Prologomon to renormalisation; analytic aspects (draft in progress)

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PART I: Regularisation procedures

Abstract: We present a unified mathematical framework to describe various regularisation techniques used both in mathematics and physics when making sense of a priori divergent sums or integrals. This presentation involves classical pseudodifferential symbols in an essential way.

1. Regularised evaluators
2. A first characterisation of the noncommutative residue
3. A first characterisation of the cnonical integral
4. Translation invariant linear forms on symbols
5. The canonical integral on non integer order symbols
6. The canonical discrete sum on non integer order symbols
7. An alternative characterisation of the noncommutative residue
8. Holomorphic regularisation schemes
9. Regularised discrete sums on symbols
10. The zeta function

1 Regularised evaluators

"Evaluating" meromorphic functions in one variable at zero with poles at that point requires regularising. We describe and compare various regularisation methods one can use to extend ordinary evaluators at zero to certain algebras of meromorphic functions. The Mellin transform proves to be a useful tool in that context.

1.1 Minimal substraction scheme

Let $\mathrm{Mer}_0^k(\mathbb{C})$ be the germ of meromorphic functions at zero [1] with poles at zero of order no larger than k, and let $\mathrm{Hol}_0(\mathbb{C})$ be the germ of holomorphic functions at zero.

Definition 1 *We call* regularised *evaluator at zero any linear map* $\lambda : \mathrm{Mer}_0(\mathbb{C}) \to \mathbb{C}$ *which extends the following evaluator at zero on holomorphic functions at zero:*

$$\begin{aligned} \mathrm{ev}_0 : \mathrm{Hol}_0(\mathbb{C}) &\to \mathbb{C} \\ f &\mapsto f(0) \end{aligned}$$

One way of building a regularised evaluator at zero is to compose the evaluator with an appropriate Rota-Baxter operator.

Definition 2 *A linear operator $R : \mathcal{A} \to \mathcal{A}$ on an (not necessarily associative) algebra $\mathcal{A}$ over a field F is a called Rota-Baxter of weight $\lambda \in F$ if it satifies the following Rota-Baxter relation:*

$$R(a)\,R(b) = R(R(a)\,b) + R(a\,R(b)) + \lambda\,R(ab).$$

Remark 1 *If $\lambda \neq 0$, replacing R by $\lambda^{-1} R$ gives rise to a Rota-Baxter operator of weight* 1.

If $f(z) = \sum_{i=k}^{\infty} a_i z^i$ we set $\mathrm{Res}_0^j(f) := a_{-j}$ called the j-th residue of f at zero.

Proposition 1 *Let $\mathrm{Mer}_0(\mathbb{C}) := \cup_{k=0}^{\infty} \mathrm{Mer}_0^k(\mathbb{C})$. The map*

$$\begin{aligned} R_+ : \mathrm{Mer}_0(\mathbb{C}) &\to \mathrm{Hol}_0(\mathbb{C}) \\ f &\mapsto \left(z \mapsto f(z) - \sum_{j=1}^{k} \frac{\mathrm{Res}_0^j(f)}{z^j} \right) \quad \text{if} \quad f \in \mathrm{Mer}_0^k(\mathbb{C}) \end{aligned}$$

satisfies the following property:

$$R_+(f\,g) = R_+(f)\,R_+(g) + R_+(f\,R_-(g)) + R_+(g\,R_-(f)).$$

Equivalently, both the map R_+ and the map

$$\begin{aligned} R_- = 1 - R_+ : \mathrm{Mer}_0(\mathbb{C}) &\to \mathrm{Mer}_0(\mathbb{C}) \\ f &\mapsto \left(z \mapsto \sum_{j=1}^{k} \frac{\mathrm{Res}_0^j(f)}{z^j} \right) \quad \text{if} \quad f \in \mathrm{Mer}_0^k(\mathbb{C}) \end{aligned}$$

are Rota-Baxter maps of weight -1 on $\mathrm{Mer}_0(\mathbb{C})$.

Proof: The maps R_+ and R_- are clearly linear. The following straightforward identity

$$R_+(f\,g) = R_+(f)\,R_+(g) + R_+(f\,R_-(g)) + R_+(g\,R_-(f)) \tag{1.1}$$

[1] i.e equivalence classes of meromorphic functions defined on a neighborhood of zero for the equivalence relation $f \sim g$ if f and g coincide on some open neighborhood of zero.

