P) := \phi^*(c_k)$. The relation to the Chern classes described at the end of the previous chapter will become clear once we have set up a correspondence between vector bundles and principal bundles; via this correspondence, the Chern class $c_k(P)$ can be seen as a Chern class on a complex rank n vector bundle E.

3.2 From group actions to principal bundles

Useful references in view of the applications we have in mind for quantum field theory are [3], [8], [14], [28], [12], [51]. Foundations for this type of slice theorem were set up in [43].

Let G be a Banach Lie group acting on the right on a Banach manifold X via a smooth action:

$$\begin{aligned} \Theta : G \times X &\to X \\ (g, x) &\mapsto x.g := R_g(x) \end{aligned}$$

i.e. $R_{g \cdot g'} = R_g \circ R_{g'}$ for $g, g' \in G$.
The action is *proper* provided the map

$$\begin{aligned} \Xi : G \times X &\to X \times X \\ (g, x) &\mapsto (x \cdot g, x) \end{aligned}$$

is proper, i.e. preimages of compact sets have compact closure.

If G is a compact Lie group the action is proper. To see this, we show that one can extract a convergent subsequence from any sequence $(x_n, g_n) \in G \times X$ such that $\Xi((x_n, g_n)) = (x_n \cdot g_n, x_n) \in K$ where K is a compact subset of $X \times X$. K being compact so is its projection onto the second component and we can extract from (x_n) a convergent subsequence $(x_{\phi(n)})$. G being compact, there is a subsequence of $(g_{\phi(n)})$ which we denote by the same symbol for simplicity, converging to some $g \in G$. The subsequence $(x_{\phi(n)}, g_{\phi(n)})$ therefore does the job.

The action is *free* provided it has no fixed points:

$$\exists x \in X, \quad x \cdot g = x \Rightarrow g = e.$$

The action is *isometric* provided it leaves the metric (given by inner products $\langle \cdot, \cdot \rangle_x$ on the fibre $T_x X$) invariant:

$$\langle DR_g U, DR_g V \rangle_{x \cdot g} = \langle U, V \rangle_x \quad \forall U, V \in T_x X.$$

The metric is said to be *compatible* with the group action.

Notation: For $x \in X$ we define the map

$$\begin{array}{rcl} \theta_x : G & \to & O_x \\ g & \mapsto & x \cdot g \end{array}$$

that sends an element of G to an element of the orbit O_x of x.
The freedom of the action Θ corresponds to the injectivity of θ_x for any $x \in X$.

Theorem: Let G be a Hilbert Lie group acting on the right on a (strong) Riemannian (Hilbert) manifold X via an isometric action:

$$\Theta : G \times X \to X$$

which is smooth, free and proper.
Provided for any $x \in X$ the tangent map $\tau_x := D_e \theta_x$ has a closed range, then

1) the orbits are closed submanifolds of X and $\theta_x : G \to O_x$ is a diffeomorphism of manifolds,

2) the quotient space X/G has a smooth Hilbert structure,

3) the projection $\pi : X \to X/G$ yields a principal fibre bundle.

Remark: In the finite dimensional case, there is no need for the splitting condition on τ_x which is automatically fulfilled. As we shall see in the next chapter, in the more general Hilbert setting, a Fredholm operator τ_x fulfills the additional requirement that the range be closed.

Proof: Let us make two preliminary trivial but useful remarks.

- Since $R_g \circ R_{g^{-1}} = Id = R_{g^{-1}} \circ R_g$ we have $DR_g \circ DR_{g^{-1}} = Id = DR_{g^{-1}} \circ DR_g$ so that DR_g is invertible and $DR_g^{-1} = DR_{g^{-1}}$.

- $D_g \theta_x \circ DR_g = DR_g \circ \tau_x$ since for $u \in Lie(G)$ we have:

$$\begin{aligned} D_g \theta_x DR_g u &= \frac{d}{dt}_{|t=0} (x e^{tu \cdot g}) \\ &= DR_g \frac{d}{dt}_{|t=0} (x e^{tu}) \\ &= DR_g \tau_x u. \end{aligned}$$

i) Θ being proper, θ_x is a closed mapping. Indeed, if $\theta_x(g_n)$ converges to y,then $(x, g_n \cdot x)$ converges to (x, y) and the properness of the action implies the existence of a subsequence $g_{\phi(n)}$ converging to some $g \in G$. The action being continuous, $g_{\phi(n)} \cdot x$ converges to $y = g \cdot x$. Thus θ_x is a homeomorphism onto its range O_x.

   Let us check that $D_g\theta_x$ is injective. Otherwise, there is some $u \neq 0 \in Lie(G)$ such that $D_g\theta_x(u \cdot g) = 0$. But since $D_g\theta_x = DR_g \circ \tau_x \circ DR_g^{-1}$, this would imply that $\tau_x u = 0$. Then, for any $t_0 \in I\!R$

$$\begin{aligned} \frac{d}{dt}_{|t=t_0} x \cdot e^{tu} &= \frac{d}{dt}_{|t=0} x \cdot e^{tu} \cdot e^{t_0 u} \\ &= DR_{g_0} \frac{d}{dt}_{|t=0} x \cdot e^{tu} \\ &= DR_{g_0}(\tau_x u) \\ &= 0 \end{aligned}$$

   where we have set $g_0 := \exp t_0 u$. This would imply that $\theta_x(g_t)$ is constant which contradicts the freedom of the action.

   Let us check that the range of the map $D\theta_x$ is closed. This follows from $R(D_g\theta_x) = DR_g R(\tau_x)$ (see the second preliminary remark) combined with the fact that τ_x has closed range. Moreover, since R_g is an isometry it preserves orthogonality and
   $R(D_g\theta_x)^\perp = DR_g\,(R(\tau_x))^\perp$ so that we have the following orthogonal splitting:

$$T_{x \cdot g}X = R(D_g\theta_x) \oplus DR_g\,(R(\tau_x))^\perp.$$

   Thus θ_x is an injective immersion which is also a homeomorphism onto its image. The inverse mapping theorem then implies it is a diffeomorphism $G \simeq O_x$ and the orbit O_x is a submanifold of X.
   The tangent space of O_x at point $y = x \cdot g$ is $R(D_g\theta_{xg})$. This finishes the proof of point 1) of the Theorem.

ii) We now check points 2) and 3). Let U_x be an open neighborhood of x in $R(\tau_x)^\perp$, small enough to build the submanifold $S_x := \exp_x(U)$ of X using the exponential map $\exp_x : U \to V_x \subset X$ at point x, where U is an open neighborhood of the zero section of the tangent bundle TX and V_x an open neighborhood of $x \in X$. Since the exponential map defines a local diffeomorphism, S_x inherits a manifold structure which by construction has tangent space at point x given by $N_x(O_x) := R(\tau_x)^\perp$ where N stands for normal, $N_x(O_x)$ being the fibre above x of the normal bundle to the orbit O_x.

   The action being continuous, free and proper, one can chose U small enough so that

$$(S_x)\,.g \cap S_x \neq \phi \Leftrightarrow g = e. \tag{1}$$

   Indeed, otherwise, we could find a sequence $(u_n) \in N_x(O_x)$ with norm $\|u_n\| \leq \frac{1}{n}$ such that both (x_n) and $(x_n \cdot g_n)$ converge to some x. But in that case, the properness of the action yields the existence of a subsequence $(g_{\phi(n)})$ converging to some $g \in G$. The continuity of the action then implies that in the limit $x \cdot g = g$. But the action being free, this implies in turn that $g = e$.

   It follows from (1) that the local slice S_x is in one to one correspondence with a subset $U_{\bar{x}}$ of the quotient space $X/G := B$. Equipping B with the quotient topology turns the projection map $\pi : X \to B$ into a continuous map and yields a homeomorphism $\pi : S_x \to U_{\bar{x}}$. The manifold structure on S_x then yields a local chart over the neighborhood $U_{\bar{x}}$ of $\bar{x} \in B$. Patching up such local trivializations yields a smooth atlas on B with transition maps obtained from the exponential maps.

This quotient manifold inherits a metric structure from the G-invariant structure on X. Given $\bar{U}, \bar{V} \in T_{\bar{x}}B$ we set:

$$\langle \bar{U}, \bar{V} \rangle_{\bar{x}} := \langle U, V \rangle_x$$

for any x in the fibre above $\bar{x}$ and any $U, V \in T_xX$ such that $D\pi(U) = \bar{U}, D\pi(V) = \bar{V}$. Since the metric is G-invariant, this is independent of the choice of x and of U and V.
The above local charts induce local trivializations for the projection $\pi : X \to B$ so that we can equip X with a G-principal bundle structure over B. Locally we have:

$$X_{|U_{\bar{x}}} \simeq S_{\bar{x}} \times G.$$

■

3.3 From vector bundles to principal bundles and back

Useful references are [6], [27].

To a smooth vector bundle $E \to B$ based on a manifold B with typical fibre a Banach space V, we associate a principal bundle $GL(E) \to B$ called the associated *frame bundle* with structure group $G := GL(V)$ and fibre above $b \in B$ given by:

$$GL_b(E) := \{L_b : V \to E_b, \quad \text{continuous and one to one}\}.$$

(Recall from the open mapping theorem that it is a homeomorphism).
Letting

$$\begin{array}{rcl} (\phi, \Phi) : E_{|U} & \to & U \times V \\ (b, u) & \mapsto & (\phi(b), \Phi_b u) \end{array}$$

be a local trivialization of E above an open subset U of B, a local trivialization $(\phi, \bar{\Phi})$ of $GL(E)$ above U is given by

$$\begin{array}{rcl} GL(E)_{|U} & \to & U \times GL(V) \\ (b, L_b) & \mapsto & \left(\phi(b), \bar{\Phi}_b(L_b) := \Phi_b \circ L_b\right). \end{array}$$

Given two local trivializations (ϕ, Φ) and (ψ, Ψ) of E and hence two induced trivializations $(\phi, \bar{\Phi})$, $(\psi, \bar{\Psi})$ of $GL(E)$ we can build a map:

$$\begin{array}{rcl} \rho : U & \to & GL(V) \\ b & \mapsto & \bar{\Psi}_b \circ \bar{\Phi}_b^{-1} : L \to \Psi_b \circ \Phi_b^{-1} \circ L_b \end{array}$$

A Banach vector bundle E of class C^k (resp. C^∞) is trivial if and only if $GL(E)$ admits a global section of class C^k (resp. C^∞). Indeed, a global α section of class C^k (resp. C^∞) yields a diffeomorphism:

$$\begin{array}{rcl} E & \to & B \times V \\ (b, u_b) & \mapsto & (b, \alpha(b)(u_b)). \end{array}$$

If E is a rank n vector bundle, a global section of class C^k (resp. C^∞) of the principal bundle $GL(E)$ corresponds to a family of frames $(e_1(b), \cdots, e_n(b))$ of class C^k (resp. C^∞) parametrized by B and the section α yields the coordinates $\alpha_i(b), i = 1, \cdots, n$ of the vector u_b in the basis $(e_1(b), \cdots, e_n(b))$ of the fibre E_b above b.

When $E = TM$, the tangent bundle to a manifold M of class C^k (resp. C^∞), the existence of a global section of $GL(E)$ is a constraint on the manifold M and we say that M is C^k- (resp. C^∞-) *parallelizable*. If we only require this section to be continuous, it is a topological constraint. A Lie group is clearly C^∞-paralellizable since left (or right) action $L_g : h \mapsto g \cdot h$ (or $R_g : h \mapsto h \cdot g$) of the group on itself induces a smooth parallelization $L_g : Lie(G) \to T_gG$ (resp. $R_g : Lie(G) \to T_gG$).

A result by Kuiper [25] tells us that given a C^k (resp. smooth) Hilbert vector bundle $E \to B$, the associated frame bundle $GL(E)$ admits a global C^0 section. Thus any Hilbert manifold is C^0-parallelizable.

Conversely, given a principal bundle with structure group G and a representation $\rho : G \to Diff(V)$ on a Banach space V, we can build the *associated vector bundle*:

$$P \times_\rho V := P \times V/_\sim$$

where $\sim$ is the equivalence relation defined by

$$(p, v) \sim (p', v') \Leftrightarrow \exists g \in G, p = p' \cdot g \quad \text{and} \quad v' = \rho(g)v$$

so that $(p \cdot g, v)$ and $(p, \rho(g)v)$ get identified. Locally, above an open subset U of the base manifold, we have:

$$P \times_\rho V_{|_U} \simeq (U \times G) \times_\rho V \simeq U \times V.$$

In particular, a vector bundle E with typical fibre V is associated to its frame bundle $GL(E)$ with structure group $GL(V)$:

$$GL(E) \times_\rho V = E$$

where ρ is the natural the action of $GL(V)$ on V.

3.4 Connections on a principal bundle

Useful references are [6], [27], [34], [52].

Given a principal bundle $\pi : P \to B$ with structure group G, and the induced map $D\pi : TP \to TB$, we call a vector field ξ *vertical* provided $D\pi(\xi) = 0$. Let us denote by VTP the subbundle of vertical vector fields with fibre above $p \in P$ given by

$$VT_pP := \{v \in T_pP, D_p\pi(v) = 0\}.$$

The map induced by the action of G on P:

$$\begin{aligned} \tau_p : Lie(G) &\to T_pP \\ u &\mapsto \frac{d}{dt}_{|t=0} (p \cdot \exp(tu)) \end{aligned}$$

introduced previously, gives rise to a vertical vector field:

$$p \mapsto \xi(p) := \bar{u}_p := \tau_p(u),$$

called the *canonical vector field* associated to u. $\tau_p : u \to \bar{u}_p$ is an isomorphism of $Lie(G)$ onto VT_pP.

Definition: A *connection* on the principal bundle P with structure group G is a smooth splitting

$$T_pP := VT_pP \oplus HT_pP, \quad \forall p \in P$$

with an *equivariance property* $HT_{p\cdot g}P = DR_g(HT_pP) \quad \forall p \in P, \forall g \in G$. HTP is called a *horizontal* distribution and HT_pP the horizontal tangent space to P at point p. A horizontal distribution HTP turns the map $D\pi : HTP \to TB$ into an ismorphism and we call $\tilde{\xi}$ the *horizontal lift* of a vector field ξ on B.
Equivalently, a connection on the principal bundle P is given by a Lie algebra valued one form on P, $\omega \in \Omega^1(P, Lie(G))$ such that:

(i) $\omega_{\bar{u}_p} = u \quad \forall u \in Lie(G)$

(ii) $\omega_{p\cdot g} = R_g^* \omega_p \quad \forall p \in P$ where $R_g : p \to p \cdot g$ corresponds to the action of G on P.

The two formulations are equivalent. Indeed, given a smooth horizontal distribution HTP, any tangent vector field ξ splits in a unique way $\xi = \xi^v \oplus \xi^h$ into a vertical part $\xi^v := \tau_p u$ for some unique $u \in Lie(G)$, and a horizontal part ξ^h. $\omega(\xi) := u$ defines a unique $Lie(G)$ valued one form ω on P satisfying requirements (i) and (ii). The curvature of the connection reads $\Omega := d\omega + \omega \wedge \omega$. it measures in how far the splitting $TP = VTP \oplus HTP$ does not respect the Lie algebra structure on vector fields, $\Omega(U, V) = [\tilde{U}, \tilde{V}] - [\tilde{U, V}]$ where $\tilde{U}$, $\tilde{V}$ are the horizontal lifts of the vector fields U and V.

Conversely, such a one form ω defines a vector bundle $HTP := \{\xi \in TP, \omega(\xi) = 0\}$ which has the required invariance property by property (ii).

There is a natural horizontal distribution $HTP = (VTP)^\perp$ whenever there is a (strong) Riemannian metric compatible with the action of the structure group G on P.
There is a one to one correspondence between covariant derivations defined on vector bundles E and connections defined on principal bundles in the following sense.

A connection ω on the principal bundle $\pi : P \to B$, seen as a horizontal distribution on P, yields a covariant derivation on the associated vector bundle E. Let $X \in T_bB, b \in B$ and let $\tilde{X}$ be its horizontal lift. Letting $\rho : G \to Aut(V)$ be an action of the gauge group G of P on a Banach vector space V, a section σ of the associated vector bundle $P \times_G V$ can be seen locally as a map from an open subset of B to the vector space V so that it makes sense to set

$$\nabla_X \sigma := \left(p, \tilde{X}\sigma\right).$$

Notice that we implicitly have used the equivariance of the horizontal distribution in this definition. ∇ yields a connection on the associated vector bundle $E := P \times_G V$.

The covariant derivation ∇ is compatible with the metric whenever the horizontal distribution is given by the orthogonal supplement to the vertical bundle.

Conversely, a covariant derivation ∇^E on a vector bundle E with typical fibre V yields a connection on the frame bundle $GL(E)$ (recall that its structure group is $GL(V)$ with Lie algebra $Lie(G) = Hom(V)$). Let us set $P = GL(E)$, the canonical projection $\pi : P \to B$ induces a map $D_p\pi : T_pB \to T_{\pi(b)}B$ so that given a tangent vector $X \in T_pP$, we can set:

$$\omega(X) = \nabla^{Hom(E)}_{D_p\pi X},$$

where $\nabla^{Hom(E)}$ is the covariant derivation induced by ∇^E on $Hom(E)$. Here $TGL(E)$, on which the form ω is defined, is locally seen as $U \times Hom(V)$. Since $Hom(E)$ is also locally seen as $U \times Hom(V)$, it makes sense to let the covariant derivation on the r.h.s. act on a section of $TGL(E)$.

3.5 Reducing and lifting principal bundles: spin and spin^c structures

Useful references are [20], [31], [32], [35].

Definition: Let H be a closed subgroup of a Banach Lie group G. A principal bundle P based on B with structure group G *reduces* to a principal bundle with structure group H whenever there is an atlas of charts (U_i, Φ_i) for P such that the transition maps have values in the subgroup H. Let $\{1\} \to H \to \tilde{G} \to G \to \{1\}$ be an exact sequence of Banach Lie groups. A G-principal bundle P based on B *lifts* to a $\tilde{G}$-principal bundle $\tilde{P}$ whenever $\tilde{P}$ reduces to P, where we view G as a subgroup of $\tilde{G}$ via the isomorphism $\tilde{G} \simeq H \times G$.

Notice that a principal bundle reduces to a bundle with structure group $H = \{1\}$ whenever the bundle is trivial.

Reducing the structure group is a way to impose geometric constraints on the bundle. In particular, a real (resp. complex) vector bundle $E \to B$ with typical fibre V can be equipped with a (strong) Riemannian (resp. Hermitian) structure whenever the associated frame bundle $GL(E)$ with structure group $GL(V, \mathbb{R})$ (resp. $GL(V, \mathbb{C})$) reduces to the orthonormal frame bundle, a principal bundle with structure group $O(V) := \{g \in GL(V, \mathbb{R}), g^* g = I\}$ (resp. $U(V) := \{g \in GL(V, \mathbb{C}), g^* g = 1\}$). In particular, when $E := TM$ where M is an n-dimensional real (resp. complex manifold), then $V = \mathbb{R}^n$ (resp. $V = \mathbb{C}^n$) and M can be equipped with a Riemannian (resp. Hermitian) metric whenever the frame bundle $GL(M) := GL(TM)$, with structure group $GL(n, \mathbb{R})$ (resp. $GL(n, \mathbb{C})$), reduces to the orthonormal (resp. unitary) frame bundle $O(M)$ (resp. $U(M)$) with structure group $O(n)$ (resp. $U(n)$). Furthermore a real rank n Riemannian vector bundle E is orientable whenever its frame bundle $GL(M)$ reduces to a principal bundle with structure group $SO(n) := \{g \in O(n), det g > 0\}$.

Lifting principal bundles is not always possible, as we shall see shortly, when trying to define spin and spin^c structures.

Definition: Let V be a real Euclidean vector space. The algebra $C\ell(V)$ over $\mathbb{R}$ generated by V with the relations:

$$v \cdot w + w \cdot v = -2\langle v, w \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on V is called the *Clifford algebra of* V. It can also be seen as a quotient space $C\ell(V) = \mathcal{T}(V)/\{v \cdot v = -2\|v\|^2\}$ of the tensor algebra $\mathcal{T}(V) = \oplus_{k=1}^{\infty} V^{\otimes^k}$ by the relation $v \cdot v = -2\|v\|^2$. The $\mathbb{Z}$-grading on $\mathcal{T}(V)$ induces a natural $\mathbb{Z}_2$ -grading on $C\ell(V)$:

$$C\ell(V) =: C\ell_0(V) + C\ell_1(V)$$

into even and odd (Clifford) products.

Given $v \in V$, let $c(v)$ act on the exterior algebra ΛV by

$$c(v)\alpha := \varepsilon(v)\alpha - i(v)\alpha$$

where ε is the adjoint of the contraction operator i on exterior forms introduced in Section 2.4. Since it satisfies the relation

$$c(v)c(w) + c(w)c(v) = -2\langle v, w \rangle$$

it extends to an action of $C\ell(V)$ on ΛV, which makes ΛV a $C\ell(V)$-module. The *symbol map* $\sigma : C\ell(V) \to \Lambda V$ is defined by:

$$\sigma(a) := c(a)1 \in \Lambda V.$$

Accordingly, given a Riemannian bundle E based on B with typical fibre V, one can define the bundle $C\ell(E)$ of Clifford algebras based on B with typical fibre $C\ell(V)$ defined fibrewise by $C\ell(E_b)$ where E_b is the fibre above $b \in B$ equipped with the inner product induced by the Riemannian structure.

A *Clifford module* on a Riemannian manifold M is a vector bundle $\mathcal{M} \to M$ with a Clifford action of $C\ell(M)$ on it:

$$\begin{array}{rcl} C\ell(M) \times \mathcal{M} & \to & \mathcal{M} \\ (v, \sigma) & \mapsto & c(v)\sigma. \end{array}$$

A vector field $v \in C^\infty(TM)$ acts on a form $\alpha \in \Omega(M)$ by the following Clifford action:

$$c(v)\alpha := \varepsilon(v)\alpha - i(v)\alpha,$$

which extends to an action of sections of the bundle $C\ell(TM)$ on $\Omega(M)$. Thus $\mathcal{M} := \Lambda T^*M$ the exterior bundle on M yields a Clifford module. The *symbol map* sends a section a of $C\ell(M)$ to $c(a) \in \Omega(M)$.

Going back to the algebraic setting, let us assume that V is finite dimensional. The space $C\ell^2(V) := c(\Lambda^2 V)$ is a Lie subalgebra of $C\ell(V)$ with bracket given by the commutator of $C\ell(V)$. The *spin group* Spin (V) is the group generated by elements in $C\ell_0(V)$ with norm 1. It can also be seen as the group obtained by exponentiating the Lie algebra $C\ell^2(V)$ inside the Clifford algebra $C\ell(V)$. Letting $V := \mathbb{R}^n$, we simply write $C\ell(n) := C\ell(\mathbb{R}^n)$ and Spin $(n) := \mathrm{Spin}(\mathbb{R}^n)$.

Here is a very classical result for which we do not give a proof here since it is purely algebraic and can be found in any text book on spin structures.

Proposition: If dim $V > 1$, there is an exact sequence of groups:

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(V) \to SO(V) \to 1.$$

Spin(V) is therefore a double covering of $SO(V)$.

Similarly, $\mathrm{Spin}^c(V)$ is the subgroup of $C\ell_0(V) \otimes_{\mathbb{R}} \mathbb{C}$ generated by Spin(V) and the unit circle of complex scalars. It yields a double covering of $SO(V) \times S^1$ and there is an exact sequence of groups:

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}^c(V) \to SO(V) \times S^1 \to 1.$$

Spin(V) is then naturally identified with the subgroup $\mathrm{Spin}(V) \times_{\pm 1} \{\pm 1\}$ of $\mathrm{Spin}^c(V)$ and

$$\mathrm{Spin}^c(V) \simeq \mathrm{Spin}(V) \times_{\pm 1} S^1.$$

Letting $V := \mathbb{R}^n$, we simply write $\mathrm{Spin}^c(n) := \mathrm{Spin}^c(\mathbb{R}^n)$.

Definition: An oriented Riemannian rank n-bundle $E \to B$ admits a spin (resp. spin^c) structure whenever the bundle $SO(E)$ of oriented orthonormal frames lifts to a principal bundle $P_{spin}(E)$

($\tilde{P}_{spin^c}(E)$) with structure group Spin (n) (resp. Spin $^c(n)$). In particular an n-dimensional oriented Riemannian manifold M *is spin* (resp. spin^c) whenever its frame bundle $GL(TM)$ admits a spin (resp. spin^c) structure.

The obstruction to the existence of a spin structure on a vector bundle is measured by the second *Stiefel Whitney class* in $H^2(M, \mathbb{Z}_2)$. The obstruction to the existence of a spin^c structure is weaker since such a structure exists whenever this second Stiefel-Whitney class is a reduction modulo 2 of an integral class $c \in H^2(M, \mathbb{Z}_2)$, i.e. if its third Stiefel-Withney class vanishes. In particular, any spin manifold is spin^c.

Let us make a short comment on Stiefel-Whitney classes. To any rank n real Riemannian vector bundle $E \to B$ classified by a map $f_E : B \to BO(n)$, one can associate the k-th *Stiefel-Whitney class* $w_k(E) := f_E^*(w_k) \in H^k(B, \mathbb{Z}_2)$ where $w_k \in H^k(BO(n), \mathbb{Z}_2)$ are the canonical generators of the $\mathbb{Z}_2$-polynomial ring $H^*(BO(n), \mathbb{Z}_2)$. The first Stiefel-Whitney class measures an obstruction to the orientability of a Riemannian bundle, and the second Stiefel-Whitney class measures the obstruction to the existence of a spin structure on an orientable Riemannian bundle.

Back again to the algebraic setting, let us set $\mathbb{C}\ell(n) := C\ell(n) \otimes \mathbb{C}$. Then $\mathrm{Spin}(n) \subset C\ell(n) \subset \mathbb{C}\ell(n)$ so that any complex representation of the complexified Clifford algebra $\mathbb{C}\ell(n)$ on some vector space S reduces to a complex representation of $\mathrm{Spin}(n)$. There are essentially two types of representations according to the parity of the manifold which we briefly describe in the following proposition, referring the reader to any classical text on spin structures for a proof.

Proposition: When n is odd all irreducible complex representations $\mathbb{C}\ell(n) \to Hom_{\mathbb{C}}(S, S)$ restrict to a unique irreducible representation of $Spin(n)$. When n is even, a complex representation $\mathbb{C}\ell(n) \to Hom_{\mathbb{C}}(S, S)$ yields a representation Δ_n of $\mathrm{Spin}(n)$ which decomposes into a direct sum of two inequivalent irreducible complex representations Δ_n^+ and Δ_n^- on S^+ and S^- respectively. Such representations are called *spinor representations* and the corresponding representation spaces S, S^+,S^- are called *spinor spaces.*

These spinor spaces give rise to *spinor bundles*:

$$S(E) := P_{spin}(E) \times_{\mathrm{Spin}(n)} S,$$

$$S^{\pm}(E) := P_{spin}(E) \times_{\mathrm{Spin}(n)} S^{\pm},$$

where $E \to B$ is some vector bundle with a spin structure. In fact, any *Clifford module* $\mathcal{M}$ based on a odd (resp. even) dimensional spin manifold B, i.e. any (resp. $\mathbb{Z}_2$-graded) vector bundle $\mathcal{M}$ with an (graded) action of the bundle $C\ell(B)$ of Clifford algebras on it

$$\begin{array}{rcl} C^\infty(C\ell(B)) \times C^\infty(\mathcal{M}) & \to & C^\infty(\mathcal{M}) \\ (a, s) & \mapsto & c(a) \cdot s \end{array}$$

is of the form:

$$\mathcal{M} := S(TB) \otimes W,$$

$$(\text{resp.} \quad \mathcal{M}^{\pm} := S^{\pm}(TB) \otimes W)$$

where W is an exterior vector bundle based on B.

Any complex representation $\rho : Spin(V) \to GL_{\mathbb{C}}(W)$ extends in an unique way to a representation $\tilde{\rho} : Spin^c(V) \to GL_{\mathbb{C}}(W)$. In particular the complex representations Δ_n, Δ_n^-, Δ_n^+ uniquely extend to $\tilde{\Delta}_n$, $\tilde{\Delta}_n^-$, $\tilde{\Delta}_n^+$ on $\tilde{S}$, $\tilde{S}^+$, $\tilde{S}^-$. The corresponding spinor spaces give rise to spinor bundles:

$$\tilde{S}(E) := \tilde{P}_{spin^c}(E) \times_{\mathrm{Spin}^c(n)} \tilde{S},$$

$$\tilde{S}^+_-(E) := \tilde{P}_{spin^c}(E) \times_{\mathrm{Spin}^c(n)} \tilde{S}^+_-$$

where $E \to B$ is some vector bundle with a spin^c structure.

4 Fredholm operators and elliptic operators on closed manifolds

4.1 Bounded linear operators

Useful references are [5], [9], [36], [28], [46], [44], [47], [53].

Let E be a complex Banach space with norm $\|\cdot\|_E$.

Definition: Let F be another Banach space with norm $\|\cdot\|_F$. The space of bounded linear operators from E to F

$$\mathcal{B}(E,F) := \{A : E \to F, \quad \exists C > 0, \|Au\|_F \leq C\|u\|_E \quad \forall u \in E\}$$

is equipped with the norm $|||A||| \equiv \sup_{u\in E, x\neq 0} \frac{\|Au\|_F}{\|u\|_E}$ is a Banach space. When $F = E$ we set $\mathcal{B}(E) := \mathcal{B}(E,E)$ which is an example of Banach algebra, a notion we briefly recall. But let us first give a useful example of bounded operator.

Example: Let $L : l_2 \to l_2$ be the left translation operator on the set l_2 of L^2 convergent sequences defined by

$$L((u_n)) := (v_n), \quad v_n = u_{n+1}.$$

An *algebra* $\mathcal{A}$ is a vector space equipped with a bilinear map:

$$\begin{array}{rcl} \mathcal{A} \times \mathcal{A} & \to & \mathcal{A} \\ (a,b) & \to & ab \end{array}$$

such that $a(bc) = (ab)c$. It is a *normed algebra* whenever it can be equipped with a submultiplicative norm:

$$\|ab\| \leq \|a\| \cdot \|b\|.$$

It is a *unital algebra* whenever it admits a unit 1, i.e. $1a = a1 = a \quad \forall a \in \mathcal{A}$.
A *Banach algebra* is a complete normed algebra.

A C^**-algebra* is a Banach algebra $\mathcal{A}$ equipped with an involution $* : \mathcal{A} \to \mathcal{A}$ (i.e. $*$ is a linear map satisfying $*^2 = I$) such that $\|a^*a\| = \|a\|^2$ for any $a \in \mathcal{A}$. As a consequence, the involution is isometric, i.e. $\|a^*\| = \|a\| \quad \forall a \in \mathcal{A}$.
Example: Given a locally compact Hausdorff space X, the space $C_0(X)$ of complex valued continuous functions on X vanishing at infinity, equipped with the involution $\star : f \mapsto \overline{f}$ is a commutative C^*-algebra.
Definition: The spectrum (or dual) of a C^*-algebra $\mathcal{A}$ denoted by $\hat{\mathcal{A}}$ is the set of unitary equivalence classes of non trivial irreducible $\star$-representations of $\mathcal{A}$.
Remark: When $\mathcal{A}$ is commutative, $\hat{\mathcal{A}}$ coincides with the dual of the algebra $\mathcal{A}$ i.e. with the set of

non zero characters on $\mathcal{A}$. When moreover $\mathcal{A}$ is unital, i.e. when it has a unit, then its spectrum is compact.

Gelfand's theorem states that any commutative C^*-algebra $\mathcal{A}$ is of the form $C_0(X)$, for some locally compact space X which corresponds to the spectrum of the C^*-algebra, i.e. $\mathcal{A} = C_0(\hat{\mathcal{A}})$. In other words, there is a map

$$\begin{array}{rcl} g : \mathcal{A} & \to & C_0(\hat{\mathcal{A}}) \\ x & \mapsto & g(x) := \hat{x} \end{array}$$

called the Gelfand map, which gives rise to an isometric isomorphism.

There is another characterisation of bounded operators on Hilbert spaces using the Hermitian products. Letting H_1, H_2 be two Hilbert spaces equipped with the Hermitian products $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$, an operator A lies in $\mathcal{B}(H_1, H_2)$ provided there exists a constant $C > 0$ such that for any $u \in H_1, v \in H_2$ $| < Au, v >_2 | \leq C\|u\|_1 \|v\|_2$. Indeed, if $A \in \mathcal{B}(H_1, H_2)$ this is the case, by the Cauchy-Schwarz inequality. Conversely, if $| < Au, v >_2 | \leq C\|u\|_1\|v\|_2$, then $\|Au\|_2^2 \leq C\|u\|_1\|Au\|_2$ so that $\|Au\|_2 \leq C\|u\|_1$ whenever $Au \neq 0$.

By the Riesz Lemma, the relation

$$\langle Au, v \rangle_2 := \langle u, A^* v \rangle_1 \quad \forall u \in H_1, v \in H_2$$

uniquely defines an operator $A^* \in \mathcal{B}(H_2, H_1)$ called the *adjoint* of A.

Back to the example: In the above example, it is easy to check that $LL^* = Id$ but $L^*L \neq Id$.
Example: If H is a Hilbert space, the algebra $\mathcal{B}(H)$ equipped with the star operation $L \to L^*$ is a C^*-algebra.
The *Gelfand-Naimark* theorem says that every abstract C^*-algebra with identity is isometrically $*$-isomorphic to a C^*-algebra of operators. To prove that result, one uses the *Gelfand-Naimark-Segal* or *GNS* construction which produces a representation from a state. To a state ρ on a C^*-algebra $\mathcal{A}$, i.e. a positive linear functional $\rho : \mathcal{A} \to \mathbb{C}$ ($\rho(a^*a \geq 0 \forall a \in \mathcal{A}$), one can associate a positive semi-definite bilinear form $\langle a, b \rangle = \rho(b^*a)$ with kernel $N_\rho = \{a \in \mathcal{A}, \rho(a^*a) = 0\}$ which is a linear subspace of $\mathcal{A}$ and a left ideal in $\mathcal{A}$. This bilinear form therefore induces a positive definite form $\langle \cdot, \cdot \rangle$ on $\mathcal{A}/N_\rho$ and hence a pre-Hilbert space structure on that quotient space, which by completion gives rise to a Hilbert space H_ρ. The left regular representation

$$\begin{array}{rcl} \pi_\rho : \mathcal{A} & \to & \mathcal{B}(H_\rho) \\ b & \mapsto & (a \mapsto ba) \end{array}$$

is cyclic with cyclic vector $x_\rho := 1_A + N_\rho$ and $\rho(a) = \langle \pi_\rho(a) x_\rho, x_\rho \rangle$ for any $a \in A$.

4.2 Closed graph theorem

Useful references are [8], [47].

The operators one comes across in geometry or in physics usually are non bounded and only defined on a dense domain of the Banach space.

Definition: Let E and F be two Banach space. The graph of an operator $A : D(A) \subset E \to F$ defined on a domain $D(A)$ is the set:

$$Gr(A) := \{(u, Au) \in E \times F, u \in D(A)\}.$$

It can be equipped with the (graph) norm

$$\|(u,v)\| := \|u\|_E + \|v\|_F.$$

Notice that whenever A is invertible, then the graph of A^{-1} is the symmetric of the graph of A w.r.to the diagonal axis.

Definition: The operator A is *closed* whenever its graph is *closed* for the graph norm.

When E and F are separable, there is another characterisation for closed operators.
An operator $A : D(A) \subset E \to F$ is *closed* if, given any sequence (u_n) converging to $x \in E$ such that Au_n converges in F, then the limit u lies in the domain $D(A)$ and $Au_n \to Au$.
We shall henceforth assume that the Banach spaces under consideration are separable.
It is easy to check that any bounded linear operator is closed.

Furthermore, a closed linear operator $A : D(A) \subset E \to F$ defined on a dense domain $D(A)$ of E extends in an unique way to a bounded operator on E whenever there is a constant $C > 0$ such that $\|Au\|_F \leq C\|u\|_E, \forall u \in D(A)$.

To prove this fact, all we need is to define the image of any element $u \in E$ by an extension of A. Since $D(A)$ is dense in E, u can be seen as a limit $u = \lim_{n\to\infty} u_n$ of a sequence (u_n) in $D(A)$. Since (u_n) is a Cauchy sequence, so is the sequence (Au_n) a Cauchy sequence so that it converges to some $v \in F$. The operator A being closed, this implies that u lies in the domain $D(A)$ and $Au = v$. This extended operator (also denoted by A) is clearly a bounded operator.
Moreover this extension does not depend on the choice of the sequence. For if (u'_n) is another sequence tending to u, from the inequality $\|Au_n - Au'_n\| \leq C\|u_n - u'_n\|$, it follows that $Au'_n \to Au$.
As a consequence we have:

Closed graph Theorem: Let $A : E \to F$ be a closed linear operator with domain $D(A) = E$. Then $A \in \mathcal{B}(E,F)$, i.e. A is bounded on E.

In the following, we assume the operators are closed and defined on a dense domain $D(A)$.

Proposition: The inverse of a bijective linear operator $A : D(A) \subset E \to F$ is a bounded operator $A^{-1} : F \to E$.
Proof: Since the graph of A is closed so is the graph of A^{-1} and the result follows from the closed graph theorem. ■

Definition: The *resolvent* of A is the set

$$\begin{aligned} \rho(A) &:= \{\lambda \in \mathbb{C}, A - \lambda I \quad \text{is bijective }\} \\ &= \{\lambda \in \mathbb{C}, A - \lambda I \quad \text{is bijective and} \quad (A-\lambda I)^{-1} \in \mathcal{B}(H)\}. \end{aligned}$$

The *spectrum* $\sigma(A)$ of A is the complement of the resolvent:

$$\sigma(A) := \mathbb{C}/\rho(A).$$

The *point spectrum* $\sigma_p(A)$ is the set

$$\sigma_p(A) := \{\lambda \in \mathbb{C}, \mathrm{Ker}(A - \lambda I) \neq \{0\}\}.$$

In finite dimensions, $\sigma_p(A) = \sigma(A)$, which is not generally the case in infinite dimensions.
Example: Let $H = l^2$ be the space of l^2 convergent sequences and let $L : l^2 \to l^2$ be the operator sending a sequence (u_n) to a sequence (v_n) with $v_0 = 0, v_i = u_{i-1}$. 0 does not belong to $\sigma_p(A)$ but 0 lies in $\sigma(A)$.

4.3 Adjoint of an operator

Useful references are [8], [44].

Let H_1 and H_2 be Hilbert spaces equipped with the Hermitian products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively. We extend here the notion of adjoint of an operator to unbounded operators. A preliminary step is the notion of symmetric of an operator.
Definition: The operator $A^* : D(A^*) \subset H_2 \to H_1$ is *symmetric* to an operator $A : D(A) \subset H_1 \to H_2$ if

$$\langle Au, v \rangle_2 = \langle u, A^* v \rangle - 1 \quad \forall u \in D(A), \quad \forall v \in D(A^*).$$

An operator A is *symmetric* if

$$\langle Au, v \rangle_2 = \langle u, Av \rangle - 1 \quad \forall u \in D(A), \quad \forall v \in D(A).$$

Notice that if A is symmetric, then $D(A) \subset D(A^*)$ and $A^*_{|_{D(A)}} = A$ so that its graph is contained in the graph of A^*. This is actually a characterisation:

$$A \quad \text{symmetric} \quad \Longleftrightarrow \quad \mathrm{Gr}(A) \subset \mathrm{Gr}(A^*).$$

Definition: The *adjoint* of an operator $A : D(A) \subset H_1 \to H_2$ defined on a dense domain $D(A)$ is an operator A^* defined on

$$D(A^*) := \{v \in H_2, \exists C(v) > 0, | < Au, v >_2 | \leq C(v) \|u\|_1 \quad \forall u \in D(A)\}$$

by

$$< Au, v >_2 = < u, A^* v >_1 \quad \forall u \in D(A), v \in D(A^*).$$

This defines A^* in an unique way in view of the density of the domain of A. For if v_1^* et v_2^* are elements in H_1 such that $< Au, v >_2 = < u, v_i^* >_1, i = 1, 2$ for any $u \in D(A)$, then $0 = < u, v_1^* - v_2^* >_1$ for any $u \in D(A)$ dense in E, which implies $v_1^* = v_2^*$.

Proposition: The domain $D(A^*)$ of the adjoint A^* of a closed and densely defined operator $A : D(A) \subset H_1 \to H_2$ is dense in H_1, and the adjoint is closed.
Proof: Let us assume that the domain of A^* is not dense in H_2. Then there exists a non zero vector $u \in H_2$ such that $< u, v >_2 = 0$ for any $v \in D(A^*)$. Using the closedness of A, this would imply that

$$\begin{aligned} < u, v >_2 = 0 \quad &\Leftrightarrow \quad < u, v >_2 + < 0, -A^* v >_1 = 0 \\ &\Leftrightarrow \quad (0, u) \in (Gr'(-A^*))^\perp = Gr(A) \end{aligned}$$

and hence $A(0) = u \neq 0$ which contradicts the linearity of A.
Let us check that A^* is closed. Let $(v_n, A^* v_n)$ be a Cauchy sequence in $D(A^*) \times H_1$ converging to $(y, x) \in H_2 \times H_1$. For any $u \in D(A)$, we have

$$< Au, y >_2 = \lim_{n \to \infty} < Au, v_n >_2 = \lim_{n \to \infty} < u, A^* v_n >_1 = < u, x >_1$$

so that $| < Au, y >_2 | \leq \|x\|_1 \cdot \|u\|_1$ for any $v \in D(A)$, which implies $y \in D(A^*)$ and $A^* y = u$. ∎

Whenever $E = F = H$ is a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_E = \langle \cdot, \cdot \rangle_F$, we say A is *self-adjoint* if $A = A^*$, i.e

$$< Au, v > = < u, Av > \quad \forall u \in D(A) \qquad \text{and} \quad D(A) = D(A^*).$$

Since

$$< (u, Au), (-A^*v, v) >=< u, -A^*v > + < Au, v >= 0 \quad \forall u \in D(A), \forall v \in D(A^*),$$

the graph of A^* relates to the graph of A by

$$(Gr'(-A^*))^\perp = \overline{Gr(A)}$$

where "prime" means the symmetric set w.r. to the diagonal axis. Whenever A is closed this reduces to $(\mathrm{Gr}'(-A^*))^\perp = \mathrm{Gr}(A)$.
Notice that if A is self-adjoint, then its graph coincides with the graph of A^*. This is actually a characterisation:

$$A \quad \mathrm{self-adjoint} \quad \Longleftrightarrow \quad \mathrm{Gr}(A) = \mathrm{Gr}(A^*).$$

Thus, a densely defined symmetric operator is self-adjoint whenever $\mathrm{Gr}(A^*) \subset \mathrm{Gr}(A)$.

Notice that when $E = F = H$ and A is self-adjoint then

(i)
$$\sigma_p(A) \subset I\!R.$$
Given $\lambda \in \sigma_p(A)$ and u an associated eigenvector, then $< Au, u >= \lambda < u, u >=< u, Au >= \bar{\lambda} < u, u >$. But since $\|u\| \neq 0$, it follows that $\bar{\lambda} = \lambda$ and $\lambda \in I\!R$.

(ii) Given $\lambda_i, i = 1, 2 \in \sigma_p(A)$, we have
$$\lambda_1 \neq \lambda_2 \Rightarrow Ker(A - \lambda_1 I) \perp Ker(A - \lambda_2 I).$$
Indeed, if $\lambda_1 \neq \lambda_2$ are two eigenvalues associated with the eigenvectors x_1 and x_2, then $< Ax_1, x_2 >= \lambda_1 < x_1, x_2 >=< x_1, Ax_2 >= \lambda_2 < x_1, x_2 >$, from which it follows that $< x_1, x_2 >= 0$.

Let $A : D(A) \subset H \to H$. When the space $H_p(A)$ spanned by the eigenvectors of A coincides with the total Hilbert space H, letting $\{e_n, n \in I\!N\}$ be an orthonormal basis of eigenvectors of H, the operator reads:

$$Au = \sum_n \lambda_n < u, e_n > \quad \forall u \in H.$$

Given a continuous bounded map $f : \sigma_p(A) \subset \mathbb{C} \to \mathbb{C}$, one can define the map $f(A)$ using this spectral representation of A on the domain:

$$D(f(A)) := \{u \in H, \sum_n f(\lambda_n)^2 < u, e_n >^2 < \infty\}$$

by

$$f(A)u = \sum_n f(\lambda_n) < u, e_n > e_n \quad \mathrm{for} u \in D(f(A)).$$

Let $A : D(A) \subset H_1 \to H_2$ be a closed operator defined on a dense domain $D(A)$ of a Hilbert space $(H_1, \langle \cdot, \cdot \rangle_1)$ with values in $(H_2, \langle \cdot, \cdot \rangle_2)$, then the domain $D(A^*)$ is dense in H_2 and we can define the adjoint A^{**} of A^* which coincides with A.

Let us first check that $A \subset A^{**}$. Given $x \in D(A)$, $| < x, A^*y > | = | < Ax, y > | \leq \|Ax\|\|y\|$ for any $y \in D(A^*)$ and hence $x \in D(A^{**})$. Since $< x, A^*y >=< Ax, y >$ the operators A and A^{**}

coincide on $D(A)$. Since A^{**} is closed as the adjoint of a closed operator, so is its graph and we have $Gr((A^*)^*) = (Gr'(-A^*))^\perp = Gr(A)$, for $Gr(A)$ is closed. This ends the proof of the identity $A^{**} = A$.

Another useful property for a densely defined operator $A : D(A) \subset H_1 \to H_2$ is that

$$(R(A))^\perp = Ker(A^*)$$

and hence

$$\mathrm{Ker}(A^*) \oplus \overline{R(A)} = H_2,$$

which applied to A^* yields:

$$(R(A^*))^\perp = Ker(A)$$

and hence

$$\mathrm{Ker}(A) \oplus \overline{R(A^*)} = H_1.$$

Indeed, we have $(R(A))^\perp \subset KerA^*$. For if $< y, Ax >= 0 \ \forall x \in D(A)$ then $y \in D(A^*)$ and $< A^*y, x >= 0 \ \forall x \in D(A)$. But $D(A)$ is dense in H so it follows that $y \in \mathrm{Ker}A^*$. Let us now check the other inclusion. Given $v \in \mathrm{Ker}A^*$, we have $< A^*v, u >= 0$ for any $u \in D(A)$ and hence $< v, A^{**}u >=< v, Au >= 0$ for any $u \in D(A)$ which shows that $v \in R(A)^\perp$.

Proposition: Let $A : D(A) \subset H$ be a densely defined symmetric operator. The following three statements are equivalent

1. A is self-adjoint.
2. A is closed and $\mathrm{Ker}(A^* + iI) = \mathrm{Ker}(A^* - iI) = \{0\}$.
3. $R(A + iI) = R(A - iI) = H$.

Proof:

- 1) $\Rightarrow$ 2). If $A^*u = iu$ for some non zero vector u in $D(A) = D(A^*)$, then

  $$i\langle u, u\rangle = \langle iu, u\rangle = \langle Au, u\rangle = \langle u, Au\rangle = \langle u, iu\rangle = -i\langle u, u\rangle$$

  so that $\|u\| = 0$ and $u = 0$.

- 2) $\Rightarrow$ 3). Since $\overline{R(A + iI)} \oplus \mathrm{Ker}(A^* + iI) = H$, $\mathrm{Ker}(A^* + iI) = 0$ implies that $R(A + iI)$ is dense in H. To prove that the range actually coincides with H, we therefore need to check it is closed. We first observe that

  $$\begin{aligned} \|(A + iI)u\|^2 &= \langle (A + iI)u, (A + iI)u\rangle \\ &= \langle (A - iI)(A + iI)u, u\rangle \\ &= \langle (A^2 + I)u, u\rangle \\ &= \|Au\|^2 + \|u\|^2. \end{aligned}$$

  In particular,

  $$\|(A + iI)u\|^2 \geq \|Au\|^2 \quad \text{and} \quad \|(A + iI)u\|^2 \geq \|u\|^2 \quad \forall u \in D(A).$$

  Let (u_n) be a sequence in $D(A)$ such that $((A + iI)u_n)$ converges to $v \in H$. The sequence $((A+iI)u_n)$ being a Cauchy sequence, by the above inequalities so are (u_n) and (Au_n) Cauchy sequences which therefore converge respectively to some vectors u and v in H. The operator A being closed by assumption, u lies in $D(A)$ and $v = Au$ so that the sequence $((A + iI)u_n)$ converges to $v + iu = (A + iI)u$. This shows that the range of $(A + iI)$ is closed; it therefore coincides with H. Similarly, we show that $R(A - iI) = H$.

- 3) $\Rightarrow$ 1). Since A is symmetric, all we need to show is that $D(A^*) \subset D(A)$. Let $v \in D(A^*)$; we want to show that v lies in $D(A)$. Since $R(A - iI) = H$, there is a vector $u \in D(A)$ such that $(A^* - iI)v = (A - iI)u$. On the other hand, $D(A) \subset D(A^*)$ so $v - u \in D(A^*)$ and the operator A being symmetric we have $A_{|_{D(A)}} = A^*_{|_{D(A)}}$. Hence $(A^* - iI)(v - u) = 0$ and $v - u$ lies in $\mathrm{Ker}(A^* - iI)$. On the other hand, combining the splitting $\overline{R(A + iI)} \oplus \mathrm{Ker}(A^* + iI) = H$ with the assumption $R(A + iI) = H$ shows that $\mathrm{Ker}(A^* + iI) = H$. Thus, $v = u$ so v lies in $D(A)$ which implies $D(A) = D(A^*)$.

■

4.4 Compact operators

Useful references are [9], [36].

Let E, F be two separable Banach spaces.

Definition: An operator $A \in \mathcal{B}(E, F)$ is *compact* whenever the range $A(B_E(0,1))$ of the unit ball $B_E(0,1)$ of E has compact closure.
Equivalently, an operator $A \in \mathcal{B}(E, F)$ is *compact* whenever given any bounded sequence (x_n) in E, one can extract from the sequence (Ax_n) a convergent sequence in F.

Let $\mathcal{K}(E, F)$ denote the set of compact operators from E to F. When $E = F$ we set $\mathcal{K}(E) := \mathcal{K}(E, E)$.

$\mathcal{K}(E)$ is a two sided ideal in $\mathcal{B}(E)$. For if $A \in \mathcal{K}(E)$, $B \in \mathcal{B}(E)$ and a bounded sequence (u_n) in E, (Bu_n) is also bounded. A being compact we can extract from (ABu_n) a convergent subsequence $(ABu_{\phi(n)})$ which shows that AB is compact. Similarly we show that the product BA is compact. Indeed, A being compact, we can extract a subsequence $(u_{\phi(n)})$ of (u_n) such that $(Au_{\phi(n)})$ converges and B being bounded, $(BAu_{\phi(n)})$ therefore converges. This shows that BA is compact.

Let us check that $\mathcal{K}(E)$ is closed in $\mathcal{B}(E)$. Let A_n be a sequence of compact operators converging to A. To show the compactness of $\overline{A(B(0,1))}$ and hence of A, it is sufficient to show we can cover the image ball $A(B(0,1))$ by a finite number of balls with given radius. Let $\varepsilon > 0$ and $n \in I\!N$ such that $|||A_n - A||| < \frac{\varepsilon}{2}$. Since A_n is compact, we can cover $A_n(B(0,1))$ by a finite number N of balls with radius $\varepsilon/2$, $A_n(B(0,1)) \subset \bigcup_{i=1}^N B(h_i, \varepsilon/2)$. This induces a covering of $A(B(0,1))$ by a finite number of balls of radius ε and ends the proof of the closedness of the set of compact operators. Hence $\mathcal{K}(E)$ becomes a Banach algebra for the operator norm $\| \cdot \|$.

When $E = H$, $\mathcal{K}(H)$ is a $*$-ideal i.e. $A \in \mathcal{K}(H) \Rightarrow A^* \in \mathcal{K}(H)$. Indeed, let A be compact and let us assume that A^* is not compact. Then there is a sequence (u_n) in the unit ball $B(0,1)$ such that $\|A^*u_n - A^*u_m\| \geq \varepsilon > 0$ for any $n, m \in I\!N$. Let $v_n = A^*u_n$, then

$$< Av_n - Av_m, u_n - u_m >= \|A^*u_n - A^*u_m\|^2 \geq \varepsilon^2$$

so that by the Cauchy-Schwartz inequality and using the fact that $\|u_n\| \leq 1$, we get

$$\varepsilon^2 \leq \|Av_n - Av_m\| \|u_n - u_m\| \leq 2\|Av_n - Av_m\|$$

in which case (Av_n) would not have a convergent subsequence. This would contradict the compactness of A.

Thus $\mathcal{K}(H)$ yields another example of C^*-algebra. Notice that unlike $\mathcal{B}(H)$ it does not contain a unit element.

An example of a compact operator is provided by Sobolev inclusions on closed manifolds (see e.g. [1], [17]). Given a closed manifold M and a vector bundle E based on M, using a partition of the unity, one can define for any $s \in I\!R$, the H^s Sobolev closure $H^s(E)$ of the space $C^\infty(E)$ of smooth sections of E (se e.g. [17]). For $t < s$, the inclusion $i : H^s(E) \to H^t(E)$ is a compact operator.

4.5 Finite rank operators

A useful reference is [9].

Since the compactness of the unit ball in a normed space implies that this space is finite dimensional, a natural question is whether one can approximate compact operators by operators with finite dimensional range. A bounded operator $A \in \mathcal{B}(E, F)$ has *finite rank* if its range is finite dimensional. The dimension of the range is called the *rank* of the operator.
A simple example is given by finite rank projection operators. Let (e_k) be a complete orthonormal basis of a separable Hilbert space H, the projection operator P_k defined by $P_k e_j \equiv e_j$ if $j \leq k$, $P_k e_j \equiv 0$ otherwise, is a finite rank operator since its range has dimension k.

We now relate compact operators to finite rank ones.
Any finite rank operator is compact, since the closure of the range $A(B(0,1))$ of the unit ball by a finite rank operator A is compact as a closed and bounded subset of a finite dimensional space. Hence, any limit (in the operator norm of bounded operators) of finite rank operators is compact since the set of compact operators is closed in the set of bounded operators.

Conversely, when F is a Hilbert space, let us show that any compact operator can be obtained as a limit (in the operator norm of bounded operators) of finite rank operators. Let A be a compact operator; the closure $\overline{A(B(0,1))}$ of the range $A(B(0,1))$ of the unit ball is compact. Given $\varepsilon > 0$, it can therefore be covered by a finite number of balls $B(h_i, \frac{\varepsilon}{2})$ centered at h_i with given radius ε in such a way that $\bar{A}(B(0,1)) \subset \cup_{i=1}^N B(h_i, \frac{\varepsilon}{2})$. Let F denote the subspace generated by $h_i, i = 1, \cdots, N$ and let P_F denote the orthogonal projection (hence the need for a Hilbert structure on the target space) onto F. Then $A_\varepsilon \equiv P_F A$ has finite rank. Let us check that $|||A_\varepsilon - A||| \leq \varepsilon$. Since $x \in B(0,1)$ there is some $i_0 \in \{1, \cdots, N\}$ such that $\|Ah - h_{i_0}\| < \frac{\varepsilon}{2}$. Since $\|P_F\| \leq 1$, this implies that $\|P_F Ah - P_F h_{i_0}\| < \frac{\varepsilon}{2}$ and hence $\|P_F Ah - h_{i_0}\| < \frac{\varepsilon}{2}$. It follows that $\|P_F Ah - Ah\| < 2\varepsilon$ for any $h \in B(0,1)$ and hence $|||A_\varepsilon - A||| < \varepsilon$, which shows that the compact operator A could be approximated by finite rank ones A_ε.

Here again, an example can be found on l_2 sequences. To a sequence (α_n) of real numbers converging to 0, we can associate a compact operator

$$\begin{aligned} A : l_2 &\to l_2 \\ (u_n) &\mapsto (\alpha_n u_n). \end{aligned}$$

It is compact as limit of operators of finite rank A_k defined by

$$A_k((u_n)) = (\alpha_0 u_0, \cdots, \alpha_k u_k, 0, \cdots, 0).$$

4.6 Fredholm operators

Useful references are [17], [31], [40], [44].

Let E and F be two separable Banach spaces .
An operator $A \in \mathcal{B}(E, F)$ is *Fredholm* whenever it is invertible "up to a compact operator", i.e. whenever there are operators $B \in \mathcal{B}(F, E)$ and $C \in \mathcal{B}(F, E)$ such that $BA - I$ et $AC - I$ are compact.

Example:

$$\begin{array}{rcl} A : l^2 & \to & l^2 \\ (u_n) & \mapsto & (\frac{n}{\sqrt{n^2+1}} u_n) \end{array}$$

is a Fredholm operator.

Any operator $I - K$ with $K \in \mathcal{K}(E)$, is Fredholm. Note that if $A \in \mathcal{B}(E, F)$ is a Fredholm operator, then $A + K$ is also Fredholm for any operator $K \in \mathcal{K}(E, F)$. The set $\mathcal{F}(E, F)$ of Fredholm operators is open in $\mathcal{B}(E, F)$.
If E and F are Hilbert spaces, since the adjoint of a compact operator is compact, the adjoint of a Fredholm operator is Fredholm.

Proposition: Let H_1 and H_2 be two separable Hilbert spaces and let $A \in \mathcal{B}(H_1, H_2)$. The following conditions are equivalent:

i) A is Fredholm

ii) $\mathrm{Ker} A$ and $\mathrm{Ker} A^*$ are finite dimensional and $\mathrm{R}(A)$ et $\mathrm{R}(A^*)$ are closed.

iii) The kernels $\mathrm{Ker} A$ et $\mathrm{Ker} A^*$ are finite dimensional and

$$H_1 = \mathrm{Ker}(A) \oplus \mathrm{R}(A^*),$$

$$H_2 = \mathrm{Ker}(A^*) \oplus \mathrm{R}(A)$$

where the sums are orthogonal.

Proof:

(i) $\Rightarrow$ (ii): We show that $\mathrm{Ker}(A)$ is finite dimensional. Let (u_n) be a sequence in the unit sphere of $\mathrm{Ker}(A)$ so that $\|u_n\| = 1$. Then $u_n = (I - BA)u_n$ and since $I - BA$ is compact, we can extract from (u_n) a convergent subsequence, which proves that the unit ball of $\mathrm{Ker}(A)$ is compact and hence by the Bolzano-Weierstrass theorem (hence the need for separability) that $\mathrm{Ker} A$ is finite dimensional.

Let us now check that the range of A is closed. Let (u_n) be a sequence in $D(A)$ and let us assume that $v_n := Au_n \to v$. Without any restriction, we can also assume that (u_n) lies in $\mathrm{Ker} A^\perp$.
Let us first assume that (u_n) is bounded. Since $u_n = Bv_n + (I - BA)u_n$, $I - BA$ being compact, we can extract a convergent subsequence $(u_{\phi(n)})$ such that $(I - BA)(u_{\phi(n)})$ converges to some w. B being compact and hence bounded, the sequence Bv_n tends to Bv so that that $u_{\phi(n)}$ tends to some $u := Bv + w$. Thus $v = \lim_n v_{\phi(n)} = \lim_n Au_{\phi(n)} = Au$ lies in the range of A so that $\mathrm{R}(A)$ is closed.
If now the sequence (u_n) is non bounded, then $\|u_n\|$ tends to $+\infty$ when $n \to +\infty$. Applying the result obtained in the bounded case to $u'_n \equiv \frac{u_n}{\|u_n\|}$ yields a subsequence $(u'_{\phi(n)}) \in \mathrm{Ker} A^\perp$ that converges to u' such that $\|u'\| = 1$ and $Au' = \lim_n \frac{Au_n}{\|u_n\|} = \lim_n \frac{v_n}{\|u_n\|} = 0$. Since A is closed $\mathrm{Ker} A$ is also closed and $u' \in \mathrm{Ker} A$ which leads to a contradiction. Since the adjoint of a Fredholm operator is Fredholm, $KerA^*$, resp. $\mathrm{R}(A^*)$ are finite dimensional, resp. closed.

(ii) $\Rightarrow$ (iii): Given a closed operator A and densely defined on H_1, we know that

$$\mathrm{Ker}A^* + \overline{\mathrm{R}(A)} = H_2 \tag{2}$$

and

$$\mathrm{Ker}A^{**} + \overline{\mathrm{R}(A^*)} = H_1 \tag{3}$$

Since $\mathrm{R}(A)$ and $\mathrm{R}(A^*)$ are closed it follows that

$$\mathrm{Ker}A^* + \mathrm{R}(A) = H_2$$

and

$$\mathrm{Ker}A + \mathrm{R}(A^*) = H_1.$$

(iii) $\Rightarrow$ (i): A is bijective from $\mathrm{Ker}A^\perp = R(A^*)$ onto $R(A) = Ker(A^*)^\perp$, so we can find two operators C defined on $\mathrm{R}(A)$ and D defined on $\mathrm{R}(A^*)$ such that $CA = I/\mathrm{Ker}A^\perp$ and $AD = I/(\mathrm{Ker}A^*)^\perp$. Let us denote by the same symbols C resp. the extension of C by 0 on $\mathrm{Ker}A^*$, resp. of D by 0 on $\mathrm{Ker}A$. They are bounded operators by the closed graph theorem and by construction we have $I - CA = \pi_{\mathrm{Ker}A}$ and $I - AD = \pi_{\mathrm{Ker}A^*}$ where π_F is the orthogonal projection on the vector space F. Since these two projections have finite rank, it follows that A is Fredholm.

■

We henceforth assume that the Hilbert spaces are separable.
Given a Fredholm operator $A : H_1 \to H_2$, the *index of* A is the positive integer given by

$$\begin{aligned} \mathrm{ind}(A) &:= \dim\, \mathrm{Ker}(A) - \dim\, \mathrm{Ker}(A^*) \\ &= \dim\, \mathrm{Ker}(A) - \mathrm{codim}\, \mathrm{R}(A) \end{aligned}$$

since the range of A and the kernel of A^* are topological complements in H_2.
It follows from the definition that $\mathrm{ind}(A^*) = -\mathrm{ind}(A)$.

Further important properties of the index are

Proposition:

(i) Given two Fredholm operators A and B then their product AB is Fredholm and

$$\mathrm{ind}(AB) = \mathrm{ind}(A) + \mathrm{ind}(B).$$

(ii) The index map $ind : \mathcal{F}(H_1, H_2) \to \mathbb{Z}$ is continuous and locally constant on the set of Fredholm operators.

Partial proof: Let us show the relation between the indices in (i), leaving the proof of the Fredholm property of the product as an exercise. We write

$$\mathrm{Ker}(AB) = \mathrm{Ker}(B) \oplus B^{-1} \left(R(B) \cap \mathrm{Ker}(A) \right) |_{\mathrm{Ker}(B)^\perp}$$

$$\begin{aligned} \mathrm{Ker}(B^* A^*) &= \mathrm{Ker}(A^*) \oplus (A^*)^{-1} \left(R(A^*) \cap \mathrm{Ker}(B^*) \right)_{\mathrm{Ker}(A^*)^\perp} \\ &= \mathrm{Ker}(A^*) \oplus (A^*)^{-1} \left(\mathrm{Ker}(A)^\perp \cap \mathrm{R}(B)^\perp \right)_{\mathrm{Ker}(A^*)^\perp}. \end{aligned}$$

Hence

$$
\begin{aligned}
\operatorname{ind}(AB) &= \dim \operatorname{Ker}(AB) - \dim \operatorname{Ker}(B^*A^*) \\
&= \dim \operatorname{Ker}(B) + \dim \left(R(B) \cap \operatorname{Ker}(A)\right) \\
&- \dim \operatorname{Ker}(A^*) - \dim \left(\operatorname{Ker}(A)^{\perp} \cap \mathrm{R}(B^*)\right) \\
&= \dim \operatorname{Ker}(B) + \dim \operatorname{Ker}(A) \\
&- \dim \left(\operatorname{Ker}(B^*) \cap \operatorname{Ker}(A)\right) - \dim \operatorname{Ker}(A^*) \\
&- \dim \left(\operatorname{Ker}(A)^{\perp} \cap \operatorname{Ker}(B^*)\right) \\
&= \dim \operatorname{Ker}\ (A) + \dim \operatorname{Ker}(B) - \dim \operatorname{Ker}(A^*) - \dim \operatorname{Ker}(B^*) \\
&= \operatorname{ind}(A) + \operatorname{ind}(B)
\end{aligned}
$$

To show (ii), let $T : H_1 \to H_2$ be a fixed Fredholm operator and let us consider the spaces $\tilde{H}_1 := \operatorname{Ker}(T^*) \oplus H_1$ and $\tilde{H}_2 = \operatorname{Ker}(T) \oplus H_2$ together with the map

$$
\begin{array}{rcl}
Hom(H_1, H_2) & \to & Hom(\tilde{H}_1, \tilde{H}_2) \\
S & \mapsto & \tilde{S}(u, h) := \pi_T h \oplus (u + S(h))
\end{array}
$$

where π_T denotes the orthogonal projection onto $\operatorname{Ker}T$. The map $S \to \tilde{S}$ is continuous since $\|\tilde{S} - \tilde{T}\| = \|S - T\|$. We check that $\tilde{S} \in Iso(\tilde{E}, \tilde{F})$. Since $\tilde{T}$ is an isomorphism and since $Iso(\tilde{H}_1, \tilde{H}_2)$ is open in $Hom(\tilde{H}_1, \tilde{H}_2)$, so is $\tilde{S}$ an isomorphism for some small perturbation S of T. Hence S is Fredholm and since $\operatorname{ind}(\tilde{S}) = 0$ we have $\operatorname{ind}(S) = \operatorname{ind}(T)$ which shows that the index is locally constant. ∎

Let us illustrate the notion of index by the example quoted at teh beginning of this chapter. Let $A : l_2 \to l_2$ be the left translation operator on the set l_2 of L^2 convergent sequences defined by

$$A((u_n)) := (v_n), \quad v_n = u_{n+1}.$$

A is onto and has one dimensional kernel. Its adjoint A^* defined on l_2 by

$$A^*((u_n)) := (v_n), \quad v_0 = 0, v_n = u_{n-1} \quad \text{if} \quad n \neq 0$$

is one to one with range given by the closed set of sequences with vanishing first term. The operator A is therefore Fredholm with index 1. It is easy to check that the index of A^n is n and the index of $(A^*)^n$ is $-n$, which shows that the index is surjective onto $\mathbb{Z}$.

4.7 Differential operators; Laplacians

Useful references are [6], [15], [20], [31], [45].

We need some notations:
For a multi index $\alpha := (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, let us set $|\alpha| := \sum_{k=1}^n \alpha_k$ and for $\xi \in \mathbb{R}^n, \xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.
We also set $D_x^\alpha := (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ with $\frac{\partial^{|\alpha|}}{\partial x^\alpha} := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$.
With these notations we have $\mathcal{F}\left(D^\alpha f\right)(\xi) = \xi^\alpha \hat{f}(\xi)$ for any Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$ where $\mathcal{F}$ also denoted by $\hat{\ }$ denotes the Fourier transform $\hat{f}(\xi) := (2\pi)^{-\frac{n}{2}} \int e^{ix\cdot\xi} f(x) dx$.
In what follows, $K := \mathbb{R}$ or $K := \mathbb{C}$.

A *differential operator* of order m on an open subset U of $\mathbb{R}^n$ is a linear map

$$A : C^k(U, K^p) \to C^{k-m}(U, K^q)$$

of the form

$$A = \sum_{|\alpha| \le m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha} \tag{4}$$

where $a_\alpha(x)$ is a (q,p) matrix of smooth K-valued functions with $a_\alpha \neq 0$ for some α and such that $|\alpha| = m$ (i.e. A differentiates m times).

The Laplacian $\Delta := -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ on $U := I\!R^n$ provides an example of differential operator of order 2.

A change of coordinates $\tilde{x} = \tilde{x}(x)$ on U gives for any $j \in \{1, \cdots, n\}$:

$$\frac{\partial}{\partial x_j} = \sum_{k=1}^n \frac{\partial \tilde{x}_k}{\partial x_j} \frac{\partial}{\partial \tilde{x}_k}.$$

As a consequence, in the new coordinates, the operator A reads

$$A = \sum_{|\alpha| \le m} \tilde{a}_\alpha(\tilde{x}) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

for some other (q,p) matrix $\tilde{a}_\alpha$ of smooth K-valued functions on U, which shows that the operator can be described in a similar way in these new coordinates. If we have $a_\alpha(x) = 0$ for all $|\alpha| = m$, then $\tilde{a}_\beta(\tilde{x}) = 0$ for all $|\beta| = m$ so that the order is conserved under a change of coordinates.

Given a smooth manifold M of dimension n, it therefore makes sense to define a differential operator $A : C^\infty(M, I\!R) \to C^\infty(M, I\!R)$ of order m as a linear operator which has the above description (4) in any local chart of M since another local chart would give the same type of local description via a change of coordinates.

Let $GL_r(K)$ be the group of invertible K-valued $r \times r$ matrices. Given two maps $\tau_1 : U \to GL_p(K)$ and $\tau_2 : U \to GL_q(K)$, $\tau_2 A \tau_1$ defines another differential operator or order m on U since

$$\tau_2 A \tau_1 = \sum_{|\alpha| \le m} \tau_2 a_\alpha(x) \tau_1 \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

is of the same type as (4). Letting $\pi_E : E \to M$ and $\pi_F : F \to M$ be two vector bundles based on M of rank p, q respectively, this shows that the shape of the operator described in (4) is invariant under a change of trivialization $\tau_E : U \to GL_p(K)$ on E and $\tau_F : U \to GL_q(K)$ on F.

It therefore makes sense to set the following

Definition: A *differential operator* of order m on a smooth manifold M is a linear map $A : C^\infty(E) \to C^\infty(F)$, where E, F are two vector bundles based on M of rank p, q over K respectively, such that each point x of M has a neighborhood U with coordinates $(x_1, \cdots, x_n)$ over which there is a local trivialization $E_{|_U} \simeq U \times K^p$ and $F_{|_U} \simeq U \times K^q$ in which the operator A reads

$$A = \sum_{|\alpha| \le m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

The Laplace Beltrami operator: Let (M, g) be a Riemannian manifold with Riemannian metric

g. Here we take $E = F = M \times \mathbb{C}$. In a local chart, the metric reads $g(x) = g_{ij}(x)dx_i dx_j$; let (g^{ij}) denotes the inverse matrix of (g_{ij}) and $\det g$ the determinant of (g_{ij}). The Laplace-Beltrami operator is defined by

$$\begin{aligned}\Delta_g &:= -\frac{1}{\det g}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\sqrt{\det g}\, g^{ij}\frac{\partial}{\partial x_j}\\ &= -\sum_{i,j=1}^{n} g^{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \text{terms of lower order.}\end{aligned}$$

Notice that setting $g_{ij} = \delta_{ij}$, the Kronecker symbol of $\{i,j\}$ yields back the local expression of the Laplacian on $\mathbb{R}^n$.
A C^2-function is *harmonic* if $\Delta_g f = 0$. Equivalently, it minimises the energy functional

$$E(f) = \int_M \|df\|_x^2 \, d\mathrm{vol}(x)$$

where $d\mathrm{vol}(x)$ is the volume form associated to the Riemannian structure and $\|\cdot\|_x$ is the norm on 1-forms induced by the metric via the inner product on one forms. When M is compact, as we shall see later in these notes, this is a finite dimensional space as the kernel of elliptic operator.

Laplacian on sections of a vector bundle: Let $E \to M$ be a vector bundle based on a Riemannian manifold M and let E be equipped with a connection ∇^E. The Levi-Civita connection ∇ on M yields a connection ∇^{T^*M} on T^*M which, when combined with ∇^E, yields a connection $\nabla^{T^*M\otimes E} = \nabla^{T^*M} \otimes 1 + 1 \otimes \nabla^E$ on $T^*M \otimes E$. Composed with ∇^E, this yields an operator $\nabla^{T^*M\otimes E}\nabla^E : C^\infty(E) \to C^\infty(T^*M \otimes T^*M \otimes E)$. Using the metric on $C^\infty(TM \otimes TM)$, by contraction, one can build its trace to obtain a second-order differential operator

$$\Delta^E := -\mathrm{tr}(\nabla^{T^*M\otimes E}\nabla^E)$$

called a Laplacian on $C^\infty(E)$.

When $E := M \times K$, and $\nabla^E = \nabla$, the Levi-Civita connection on M, it yields back the Laplace-Beltrami operator and we have:

$$\Delta_g = -\mathrm{div} \circ \nabla = \nabla^* \nabla$$

where div denotes the divergence defined by:

$$-\langle \mathrm{div} U, f\rangle = \langle U, \nabla f\rangle \quad \forall f \in C^\infty(M), U \in C^\infty(TM)$$

and ∇^* the adjoint (in the operator sense) of the connection ∇. Indeed, in local coordinates the divergence reads $\mathrm{div} U = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_j}(U^j\sqrt{g})$, which combined with the local formula $(\nabla f)^j = \sum_{i,j=1}^n g^{ij}\partial_i f$ yields the first identity.

More generally, take $E = \Lambda T^*M$ equipped with the connection ∇^E induced by the Levi-Civita connection on M, then

$$\Delta^{\Lambda T^*M} = -\mathrm{tr}(\nabla^{T^*M\otimes \Lambda T^*M}\nabla^{\Lambda T^*M}) = \nabla^*\nabla$$

where we have set for short $\nabla = \nabla^{\Lambda T^*M}$ and ∇^* its adjoint.
We shall come across another second order elliptic operator $(d + d^*)^2$ acting on forms with the same leading symbol $\sigma(x,\xi) = |\xi|^2$ as $\Delta^{\Lambda T^*M}$, which relates to $\Delta^{\Lambda T^*M}$ by the Bochner-Weizenböck formula.

4.8 Dirac operators

Following Dirac, we look for a differential operator D^E whose square is a Laplacian Δ^E. When E is the trivial bundle $E = I\!R^n \times \mathbb{C}$, Δ^E is the ordinary Laplacian $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial^2 x_i}$ on $I\!R^n$. Looking for an operator $D = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i}$ such that $D^2 = \Delta$ leads to the Clifford relations of Section 3.5, namely:

$$c_i c_j + c_j c_i = -2\delta_{ij}.$$

$D = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i}$ then yields a first order differential oprator which provides a square root of the Laplacian. Extending this construction to search for a square root of a Laplacian Δ^E acting on sections of a vector bundle E, requires the use of a Clifford connection. We use here the notations of section 3.5. Let M be a Riemannian manifold and let $E \to M$ be a Clifford module based on M. A connection ∇^E on E is called a *Clifford connection* if it commutes with the Clifford multiplication on E:

$$[\nabla^E, c(a)] = c(\nabla a) \quad \forall a \in C^\infty(C(M)),$$

where $C(M)$ is the bundle of Clifford algebras on M. Here ∇ is the Levi-Civita connection on M.

Examples:

- **The exterior power of the cotangent bundle:** Let $E = \Lambda T^*M$ be equipped with the connection ∇^E induced by the Levi-Civita connection on M. Then ∇^E is a Clifford connection for the Clifford action $c(v) = \varepsilon(v) - i(v) \quad \forall v \in C^\infty(TM)$ on ΛT^*M seen as a Clifford module.
- **Spinor bundles:** Let $E = S \otimes W$ where S is the spinor bundle on M. The Clifford module acts on E via a Clifford multiplication c. Since $Spin(V)$ is a finite covering of $SO(V)$, we can lift the Levi-Civita connection on $SO(TM)$ to a connection on the spinor bundle S. Combined with a connection ∇^W on W, it yields a Clifford connection $\nabla^{S\otimes W}$ on $S \otimes W$.
- **Spin^c bundles:** Let M be a Spin^c-manifold, so that the orthonormal frame bundle $SO(TM) \to M$ lifts to some Spin$^c(V)$-bundle where V is the model space for M. Since Spin$^c(V) \to SO(V)$ is not a finite covering, the Levi-Civita connection on M does not automatically lift to a connection on Spin$^c(V)$. We need additional information, namely a connection on the $SO(V) \times S^1$-bundle obtained from the quotient of Spin$^c(V)$ by $\pm$, which lifts to a connection on Spin$^c(V)$. This connection $\tilde{\nabla}$ is obtained from combining the Levi-Civita connection on M with a connection on the $U(1)$ bundle obtained from Spin$^c(V)$ by dividing out by $Spin(V)$.

From a Clifford connection ∇^E one can build a first order differential operator

$$D^E := \sum_{i=1}^n c(e^i)\nabla^E_{e_i}$$

called a *Dirac operator.*

Examples:

- **A Dirac operator associated to the de Rham operator:** In the first of the above examples, using the fact that the Levi-Civita connection ∇ relates to the exterior differential by $d = \varepsilon \circ \nabla$ and to its adjoint by $d^* = -i \circ \nabla$, we can write $d + d^* = c \circ \nabla$ where $c := \varepsilon - i$ defines a Clifford multiplication on $\Omega(M)$. The operator $d + d^*$ can therefore be interpreted as a Dirac operator acting on sections of the Clifford module ΛT^*M:

$$D^{T^*M} := \sum_{i=1}^n c(e^i)\nabla^E_{e^i} = d + d^*.$$

- **The twisted Dirac operator:** When M is a spin manifold, S the spinor bundle and W an exterior vector bundle based on M with connection ∇^W, the operator

$$D^{S\otimes W} := \sum_{i=1}^{n} c(e^i)\nabla_{e_i}^{S\otimes W}$$

is called a *twisted Dirac operator*. In the absence of exterior bundle W, i.e. when $E = S$, it is often denoted by D and simply called the *Dirac operator on M*. When the dimension of M is odd, $D^{S\otimes W}$ is an essentially self-adjoint Dirac operator (on the adequate domain), when the dimension is even, $E = E^+ \oplus E^- = S^+ \otimes W \oplus S^- \otimes W$ is $\mathbb{Z}_2$-graded and D is odd for this grading, i.e. $D^{S\otimes W} = \begin{pmatrix} 0 & \left(D^{S\otimes W}\right)^- \\ \left(D^{S\otimes W}\right)^+ & 0 \end{pmatrix}$.

- **Dirac operators for Spin^c-structures:** To a connection $\tilde{\nabla}$ on a Spin^c-bundle, there is also an associated Dirac operator :

$$\tilde{D} = \sum_{i=1}^{n} c(e_i)\tilde{\nabla}_{e_i}.$$

In general, the square of a Dirac operator D^E only coincides with the Laplacian Δ^E only up to a zeroth order differential operator as we shall see from the Lichnerowicz and Bochner-Weitzenböck formulae below.

- The square of $D^{\Lambda T^*M}$ yields the *Hodge Laplacian*

$$\Delta = d^*d + dd^*$$

acting on forms on a smooth manifold M. It relates to the Laplacian $\Delta^{\Lambda T^*M}$ by a *Bochner-Weizenböck* relation:

Proposition

$$(d+d^*)^2 = \Delta^{\Lambda T^*M} + \sum_{i<j} c(dx_i)c(dx_j)\Omega(e_i, e_j)$$

where $\Omega(u, v) := [\nabla_u, \nabla_v] - \nabla_{[u,v]}$ is the curvature tensor on ΛT^*M equipped with the connection induced by the Levi-Civita connection.

Proof: Since the operators involved in the equality to be proven are differential operators and since the curvature operator is tensorial, the proof can be carried out choosing an orthornormal tangent frames $(e_1(x), \cdots, e_n(x))$ at a given point x and does not depend on the way we extend it to a field of orthonormal frames in a neighborhood of x. We choose to extend it to a field of orthonormal frames $(e_1, \cdots, e_n)$ such that $(\nabla e_j)_x = 0$ at point $x \in M$.

$$\begin{aligned}
(d+d^*)^2\alpha &= \sum_{i,j=1}^{n} c(dx_i)\nabla_i(c(dx_j)\nabla_j\alpha) \\
&= \sum_{i,j=1}^{n} c(dx_i)c(dx_j)\nabla_i(\nabla_j\alpha) \\
&= -\nabla_i \sum_{i=1}^{n} \nabla_i\nabla_j\alpha + \sum_{i<j}^{n} c(dx_i)c(dx_j)\left(\nabla_i\nabla_j - \nabla_j\nabla_i\right)\alpha \\
&= \Delta^{\Lambda T^*M}\alpha - \sum_{i<j} c(dx_i)c(dx_j)\Omega(e_i, e_j)\alpha.
\end{aligned}$$

- In order to relate the square of the twisted Dirac operator $D^{S\otimes W}$ on a spin manifold with $\Delta^{S\otimes W}$ we need the notion of twisting curvature. Let us set $E = S\otimes W$ equipped with the connection ∇^E as before. We first observe that the curvature Ω^E of ∇^E decomposes under the isomorphism $End(E) \simeq C(M)\otimes End_{C(M)}(E)$ as follows

$$\left(\nabla^E\right)^2 = R^E + F^{E/S}$$

where R^E is a $C(M)$- valued 2-form on M induced by the action on E of the Riemannian curvature R of M given by the formula

$$R^E(e_i,e_j) = \frac{1}{4}\sum_{k,l=1}^{n}\langle\Omega(e_i,e_j)e_k,e_l)c(e^k)c(e^l)\rangle$$

where $e_i, i=1,\cdots,n$ is an orthonormal frame of the tangent bundle TM and $e^i, i=1,\cdots,n$ the dual frame and where Ω is the curvature of ∇ as in section 2.5. The remaining two form $F^{E/S}$ is called the *twisting curvature*. When $E = S\otimes W$, the twisted curvature $F^{E/S}$ coincides with the curvature Ω^W of ∇^W.

The square of a Dirac operator D^E differs from the Laplacian Δ^E by a term involving the scalar curvature of M and the twisting curvature of the Clifford connection ∇^E as can be seen from the *Lichnerowicz formula*:

Proposition:

$$\left(D^E\right)^2 = \Delta^E + c(F^{E/S}) + \frac{s}{4}$$

where s is the scalar curvature of M and

$$c(F^{E/S}) = \sum_{i<j} F^{E/S}(e_i,e_j)c(e^i)c(e^j).$$

When $E = S\otimes W$, the Lichnerowicz formula reads:

$$\left(D^W\right)^2 = \Delta^W + \sum_{i<j}\Omega^W(e_i,e_j)c(e^i)c(e^j) + \frac{s}{4}$$

where Δ^W is the Laplacian built from the connection $\nabla^E = \nabla\otimes 1 + 1\otimes\nabla^W$.

The Lichnerowicz formula further reduces to:

$$D^2 = \Delta + \frac{s}{4}$$

for the ordinary Dirac operator D on M (with no exterior bundle).

4.9 Elliptic operators; generalised Laplacians

Letting x and $\tilde{x} = \tilde{x}(x)$ be two systems of coordinates on an open subset U of M, the Schwartz property by which one can exchange partial differentiations yields for $|\alpha| = m$:

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \sum_{|\beta|=m}\left[\frac{\partial\tilde{x}}{\partial x}\right]_\beta^\alpha \frac{\partial^{|\beta|}}{\partial\tilde{x}^\beta}$$

where $\left[\frac{\partial\tilde{x}}{\partial x}\right]$ is the symmetrization of the m-th tensor product of the matrix $\left(\frac{\partial\tilde{x}}{\partial x}\right)$. Hence for any $|\alpha| = m$, the matrices $a_\alpha(x)$ transforms to:

$$\tilde{a}_\alpha(\tilde{x}) = \sum_{|\beta|=m} a_\beta(x)\left[\frac{\partial\tilde{x}}{\partial x}\right]_\beta^\alpha.$$

Thus the expression $\sum_{|\alpha|=m} a_\alpha(x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ extends to a section of $\otimes_{sym}^m TM \otimes Hom(E,F)$ called the *leading symbol* of A and denoted by $\sigma_L(A)$.

In the same way that we identify the symmetric tensor power $\otimes_{sym}^m V$ of a vector space V to the set of homogeneous polynomials of degree m on the dual space V^*, we can see the leading symbol $\sigma_L(A)(x)$ at point x of a differential operator $A = \sum_{\alpha|\leq m} a_\alpha(x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ as a homogeneous polynomial of degree m in the variable $\xi \in T_x^*X$:

$$\sigma_L(A)(x)(\xi) = i^m \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha,$$

where we have replaced $\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ by $i^\alpha \xi^\alpha$.

It is easy to check that

$$\sigma_L(AB) = \sigma_L(A)\sigma_L(B)$$

for two differential operators A and B and whenever the bundle E is Hermitian

$$\sigma_L(A^*) = \sigma_L(A)^*.$$

We call a differential operator of order m *elliptic* whenever its leading symbol $\sigma_L(A)(x)(\xi)$ in $Hom(E_x, F_x)$ is invertible for any non zero $\xi \in T_x^*M$ and any $x \in M$.

From the above properties of the leading symbol, it follows that if A is elliptic then so is A^*, and hence so is A^*A.
In fact the injectivity of the leading symbol of A is enough to have the ellipticity of A^*A since $\sigma_L(A)(x)(\xi)$ implies that $\sigma_L(A^*)(x)(\xi) = (\sigma_L(A)(x)(\xi))^*$ is onto and hence that $\sigma_L(A^*A)$ is bijective as a product of $(\sigma_L(A)(x)(\xi))^*$ and $\sigma_L(A)(x)(\xi)$.

Generalised Laplacians: The Laplace-Beltrami operator has leading symbol given by:

$$\sigma_L(\Delta_g)(x)(\xi) = \sum_{i,j=1}^n g^{ij}(x)\xi_i\xi_j = \|\xi\|^2$$

where the norm is defined by the scalar product on the cotangent space induced by the metric on M. More generally, we call *generalised Laplacian* on a vector bundle E a second order differential operator H such that $\sigma_L(H)(x)(\xi) = \|\xi\|^2$. The Laplacian acting on sections of E described above is a generalised Laplacian.

A generalised Laplacian H is elliptic since $\sigma_L(H)(x)(\xi) = \|\xi\|^2$ is invertible whenever $\xi \neq 0$.

The leading symbol of a Dirac operator is given by $\sigma_L(D)(x)(\xi) = ic(\xi)$. One can easily check that its square yields back the leading symbol of the Laplacian since $c(\xi)^2 = \|\xi\|^2$; it is therefore a first order differential operator whose square is a generalised Laplacian.

The total symbol: The Fourier transform description of differential operators, called the *momentum space* description, is often useful in physics. Given a point $x \in M$ and a local trivialization on a subset U containing x, we identify T_x^*M with $I\!R^n$, x with a point in $I\!R^n$, and write a local section u of the trivialised bundle E over U:

$$u(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{I\!R^n} e^{i\langle x,\xi\rangle_x} \hat{u}(\xi) d\xi$$

where $\langle x, \xi\rangle_x$ is the inner product on $I\!R^n$ induced by the Riemannian metric on X. Then a straightforward calculation yields:

$$(Au)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{I\!R^n} e^{i\langle x,\xi\rangle_x} \sigma(A)(x,\xi)\hat{u}(\xi) d\xi$$

where

$$\sigma(A)(x)(\xi) := \sum_{|\alpha|\leq m} i^{|\alpha|} a_\alpha(x)\xi^\alpha$$

is the *total symbol* of A. Unlike the leading symbol, the total symbol is only locally defined.

For a differential operator, the total symbol is polynomial in ξ; allowing non polynomial functions with some growth conditions leads to pseudo-differential operators.

4.10 Pseudo-differential operators on manifolds

Useful references are [31], [49], [50].

In what follows we shall only briefly sketch the definitions and properties which can be of use to us later on.

A *pseudo-differential* operator of order m on a closed Riemannian manifold M is a linear operator $A : C^\infty(E) \to C^\infty(F)$ where E, F are two vector bundles based on M of rank p, q respectively, such that each point x of M has a neighborhood U with coordinates $(x_1, \cdots, x_n)$ over which there are local trivializations $E_{|_U} \simeq U \times \mathbb{C}^p$ and $F_{|_U} \simeq U \times \mathbb{C}^q$ in which the operator A reads

$$Au(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{I\!R^n} e^{i\langle x,\xi\rangle_x} a(x,\xi)\hat{u}(x) d\xi,$$

where the $p \times q$-matrix valued function $a(x, \xi)$ (called the *symbol of* A) obeys the following growth condition. For any multiindices α, β there is a constant $C_{\alpha,\beta}$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1 + \|\xi\|)^{m-|\beta|} \quad \forall (x,\xi) \in T^*X.$$

Let Sym^m be the set of symbols satisfying this requirement.
One can check that this condition is satisfied whenever a is polynomial of degree m in ξ.
Also the inverse $(1 + \Delta_g)^{-1}$ is a pseudo-differential operator of order -2.

A useful notation is the following; The symbol a is said to have the formal development

$$a \sim \sum_{j=1}^{\infty} a_j, \quad a_j \in Sym^{m_j}$$

if for each integer m there exists some integer K for which $a - \sum_{j=1}^K a_j \in Sym^{-m} \quad \forall k \geq K$. A symbol a is *classical* whenever the a_{m_j}'s can be chosen positively homogeneous of degree m_j in ξ, i.e.

$a_{m_j}(x, t\xi) = t^{m_j} a_{m_j}(x, \xi) \quad \forall t > 0.$

A pseudo-differential operator A that has order smaller than any negative integer is called a *smoothing* operator.

A typical example of smoothing operator is provided by the heat operator $e^{-t\Delta_g}$, $t > 0$ where Δ_g is the Laplace-Beltrami operator defined above. The exponential is defined as $f(\Delta_g)$ where $f(x) = e^{-tx}$ using spectral representation of self-adjoint operators.

Letting as before $H^s(E)$ denote the H^s Sobolev closure of the space $C^\infty(E)$ of smooth sections of E, we have the following fundamental result:

Proposition: Let $E \to M$ and $F \to M$ be two vector bundles based on a closed manifold M. A pseudo-differential operator $A : C^\infty(E) \to C^\infty(F)$ of order m extends to a bounded operator

$$A : H^s(E) \to H^{s-m}(F).$$

Idea of proof: Using a partition of unity on the base manifold M, one can reduce the proof to the case of an operator acting on functions with compact support in $\mathbb{R}^n$ where n is the dimension of M; we follow the steps of Proposition 3.2. in chapter III of [31]. Given two smooth functions u, v with compact support in $\mathbb{R}^n$ we can write

$$\begin{aligned}
\langle Au, v\rangle &= \int_{\mathbb{R}^n} \hat{Au}(\eta)\hat{v}(\eta)d\eta \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} e^{i\langle \xi-\eta\rangle} a(x,\xi)dx\right]\hat{u}(\xi)\hat{v}(\eta)d\xi d\eta.
\end{aligned}$$

Setting $U(\xi) := \hat{u}(\xi)(1 + |\xi|)^{s+m}$ and $V(\eta) = \hat{v}(\eta)(1 + |\eta|)^{-s}$ yields:

$$\begin{aligned}
|\langle Au, v\rangle| &\leq \int\int \Psi(\xi,\eta)U(\xi)V(\eta)d\xi d\eta \\
&\leq \left(\int\int \Psi(\xi,\eta)U^2(\xi)d\xi d\eta\right)^{\frac{1}{2}}\left(\int\int \Psi(\xi,\eta)V^2(\xi)d\xi d\eta\right)^{\frac{1}{2}} \\
&\leq C\|u\|_{s+m}\|v\|_{-s},
\end{aligned}$$

where we have set: $\Psi(\xi, \eta) := |\int e^{i\langle x,\xi-\eta\rangle} a(x, \xi)dx|(1 + |\xi|)^{-s-m}(1 + |\eta|)^s$ which, according to the growth assumptions is bounded from above by $C(1 + |\xi - \eta|)^{-t+|s|}$ for some positive constant C. ■

Corollary: Under the assumptions of the above proposition, the operator

$$A : H^s(E) \to H^s(F)$$

is compact whenever its order is negative. In particular, any smoothing operator is compact.

Proof: The first assertion follows from the compactness of the inclusion $i : H^t(F) \to H^s(F)$ for $t > s$. Indeed, setting $t := s - m$, we find that $A : H^s(E) \to H^{s-m}(F)$ composed with the inclusion i is compact as the composition of a bounded and a compact operator, which yields the compactness of $A : H^s(E) \to H^s(F)$. The rest of the proposition easily follows. ■

The notion of ellipticity extends to pseudo-differential operators. Namely, a pseudo-differential operator of order m is elliptic if there exists a constant $c > 0$ such that, for all ξ with large enough module $|\xi| \geq c$, the inverse $a(x, \xi)$ exists and satisfies

$$|a(x, \xi)^{-1}| \leq c(1 + \|\xi\|)^{-m}.$$

Elliptic differential operators yield examples of elliptic pseudo-differential operators. We are now ready to state an important result, referring the reader to e.g. [31] for a proof.

Proposition: Let $E \to M$, $F \to M$ be two vector bundles based on a closed manifold M. Any elliptic pseudo-differential operator $A : C^\infty(E) \to C^\infty(F)$ admits a parametrix, i.e. an inverse map up to a smoothing operator. In other words, there is a pseudo-differential operator $B : C^\infty(M, F) \to C^\infty(M, E)$ such that $AB - I_F$ and $BA - I_E$ are smoothing operators where I_E, resp. I_F is the identity map on $C^\infty(E)$, resp. $C^\infty(F)$.

Corollary: Let $E \to M$, $F \to M$ be two vector bundles based on a *closed manifold* M. An elliptic pseudo-differential operator $A : C^\infty(E) \to C^\infty(F)$ of order a extends to a Fredholm map:

$$A^s : H^s(E) \to H^{s-a}(F).$$

In particular,

- Ker A and KerA^* are finite dimensional vector spaces. They are subspaces of $C^\infty(E)$ and $C^\infty(F)$ respectively.
- *Decomposition theorem*

$$H^s(E) = \mathrm{Ker}(A) \oplus \mathrm{R}(A^*) \quad \forall s \in I\!R,$$
$$H^s(F) = \mathrm{Ker}(A^*) \oplus \mathrm{R}(A) \quad \forall s \in I\!R,$$

  where the sums are orthogonal w.r.to the H^s-inner product,

$$C^\infty(E) = \mathrm{Ker}(A) \oplus \mathrm{R}(A^*),$$

$$C^\infty(F) = \mathrm{Ker}(A^*) \oplus \mathrm{R}(A)$$

  where the sums are orthogonal w.r.to the L^2-inner product.

Idea of the proof: It is based on the results of section 3.6.

- The fact that A induces a Fredholm operator follows from its invertibility "up to a smoothing operator" using one of the characterizations of Fredholm operators, as bounded operators which are invertible, "up to a compact operator".
- The fact that Ker (A) and Ker(A^*) are finite dimensional vector spaces and the Sobolev decomposition theorems follow directly from the properties of Fredholm operators (see section 3.6).
- Since $\mathrm{Ker}(A) = \bigcap_s \mathrm{Ker} A^s$ and and $\mathrm{Ker}(A^*) = \bigcap_s \mathrm{Ker}\,(A^*)^s$, these kernels are subspaces of $C^\infty(E)$ and $C^\infty(F)$.
- The decompositions of the spaces of smooth sections then follow from the Fredhom property of the various extensions of A to Sobolev spaces.

4.11 The Hodge de Rham descomposition theorem

From the Bochner-Weitzenböck formula, we know that the Hodge Laplacian $\Delta = (d + d^*)^2$ differs from the Laplacian $\Delta^{\Lambda T^*M}$ by a zeroth-order differential operator. They therefore have the same leading symbol; since we know that $\Delta^{\Lambda T^*M}$ is elliptic, so is Δ. As a consequence of the above corollary applied to the restriction Δ_p of Δ to p-forms, the space of *p-harmonic forms* given by:

$$\mathcal{H}_p(M) := \{\alpha \in \Omega^p(M), \Delta_p \alpha = 0\}$$

is finite dimensional. Its dimension is called the *p-th Betti number* and denoted by $\beta_p(M)$ so that

$$\beta_p(M) = \dim(H^p(M)).$$

On the other hand, the decomposition theorem for Fredholm operators yields a *Hodge decomposition theorem*:

$$\Omega^p(M) = \mathcal{H}_p(M) \oplus R(d+d^*)_{|\Omega^p(M)} = \mathcal{H}_p(M) \oplus R(d_{p-1}) \oplus R(d^*_{p+1}),$$

the direct sums corresponding to orthogonal sums w.r.to the inner product on forms. As a consequence, we have:

$$\beta_p(M) = \dim\left(\mathcal{H}_p(M)\right).$$

The Hodge star isomorphism between p-forms and $n-p$-forms yields the isomorphism:

$$H^p(M) \simeq H^{n-p}(M)$$

and hence

$$\beta_p(M) = \beta_{n-p}(M).$$

The *Euler characteristic* which is given by the alternating sum of the Betti numbers defines a topological invariant of the manifold:

$$\xi(M) = \sum_{p=0}^n \beta_p(M).$$

There is another possible interpretation of the Euler characteristic as the index of the zero section in the tangent bundle TM. Let $f : M \to N$ be a smooth map from a closed oriented smooth m-dimensional manifold M to another closed oriented smooth n-submanifold N of an oriented manifold W of dimension $m+n$ such that f is transverse to N. A point $x \in f^{-1}(N)$ has positive or negative type according to whether the composition:

$$M_x \to W_{f(x)} \to W_{f(x)}/N_{f(x)}$$

preserves or reverses orientation. Here the first map is the tangent map $T_x f$ to f at point x. Accordingly we set $i_x(f,N) = 1$ or $i_x(f,N) = -1$. The *intersection number* of (f,N) is the integer:

$$i(f,N) := \sum_{x \in f^{-1}(N)} i_x(f,N).$$

It is invariant under homotopies of the map f.

Now if $s_0 : M \to TM$ is the zero section of the tangent bundle of M, we have:

$$\xi(M) = i(s_0, M).$$

As a consequence, since any section s of the tangent bundle is homotopic to the zero section by the map $(t,x) \mapsto ts(x)$, if the tangent bundle to M has a section which is nowhere zero then $\xi(M) = 0$. Since $\xi(S^{2n}) = 2$, every vector field on S^{2n} vanishes somewhere. In other words, a "hairy ball cannot be combed".

We shall also need so split the space of harmonic forms as the sum

$$\mathcal{H}^p(M) = \mathcal{H}^{p,sd}(M) \oplus \mathcal{H}^{p,asd}(M)$$

of the space of *self-dual harmonic p-forms*

$$\mathcal{H}^{p,sd}(M) := \{\alpha \in \mathcal{H}^p(M), \star\alpha = \alpha\}$$

and of the space of *anti self-dual harmonic p-forms*

$$\mathcal{H}^{p,asd}(M) := \{\alpha \in \mathcal{H}^p(M), \star\alpha = -\alpha\}$$

which are both trivially finite dimensional since the space of harmonic p-forms is. We set

$$\beta_p^+(M) := \dim\left(\mathcal{H}^{p,sd}(M)\right), \quad \beta_p^-(M) := \dim\left(\mathcal{H}^{p,asd}(M)\right).$$

Clearly we have:

$$\beta_p(M) = \beta_p^+(M) + \beta_p^-(M).$$

4.12 An incursion into index theory

It follows from the above proposition that an elliptic pseudo-differential operator $A : C^\infty(E) \to C^\infty(F)$ is Fredholm since it is invertible up to a compact operator, namely here a smoothing operator. It therefore has a well-defined index:

$$\mathrm{ind}(A) := \dim(\mathrm{Ker}(A)) - \dim(\mathrm{Ker}(A^*))$$

which is computed by so-called *index theorems* which express the index in terms of some chomology forms thus relating it to topological characteristic classes. Just to give a flavour, we state here (without proof) a few examples of index theorems.

- We saw that the operator $D = d + d^*$ gives rise to a Dirac operator D acting on forms on a closed manifold M. Let us equip $\Omega(M)$ with the $\mathbb{Z}_2$-grading given by the parity of the form

  $$\Omega(M) = \Omega^{ev}(M) \oplus \Omega^{odd}(M)$$

  where $\Omega^{ev}(M) = \mathrm{Ker}(I - P)$ is the algebra of forms of even degree and $\Omega^{odd} = \mathrm{Ker}(I + P)$ the space of forms of odd degree. Here P denotes the parity operator which is 1 on even forms and -1 on odd forms. The index of the Dirac operator

  $$D_P^+ := (d + d^*)_{|_{\mathrm{Ker}(I-P)}}$$

  can be expressed in terms of the Euler characteristic:

  Lemma:

  $$\mathrm{ind}(D_P^+) = \chi(M).$$

  Proof.

  $$\begin{aligned}
  \mathrm{ind}(D_P^+) &= \mathrm{dimKer}(D_P^+) - \mathrm{dimKer}(D_P^-) \\
  &= \mathrm{dimKer}((1-P)D) - \mathrm{dimKer}((1+P)D) \\
  &= \sum_p \mathrm{dimKer}(D_{|_{\Omega^{2p}(M)}}) - \sum_p \mathrm{dimKer}(D_{|_{\Omega^{2p+1}(M)}}) \\
  &= \sum_{p=0}^n (-1)^p \mathrm{dimKer}(D_{|_{\Omega^p(M)}}) \\
  &= \sum_{p=0}^n (-1)^p \mathrm{dim}\mathcal{H}^p(M) \\
  &= \sum_{p=0}^n (-1)^p \beta_p(M) \\
  &= \chi(M).
  \end{aligned}$$

An index theorem (which we do not prove here) gives a local expression of the index:

Theorem

$$\mathrm{ind}(D_P^+) = \chi(M) = \frac{1}{(2\pi)^{-\frac{n}{2}}} \int_M e(\nabla)$$

where ∇ is the Levi-Civita connection on M, $e(\nabla)$ the Euler class of M equipped with ∇.

- Let us now introduce another $\mathbb{Z}_2$-grading on $\Omega(M)$ using the *chirality operator* defined on p forms by:

$$\Gamma = (-1)^{pn+\frac{p(p-1)}{2}+l} i^{k(n)} \star$$

where $k(n) = \frac{n}{2}$ if n is even and $k(n) = \frac{n+1}{2}$ if n is odd. $\star$ is the Hodge star. Since $\Gamma^2 = I$, the space of forms splits:

$$\Omega(M) = \Omega^+(M) \oplus \Omega^-(M)$$

where we have set $\Omega^+(M) = \mathrm{Ker}(\Gamma - I)$ and $\Omega^-(M) := \mathrm{Ker}(\Gamma + I)$. If the dimension n is even, then $D = d + d^*$ anti commutes with Γ, i.e. $\Gamma D = -D\Gamma$, so that the operator

$$D_\Gamma^+ := (d + d^*)_{|\mathrm{Ker}(I-\Gamma)}$$

acts from the space $\Omega^+(M)$ to the space $\Omega^-(M)$.

We henceforth specialise to the case $n = 2k = 4l$ is a multiplie of 4, for which Γ coincides with the Hodge star operator on k forms. In particular we have:

$$\mathcal{H}^{k,sd}(M) = \{\alpha \in \Omega^k(M), \Gamma\alpha = \alpha\}$$

and

$$\mathcal{H}^{k,asd}(M) = \{\alpha \in \Omega^k(M), \Gamma\alpha = -\alpha\}$$

that are finite dimensional spaces with dimensions $\beta_k^+(M)$ and $\beta_k^-(M)$ respectively.

Proposition: In dimension $n = 2k = 4l$, the bilinear form

$$\begin{aligned} \sigma : \Omega^k(M) \times \Omega^k(M) &\to \mathbb{R} \\ (\alpha, \beta) &\to \sigma(\alpha, \beta) := \int_M \alpha \wedge \beta \end{aligned}$$

induces a non degenerate symmetric bilinear form on $H^k(M)$. Its signature, called the signature of M, is given by

$$\sigma(M) := \mathrm{sign}(\sigma) = \beta_k^+(M) - \beta_k^-(M).$$

Proof. We saw previously that $\int_M \alpha \wedge \beta = (-1)^k \int_M \beta \wedge \alpha$ on k-forms so that if k is even, it yields a symmetric bilinear form. Given two closed forms α and β, $\sigma(\alpha, \beta)$ only depends on the cohomology class of α and β; indeed

$$\begin{aligned} \int_M (\alpha + d\gamma) \wedge \beta &= \int_M \alpha \wedge \beta + \int_M d\gamma \wedge \beta \\ &= \int_M \alpha \wedge \beta + \int_M d(\gamma \wedge \beta) - \int_M \gamma \wedge d\beta \\ &= \int_M \alpha \wedge \beta \end{aligned}$$

where we have used Stoke's theorem to set the middle integral to zero and the fact that β is closed to set the last integral to zero.

The form σ is non degenerate; indeed, let us assume that $\int_M \alpha \wedge \beta = 0$ for any closed k-form β and let us show that $\alpha = 0$. Pick a harmonic k-form α as representative of a cohomology class in $H^k(M)$. Then

$$(d + d^*)\alpha = 0 \Rightarrow d\alpha = d^*\alpha = 0 \Rightarrow d(\star\alpha) = 0.$$

Hence $\star\alpha$ is also a closed k-form and we can take $\beta = \star\alpha$. This yields $\alpha \wedge \star\alpha = \|\alpha\|^2 = 0$ so that $\alpha = 0$ which ends the proof of the non degeneracy.

Let us now compute this bilinear form in different cases. First observe that for $\alpha \in \mathcal{H}^{k,sd}(M)$, $\beta \in \mathcal{H}^{k,sd}(M)$,

$$\sigma(\alpha, \beta) = \int_M \alpha \wedge \beta = \int_M \alpha \wedge \star\beta = \langle \alpha, \beta \rangle,$$

if $\alpha \in \mathcal{H}^{k,asd}(M)$, $\beta \in \mathcal{H}^{k,asd}(M)$

$$\sigma(\alpha, \beta) = \int_M \alpha \wedge \beta = \int_M \alpha \wedge \star\beta = -\langle \alpha, \beta \rangle$$

and if $\alpha \in \mathcal{H}^{k,sd}(M)$, $\beta \in \mathcal{H}^{k,asd}(M)$

$$\sigma(\alpha, \beta) = \int_M \star\alpha \wedge \beta = (-1)^k \int_M \beta \wedge \star\alpha = \langle \alpha, \beta \rangle = \int_M \star\alpha \wedge \star\beta = -\sigma(\alpha, \beta) = 0.$$

As a consequence, σ is diagonal on $\mathcal{H}^{k,sd}(M) \oplus \mathcal{H}^{k,asd}(M)$ with eigenvalues $+1, -1$ with multiplicity $\beta_k^+(M)$ and $\beta_k^-(M)$. The bilinear form σ therefore has signature

$$\mathrm{sign}(\sigma) = \mathrm{dim}\mathcal{H}^{k,sd}(M) - \mathrm{dim}\mathcal{H}^{k,asd}(M) = \beta_k^+(M) - \beta_k^-(M).$$

The following Lemma relates the index of D_Γ^+ to the signature of the manifold:

Lemma:

$$\mathrm{ind}(D_\Gamma^+) = \sigma(M)$$

Proof. We first observe that if $\alpha_i, i = 1, \cdots, \beta_j(M)$ is an orthonormal basis of $\mathcal{H}^j(M)$ with $j < k$ then $\alpha_i^+ = \alpha_i + \star\alpha_i, \alpha_i^- = \alpha_i - \star\alpha_i$, $i = 1, \cdots, \beta_j(M)$ yield an orthonormal basis of $\mathcal{H}^j \oplus \mathcal{H}^{n-j}$, where $\star$ is the Hodge star operator. Since the α_i^+ are self-dual and the α_i^- are anti self-dual, they yield an orthonormal basis respectively of $\mathcal{H}^{j,sd}(M) \oplus \mathcal{H}^{n-j,sd}(M)$ and $\mathcal{H}^{j,asd}(M) \oplus \mathcal{H}^{n-j,asd}(M)$. As a consequence,

$$\begin{aligned} & \beta_j^+(M) + \beta_{n-j}^+(M) = \dim\left(\mathcal{H}^{j,sd}(M) \oplus \mathcal{H}^{n-j,sd}(M)\right) \\ = \; & \dim\left(\mathcal{H}^{j,asd}(M) \oplus \mathcal{H}^{n-j,asd}(M)\right) = \beta_j^-(M) + \beta_{n-j}^-(M). \end{aligned}$$

Hence we have,

$$\begin{aligned} \mathrm{ind}(D_\Gamma^+) &= \mathrm{dimKer}(D_\Gamma^+) - \mathrm{dimKer}(D_\Gamma^-) \\ &= \mathrm{dimKer}(D_{|_{\Omega^{sd}(M)}}) - \mathrm{dimKer}(D_{|_{\Omega^{asd}(M)}}) \\ &= \sum_{j=0}^n \dim\left(\mathrm{Ker}\mathcal{H}^{j,sd}(M)\right) - \sum_{j=0}^n \dim\left(\mathrm{Ker}\mathcal{H}^{j,asd}(M)\right) \\ &= \sum_{j=0}^n \beta_j^+(M) - \sum_{j=0}^n \beta_j^-(M) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{k-1}\left(\beta_j^+(M)+\beta_{n-j}^+(M)\right)-\sum_{j=0}^{k-1}\left(\beta_j^-(M)+\beta_{n-j}^-(M)\right)+\beta_k^+(M)-b_k^-(M)\\
&= \beta_k^+(M)-b_k^-(M)\\
&= \sigma(M).
\end{aligned}
$$

An index theorem (which we do not prove here) gives a local expression of the index as an integral of the $\hat{L}$ genus:

Theorem Let M be an oriented Riemannian closed $n = 4l$ dimensional manifold.

$$\mathrm{ind}(D_\Gamma^+) = \frac{(-1)^l}{\pi^{2l}} \int_M L(\nabla)$$

where as before, ∇ is the Levi-Civita connection.

- Let M be an even dimensional spin manifold and let $E = S \otimes W$ be the tensor product of the spinor bundle S of M and some exterior bundle W. Since $S = S^+ \oplus S^-$ is $\mathbb{Z}_2$-graded, so is $E = E^+ \oplus E^-$. Let $D^+ : C^\infty(E^+) \to C^\infty(E^-)$ be the corresponding Dirac operator. The Atiyah-Singer index theorem (see e.g. [6], [31]) expresses this index of D^+ as the integral over the base manifold of some Chern-Weil forms, namely the $\hat{A}$-genus on M and the Chern character on W described at the end of chapter 2:

$$\mathrm{ind}(D^+) = \frac{1}{(2i\pi)^{\frac{n}{2}}} \int_M \hat{A}(\nabla)\mathrm{ch}(\nabla^W).$$

5 Configuration and moduli spaces

This chapter offers an illustration of how the various tools from geometry and operator theory presented in the previous chapters can come into play in quantum field theory. A short description of spaces of (inequivalent) configurations arising in Yang-Mills, Seiberg-Witten and string theory is given here.

5.1 The geometric setting

A classical field theory with symmetries typically leads to the following geometric setting. A gauge group $\mathcal{G}$ (a group of symmetries) acts on an (infinite dimensional) space of configurations X, and one is interested in the *moduli space* of inequivalent configurations $\mathcal{M} := X/\mathcal{G}$.

The space of inequivalent configurations can play a role to study solutions of the classical field equations –namely the Euler-Lagrange equations minimizing the classical action– (Yang-Mills equation in Yang-Mills theory and the Seiberg -Witten equations in Seiberg-Witten theory) or to investigate the quantized theory from a path integration point of view (in string theory). In both cases the non-compactness of the moduli space can come into the way; Seiberg-Witten theory offers the advantage over Yang-Mills theory that the moduli space of classical solutions is compact and Seiberg-Witten invariants are built up from integrals on the moduli space of inequivalent solutions to the Seiberg-Witten equations. In the path integral approach to quantization, when the moduli space is a finite dimensional manifold (for string theory, the Teichmüller space of inequivalent conformal structures on a Riemann surfaces is a smooth finite dimensional manifold), one can reduce path integrals on infinite dimensional configuration spaces to ordinary integrals on the moduli space.

If the action is not free, the moduli space might not be a Hausdorff space; to cure this problem, one can either reduce the group of gauge transformations or reduce the configuration space in order to get a free action and a manifold structure on the quotient space. For this reason, in Yang-Mills and Seiberg-Witten theory one restricts to irreducible configurations, whereas in string theory, one restricts the gauge group, considering only the connected component of the identity of the group of diffeomorphisms of a surface.

Recall from the (slice) theorem of section 3.2, that given a Hilbert Lie group $\mathcal{G}$, acting L^2-isometrically on a smooth Hilbert manifold X via a smooth, free and proper action, provided the tangent map $\tau_x := D_e\theta_x$ (see notations of section 3.2) has a closed range, then the quotient space $X/\mathcal{G}$ is a smooth Hilbert manifold. In practice, one typically comes across the following situation:

- $\mathcal{G}$ is modelled on some space $H^{s+k}(E)$ of Sobolev sections of some vector bundle E based on a closed manifold M of dimension n ($k > 0$, s is usually chosen large enough $s > \frac{n}{2}$ so that Sobolev sections are continuous),

- X is modelled on some space $H^s(F)$ of Sobolev sections of another vector bundle F based on M,

- for $x \in X$, $\tau_x : C^\infty(E) \to C^\infty(F)$ is a differential operator of order $k > 0$ with injective symbol.

The last proposition in section 4.8 then tells us that if τ_x were elliptic, it would extend to a Fredholm operator $\tau_x : H^{s+k}(E) \to H^s(F)$ and hence have a closed range (see section 3).

The fact that τ_x be elliptic requires that its leading symbol be an isomorphism for non vanishing ξ, but this in turn implies that E and F should have the same rank, which is not always the case in practice. However, under the weaker requirement that the leading symbol be injective for non vanishing ξ (which we recall implies that $\tau_x^*\tau_x$ is elliptic), the closedness of the range of τ_x and as a consequence, the L^2- orthogonal splitting:

$$R(\tau_x) \oplus \mathrm{Ker}(\tau_x^*) = C^\infty(F)$$

still hold. In fact this splitting holds in the H^s topology and we have:

Theorem: Let $\mathcal{G}$ be a Hilbert Lie group modelled on some space $H^{s+k}(E)$ of Sobolev sections of some Hermitian vector bundle E based on a closed manifold M of dimension n (with $s > \frac{n}{2}$, $k > 0$) acting on a Hilbert manifold X modelled on some space $H^s(F)$ of Sobolev sections of another Hermitian vector bundle F based on M. We assume that the action is L^2-isometric for the L^2 Riemannian metric on X built from inner products on the model space obtained by integrating along M the inner products on the fibres. If the action is free, proper and smooth and if moreover, for any $x \in X$, $\tau_x : C^\infty(E) \to C^\infty(F)$ is a differential operator of order $k > 0$ with injective symbol, then the moduli space $X/\mathcal{G}$ is a smooth manifold.

In applications, an additional difficulty occurs; because one restricts oneself to some Sobolev setting, the differential operator τ_x may have non smooth coefficients lying in some Sobolev space so that one then needs to adapt the classical results on differential operators with smooth coefficients (notice that a differential operator of order a with H^k-coefficients takes smooth sections to H^{k-a} sections unlike an operator with smooth coefficients which takes smooth sections to smooth sections). We shall elude this difficulty here, referring the reader to [29] for a discussion on this point.

5.2 The I.L.H. setting

In practice, one comes across Fréchet spaces of smooth sections rather than spaces of H^s-sections, namely intersections $C^\infty(E) = \bigcap_{s>0} H^s(E)$ rather than a space $H^s(E)$ for some fixed s, $\pi : E \to M$

being a vector bundle based non a closed manifold M. Omori [41] introduced the notion of I.L.H. space -the inverse limit of Hilbert spaces- by which $C^\infty(E)$ is seen as the inverse limit of the Hilbert spaces $H^s(E)$ (see [41]).

Recall that if $\{X_n, n \in I\!N\}$ is a countable family of topological spaces with continuous inclusions $X_{n-1} \subset X_n$ then the intersection $X := \bigcap_n X_n$ can be given the projective topology which corresponds to the weakest topology on X that makes the inclusions $X \to X_n$ continuous. We denote the resulting topological space, called the *inverse limit* of the X_n by $(X; X_n, n \in I\!N)$.
If for every $n \in I\!N$, the space X_n is a linear Hilbert space and the inclusion maps are linear, the resulting inverse limit $(X; X_n, n \in I\!N)$ is a linear space called an inverse limit of Hilbert spaces or an *I.L.H. vector space* for short.

A C^k- I.L.H. manifold modelled on an I.L.H. linear space $(E; E_n, n \in I\!N)$ is an I.L.H. topological space $(X; X_n, n \in I\!N)$ such that

- X_n is a C^k-Hilbert manifold modelled on E_n,
- For each $x \in X$, for any $n \in I\!N$, there is an open neighborhood $U_n(x)$ of x in X_n and homeomorphisms:
$$\Phi_n : U_n(x) \to V_n \subset E_n$$
which yield C^k coordinate systems around $x \in X_n$ and satisfy:
$$U_{n+1}(x) \subset U_n(x) \quad \text{and} \quad \Phi_{n|_{U_{n+1}(x)}} = \Phi_{n+1},$$
- $U(x) := \bigcap_n U_n(x)$ is an open neighborhood of x in $(X; X_n, n \in I\!N)$.

We have included the last condition in the definition of an I.L.H. manifold which makes it a strong I.L.H. manifold according to the usual convention, so that I.L.H. manifolds considered here are in fact *strong I.L.H. manifolds*.

A map $\phi : X \to Y$ between two C^k-I.L.H. manifolds is C^k-I.L.H. differentiable if it is the inductive limit of C^k-differentiable maps $\phi_n : X_{m(n)} \to Y_n$ for some $m(n)$ such that $\phi_{n|_{X_{m(n+1)}}} = \phi_{m(n+1)}$. It is smooth if it is C^k for all $k \in I\!N$.

There are examples for which one can choose $m(n) = n$ (e.g. the multiplication in the Weyl group, see below), but allowing $m(n) \neq n$ is necessary if we want to put an I.L.H. Lie group structure on the group of diffeomorphisms we need in the context of string theory.

An I.L.H. topological group is called an *I.L.H. Lie group* if it is a smooth I.L.H. manifold with the group operations given by smooth I.L.H. maps.
The group of smooth diffeomorphisms on a closed manifold can be equipped with an I.L.H. Lie group structure [41] even though the group of diffeomorphisms of a fixed Sobolev class is not a Hilbert Lie group, hence the relevance of the concept of I.L.H. space.

Let P, B be smooth I.L.H. manifolds modelled on E and F respectively, $\pi : P \to B$ a smooth I.L.H. map and G an I.L.H. Lie group. Then (P, B, G, π) is an I.L.H. principal bundle if and only if the transition maps are smooth I.L.H. maps.

We are now ready to state the I.L.H. version of the above slice theorem. The notion of properness extends to the I.L.H. setting in a straight forward way.

Theorem: Let $\mathcal{G}$ be an I.L.H. Lie group acting transitively on the right on a smooth I.L.H. manifold X:

$$\begin{aligned} \Theta : \mathcal{G} \times X &\to X \\ (g, x) &\mapsto x \cdot g. \end{aligned}$$

Let us assume that the I.L.H. manifolds X, resp. $\mathcal{G}$ are modelled on the I.L.H. spaces $C^\infty(E)$, resp. $C^\infty(F)$ where $E \to M$ and $F \to M$ are two Hermitian vector bundles based on some closed Riemannian manifold M. The manifold X is equipped with an L^2 Riemannian structure built from inner products on their model spaces obtained by integrating along M the inner product on the fibres.
Under the following assumptions:

- The action of $\mathcal{G}$ on X is smooth I.L.H., L^2-isometric, free and proper,

- Setting $\theta_x := \Theta(\cdot, x)$ for any $x \in X$, the map:

$$\begin{aligned} \tau_x : Lie(\mathcal{G}) &\to T_x X \\ u &\mapsto D_e \theta_x(u) \end{aligned}$$

  is an injective differential operator with injective symbol, then The quotient is a smooth I.L.H. manifold equipped with the induced L^2-structure and the canonical projection $\pi : X \to X/\mathcal{G}$ yields an I.L.H. principal fibre bundle.

Let us now turn to three examples in quantum field theory; Yang-Mills, Seiberg-Witten and string theory which use the above theorem.

5.3 Configurations in Yang-Mills gauge theory

Useful references are [10], [12], [29], [34], [40], [52].

G denotes a fixed compact connected Lie group.

Let P be be a smooth principal bundle based on a closed manifold M with structure group G. Let $adP := P \times_G Lie(G)$ be the the vector bundle based on M with typical fibre given by the Lie algebra $Lie(G)$ of G associated to the adjoint action of G on $Lie(G)$. Let us set $E := adP$ and $F := T^*M \otimes adP$, a vector bundle the sections of which are 1-forms on M with values in adP.

The space of configurations: Let $\mathcal{C}(P)$ (resp. $\mathcal{C}^s(P)$) denote the space of smooth (resp. H^s) connections on P. Since a connection on P is a G-equivariant $Lie(G)$-valued one form ω on P such that $\omega(\tilde{X}) = X \quad \forall X \in Lie(G)$ ($\tilde{X}$ is the canonical vector field generated by X), two connections differ by a horizontal one form on P and hence an adP valued one form on M. $\mathcal{C}(P)$ (resp. $\mathcal{C}^s(P)$) is an affine I.L.H. (resp. Hilbert) space with tangent vector space $C^\infty(F)$ (resp. $H^s(E)$).

The gauge group: Let $E_G := P \times_G G$ where G acts on itself by the adjoint action, then the set $C^\infty(E_G)$ (resp. $H^s(E_G)$) of smooth (resp. H^s-Sobolev) sections of E_G is an I.L.H. (resp. Hilbert) Lie group modelled on $C^\infty(E)$ (resp. $H^s(E)$). It corresponds to the group of automorphisms of P that cover the identity map on M.

The space of irreducible configurations: A connection A on P induces a covariant derivation ∇^A on adP from which one can define a differential operator of order 1:

$$\begin{aligned} d_A : \Omega^0(M, E) = C^\infty(E) &\to \Omega^1(M, E) = C^\infty(F) \\ \sigma &\mapsto \left(X \to \nabla^A_X(\sigma)\right) \end{aligned}$$

which extends to the exterior differential $d_A : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$. The operator d_A is generally not injective so that we need to restrict ourselves to *irreducible* connections, namely those for which d_A is one to one. Notice that when A is reducible, an element $u \in \mathrm{Ker} d_A$ generates gauge transformations $g_t := e^{tu}$ that leaves A fixed. The space $\bar{\mathcal{C}}(P)$ (resp. $\bar{\mathcal{C}}^s(P)$) of irreducible smooth (resp. H^s) connections on P) is also an I.L.H. (resp. Hilbert) manifold modelled on $C^\infty(F)$ (resp. $H^s(F)$).

An L^2-structure on the configuration space: Since G is compact, its Lie algebra $Lie(G)$ can be equipped with a positive definite inner product which is invariant under the adjoint action. The bundle $E = adP$ thus inherits an inner product which, combined with the Riemannian metric on M yields an inner product on $F = T^*M \otimes adP$. Hence, the configuration space $\mathcal{C}(P)$ which is modelled on $C^\infty(F)$ can be equipped with an L^2-metric obtained by integrating along M the inner product on F. This metric is invariant under the action Θ.

The gauge group action: The I.L.H. Lie group $C^\infty(E_G)$ acts smoothly on the I.L.H. space $\mathcal{C}(P)$ of smooth connections on P:

$$\begin{aligned} \Theta : C^\infty(E_G) \times \mathcal{C}(P) &\to \mathcal{C}(P) \\ (g, A) &\mapsto A \cdot g := A + g^{-1} d_A g. \end{aligned}$$

The action Θ is L^2-isometric and it induces a smooth, free and proper action on the I.L.H. space of *irreducible* configurations:

$$\begin{aligned} \bar{\Theta} : C^\infty(E_G)/\mathcal{Z} \times \bar{\mathcal{C}}(P) &\to \bar{\mathcal{C}}(P) \\ (g, A) &\mapsto a \cdot g := A + g^{-1} d_A g. \end{aligned}$$

Here $\mathcal{Z}$ is the center of $C^\infty(E_G)$ and corresponds to $C^\infty(P \times_G Z(G))$ where $Z(G)$ is the center of G.

The tangent operator τ_x: It is the tangent operator at identity to $\theta_A := \Theta(\cdot, A)$ and therefore corresponds to the first order differential operator $d_A : C^\infty(E) \to C^\infty(F)$ which has injective symbol.

The moduli space of inequivalent connections: Applying the slice theorem to the I.L.H. gauge group quotiented by its center $\mathcal{G} := C^\infty(E_G)/\mathcal{Z}$ acting on the I.L.H. manifolds of irreducible configurations $X := \bar{\mathcal{C}}(P)$ shows that the moduli space $X/\mathcal{G}$ of inequivalent irreducible connections on P is a smooth I.L.H. manifold.

5.4 Configurations in Seiberg-Witten theory

Classical references are [32], [35].

The setting is similar in spirit to the Yang-Mills setting. Here M is a closed 4-dimensional Spin^c manifold and $\tilde{P}$ the lift of the orthonormal frame bundle $SO(TM)$ to a principal Spin^c bundle based on M. Let $E := M \times I\!R$ and $F := T^*M \otimes I\!R \oplus S^+(\tilde{P})$ where $S^+(\tilde{P})$ is the spinor bundle associated to the Spin^c structure on M.

The space of configurations: Let $\mathcal{C}(\mathcal{L})$ (resp. $\mathcal{C}^s(\mathcal{L})$) be the I.L.H. (resp. Hilbert) space of $U(1)$ smooth (resp. H^s) connections on $\mathcal{L}$, the determinant line bundle $\mathcal{L}$ associated to $\tilde{P}$. The space of smooth (resp. H^s) configurations is given by:

$$\mathcal{C}(\tilde{P}) := \mathcal{C}(\mathcal{L}) \times C^\infty(S^+(\tilde{P}))$$

resp.

$$\mathcal{C}^s(\tilde{P}) := \mathcal{C}^s(\mathcal{L}) \times H^s(S^+(\tilde{P})).$$

It is a smooth I.L.H. (resp. Hilbert) manifold.

The gauge group: The group of smooth (resp. H^s) automorphisms of $\tilde{P}$ that cover the identity on the frame bundle of M is an I.L.H. (resp. Hilbert) Lie group which coincides with the group of smooth (resp. H^s) maps $C^\infty(M, S^1)$ (resp. $H^s(M, S^1)$).

The gauge group action: An element g of $C^\infty(M, S^1)$ induces a bundle map $\det g$ on the determinant bundle $\mathcal{L}$ and a bundle map $S^+(g)$ on the spinor bundle $S^+(\tilde{P})$. It acts on the space of configurations by:

$$\begin{aligned} \Theta : C^\infty(M, S^1) \times \mathcal{C}(\tilde{P}) &\rightarrow \mathcal{C}(\tilde{P}) \\ (g, (A, \psi)) &\mapsto (A, \psi) \cdot g := (\det g^* A, S^+(g^{-1})\psi). \end{aligned}$$

Irreducible configurations: Let $\bar{\mathcal{C}}(\tilde{P}) := \{(A, \psi) \in \mathcal{C}(\tilde{P}), \psi \neq 0\}$ (resp. $(\bar{\mathcal{C}}^s(\tilde{P}) := \{(A, \psi) \in \mathcal{C}^s(\tilde{P}), \psi \neq 0\})$ denote the space of irreducible configurations. It is a submanifold of $\mathcal{C}(\tilde{P})$ (resp. $\bar{\mathcal{C}}^s(\tilde{P})$) as an open subset of that I.L.H. (Hilbert) manifold. The above action is free when restricted to irreducible configurations:

$$\begin{aligned} \bar{\Theta} : C^\infty(M, S^1) \times \bar{\mathcal{C}}(\tilde{P}) &\rightarrow \bar{\mathcal{C}}(\tilde{P}) \\ (g, (A, \psi)) &\mapsto (A, \psi) \cdot g := (\det g^* A, S^+(g^{-1})\psi). \end{aligned}$$

The action Θ_A is smooth, free and proper.

An L^2-isometric action: The Riemannian metric on M induces a Hermitian product on the spinor bundle $S^+(\tilde{P})$ and hence one on the bundle F. Integrating this inner product on M, yields an L^2-metric on $\bar{\mathcal{C}}^s(\tilde{P})$ which is invariant under the Θ action.

The tangent operator τ_x: The tangent operator at Id to the map $\theta_{(A,\psi)} := \tilde{\Theta}(\cdot, (A, \psi))$ is the first order differential operator

$$\begin{aligned} \tau_{(A,\psi)} : C^\infty(E) &\rightarrow C^\infty(F) \\ f &\mapsto (2df, -f \cdot \psi) \end{aligned}$$

which has injective symbol.

The moduli space of irreducible configurations: We can apply the slice theorem to the gauge group $\mathcal{G} := C^\infty(M, S^1)$, the space of irreducible configurations $X := \bar{\mathcal{C}}(\tilde{P})$ and conclude that the moduli space $X/\mathcal{G}$ of inequivalent irreducible configurations is a smooth I.L.H. manifold.

5.5 The Teichmüller space in string theory

The geometric setting for string theory is also that of Teichmüller theory. Useful references are [4], [11], [13], [51] and many references therein.

Here M is a closed Riemann surface of genus p (which we shall assume here is larger than 1), $E := I\!R \oplus TM$ and $F := T^*M \otimes_s T^*M$ the symmetrized product of the cotangent bundle, where $\otimes_s$ denotes the symmetrized tensor product.
In what follows, s will be assumed large enough for the sections of the different bundles to be continuous.

The configuration space: Let $\mathcal{M}(M) := \{g \in C^\infty(T^*M \otimes_s T^*M), \det g > 0\}$ (resp. $\mathcal{M}^s(M) :=$

$\{g \in H^s(T^*M \otimes_s T^*M), \det g > 0\}$) be the space of smooth (resp. H^s) Riemannian metrics on M; it is an I.L.H. (resp. Hilbert) manifold modelled on $C^\infty(T^*M \otimes_s T^*M)$ (resp. $H^s(T^*M \otimes_s T^*M)$).

The gauge group: Let $\mathcal{W}(M) := \{e^\phi, \phi \in C^\infty(M, I\!R)\}$ (resp. $\mathcal{W}^s(M) := \{e^\phi, \phi \in H^s(M, I\!R)\}$) be the group of smooth (resp. H^s) *Weyl transformations*, $\mathcal{D}(M) := \{f \in C^\infty(M, M), f^{-1} \in C^\infty(M, M)\}$ (resp. $\mathcal{D}^s(M) := \{f \in H^s(M, M), f^{-1} \in H^s(M, M)\}$) the group of smooth (resp. H^s) diffeomorphisms of M. $\mathcal{W}(M)$, $\mathcal{D}(M)$ are I.L.H. Lie groups modelled on $C^\infty(M, I\!R)$, $C^\infty(TM)$ respectively. For fixed large enough s, $\mathcal{W}^s(M)$ is a Hilbert Lie group, but $\mathcal{D}^s(M)$ is not; it is a Hilbert manifold modelled on $H^s(TM)$ which is only a topological group. The I.L.H. setting is therefore useful to put a Lie group structure on diffeomrophisms.

Let $\mathcal{D}_0(M)$ denote the connected component of the identity map in $\mathcal{D}(M)$ and let

$$\mathcal{G} := \mathcal{D}_0(M) \odot \mathcal{W}(M)$$

where $\odot$ stands for the semi-direct product, namely the product with a twisted product law $(f, \phi) \odot (f', \phi') := (f \circ f', \phi \circ f' + \phi)$.

Group actions: The Weyl group $\mathcal{W}(M)$ acts on the configuration space $\mathcal{M}(M)$ by pointwise multiplication:

$$\begin{array}{rcl} \mathcal{W}(M) \times \mathcal{M}(M) & \to & \mathcal{M}(M) \\ (\phi, g) & \mapsto & e^\phi g \end{array}$$

and this I.L.H. action is smooth, free and proper. The set

$$Conf(M) := \left\{[g] := \{e^\phi \cdot g, e^\phi \in \mathcal{W}(M)\}, \quad g \in \mathcal{M}(M)\right\}$$

is an I.L.H. manifold, the manifold of *conformal structures*. It is diffeomorphic to the I.L.H. manifold

$$\mathcal{J}(M) := \{J \in C^\infty(TM \otimes T^*M), \ J \quad \text{preserves orientation} \quad \text{and} \quad J^2 = -I\}$$

of smooth almost complex structures on M .

Because the genus is assumed to be larger than 1, in each conformal class $[g]$ of $g \in \mathcal{M}^s(M)$, there is an smooth metric with curvature -1. Let us set $\mathcal{M}_{-1}(M) := \{g \in \mathcal{M}(M), s_g = -1\}$ where s_g is the scalar curvature of g.
There is a diffeomorphism of I.L.H. manifolds [51]:

$$\mathcal{M}_{-1}(M) \simeq \mathcal{J}(M) \simeq Conf(M).$$

The gauge group $\mathcal{G}$ of interest to us acts on $\mathcal{M}(M)$ by:

$$\begin{array}{rcl} \mathcal{D}_0(M) \odot \mathcal{W}(M) \times \mathcal{M}(M) & \to & \mathcal{M}(M) \\ ((f, \phi), g) & \mapsto & f^* e^\phi g \end{array}$$

and the action is smooth, free and proper (using here again the fact that the manifold has genus larger than 1).

An L^2-metric on the configuration space: In order to understand the quotient space, we use a splitting of the tangent space to the manifold of metrics. It is isomorphic to the space of covariant two tensors and splits into pure trace and traceless two covariant tensors:

$$T_g\mathcal{M}(M) \simeq C^\infty(T^*M \otimes_s T^*M) = C^\infty(M, I\!R) \cdot g \oplus C^\infty_{0,g}(T^*M \otimes_s T^*M)$$

where $C^\infty_{0,g}(T^*M \otimes_s T^*M) := \{h \in C^\infty(T^*M \otimes_s T^*M), \quad tr_g(h) := g^{ab}h_{ab} = 0\}$. This splitting is orthogonal w.r.to the inner product induced by the metric g on M on the space of smooth covariant two tensors. This inner product on $T_g\mathcal{M}(M)$ induces an L^2-metric on $\mathcal{M}(M)$ which is only invariant under $\mathcal{D}(M)$ but not under $\mathcal{W}(M)$. As we saw above, the action of the Weyl group on $\mathcal{M}(M)$ being a straightforward pointwise multiplication, the fact that it is not L^2- isometric is not a major obstruction to apply the slice theorem when taking the quotient. It is however a serious obstacle from the path integration point of view and this non invariance of the metric under Weyl transformations is a source of *conformal anomaly.*

The Faddeev-Popov operator: Using this L^2-orthogonal splitting of $T_g\mathcal{M}(M)$, the tangent map at $(Id, 1)$ to the map $\theta_g := \Theta(\cdot, g)$ reads:

$$\begin{array}{rcl} \tau_g : C^\infty(TM) \odot C^\infty(M, I\!R) & \to & C^\infty_{0,g}(T^*M \otimes_s T^*M) \oplus C^\infty(M, I\!R) \\ (u, \lambda) & \mapsto & (P_g u, \mathrm{tr}_g(\nabla_g u) + \lambda \cdot g) \end{array}$$

where $\nabla_g u$ is the symmetric covariant 2-tensor given by Lie derivative of g in the direction u, $tr_g(\nabla_g u)$ its trace w.r.to g. The operator

$$P_g u := \nabla_g u - \frac{1}{2}\mathrm{tr}_g(\nabla_g u) \cdot g,$$

called *Faddeev-Popov* operator,is the traceless part of $\nabla_g u$. It can be shown that the operator P_g has injective symbol.

The Teichmüller space: Up to the fact that the action is isometric only for $\mathcal{D}_0(M)$, a discrepancy we argued is only a minor difficulty when applying the above theorem because the action of the Weyl group is rather straightforward, we can apply the slice theorem. The quotient space, called the Teichmüller space

$$\mathcal{T}(M) := \mathcal{M}(M)/\mathcal{G}(M)$$

is a smooth finite dimensional manifold (its dimension over $I\!R$ is $6p - 6$) and we have the following diffeomorphisms of finite dimensional manifolds [4], [51]:

$$\mathcal{T}(M) \simeq \mathcal{J}(M)/\mathcal{D}_0(M) \simeq Conf(M)/\mathcal{D}_0(M).$$

Acknowledgments. I would like to express my gratitude to Alexander Cardona and Jean-Pierre Magnot for a careful reading they made of a first version of the manuscript, which was very useful to make various improvements. Many thanks also to Hernan Ocampo for his comments on a preliminary version of the text. I am indebted to Dominique Manchon for his helpful remarks which were crucial to repair some incoherences in a preliminary version of this manuscript. I am also grateful to many students, both in France and in Colombia, who by their questions and remarks during the lectures I gave on parts of these notes, helped me improve their presentation and formulation.

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