is says that the contributions to the holomorphic part of the product differ from the product of the holomorphic parts of f and g by contributions involving residues of f through $R_-(f)$ or residues of g through $R_-(g)$. Setting $R_- = Id - R_+$ shows its equivalence with the Rota -Baxter relation for R_+:

$$\begin{aligned} & R_+(f\,g) = R_+(f)\,R_+(g) + R_+(f\,R_-(g)) + R_+(g\,R_-(f)) \\ \iff & R_+(f\,g) = R_+(f)\,R_+(g) + R_+(f\,(g - R_+(g))) + R_+(g\,(f - R_+(f))) \\ \iff & R_+(f\,g) = R_+(f)\,R_+(g) + 2R_+(f\,g) - R_+(f\,R_+(g))) - R_+(g\,R_+(f))) \\ \iff & R_+(f)\,R_+(g) = R_+(f\,R_+(g))) + R_+(g\,R_+(f))) - R_+(f\,g). \end{aligned}$$

Setting $R_+ = Id - R_-$ then shows the equivalence with the Rota-Baxter relation for R_+:

$$\begin{aligned} & R_+(f)\,R_+(g) = R_+(f\,R_+(g))) + R_+(g\,R_+(f))) - R_+(f\,g) \\ \iff & (f - R_-(f))\,(g - R_-(g)) = f\,(g - R_-(g)) \;- R_-(f\,(g - R_-(g)) \\ + \;& g\,(f - R_-(f)) - R_-(g\,(f - R_-(f))) - f\,g + R_-(fg) \\ \iff & fg + R_-(f)\,R_-(g) - f\,R_-(g) - g\,R_-(f) = f\,g - f\,R_-(g) - R_-(f\,g) + R_-(f\,R_-(g)) \\ + \;& g\,f - g\,R_-(f) - R_-(g\,f) + R_-(g\,R_-(f)) - f\,g + R_-(fg) \\ \iff & R_-(f)\,R_-(g) = R_-(f\,R_-(g))) + R_-(g\,R_-(f)) - R_-(f\,g). \end{aligned}$$

□

Combining the evaluation at zero with the map R_+ –called minimal substraction scheme by physicists– provides a first regularised evaluator on $\mathrm{Mer}_0^k(\mathbb{C})$ at zero:

$$\begin{aligned} \mathrm{ev}_0^{\mathrm{reg}} : \mathrm{Mer}_0^k(\mathbb{C}) &\to \mathbb{C} \\ f &\mapsto \mathrm{ev}_0^{\mathrm{reg}} f(z) := \mathrm{ev}_0 \circ R_+(f), \end{aligned} \tag{1.2}$$

i.e. a linear form that extends the ordinary evaluator $\mathrm{ev}_0(f) = f(0)$ defined on the space $\mathrm{Hol}_0(\mathbb{C})$ of holomorphic functions at zero.

Proposition 2 *Any linear form on* $\mathrm{Mer}_0^k(\mathbb{C})$ *which extends* ev_0 *is of the form:*

$$\lambda = \mathrm{ev}_0^{\mathrm{reg}} + \sum_{j=1}^k \mu_j \,\mathrm{Res}_0^j$$

for some constants $\mu_1, \cdots, \mu_k$.
In particular, all linear forms on $\mathrm{Mer}_0^1(\mathbb{C})$ *which extend* ev_0 *are of the form:*

$$\lambda = \mathrm{ev}_0^{\mathrm{reg}} + \mu\,\mathrm{Res}_0$$

for some constant μ.

Proof: A linear form λ which extends ev_0 coincides with ev_0 on the range of R_+ and therefore fulfills the following identity $\lambda \circ R_+ = \mathrm{ev}_0 \circ R_+ = \mathrm{ev}_0^{\mathrm{reg}}$. Thus, for any $f \in \mathrm{Mer}_0^k(\mathbb{C})$,

$$\begin{aligned} \lambda(f) &= \lambda\,(R_+(f)) + \lambda(R_-(f)) \\ &= \mathrm{ev}_0^{\mathrm{reg}} + \sum_{j=1}^k \lambda(z^{-j})\,\mathrm{Res}_0^j(f) \\ &= \mathrm{ev}_0^{\mathrm{reg}} + \sum_{j=1}^k c_j\,\mathrm{Res}_0^j(f) \end{aligned}$$

where we have set $c_j := \lambda(z^{-j})$. □

1.2 The Gamma function extended to negative integers

The map $\phi : x \mapsto e^{-x}$ defines a Schwartz function on $[0, +\infty[$ such that all its derivatives are also Schwartz functions on $[0, +\infty[$. The integral

$$\Gamma(s) := \int_0^\infty x^{s-1}\,\phi(x)\,dx = \int_0^\infty x^{s-1}\,e^{-x}\,dx$$

defined for $\mathrm{Re}(s) > 0$ is called the **Gamma function**.
Repeated integration by parts on $\mathrm{Re}(s) > 0$ yields:

$$\Gamma(s+k) = s(s+1)\cdots(s+k-1)\,\Gamma(s) \quad \text{if} \quad \mathrm{Re}(s) > 0.$$

A similar formula yields a meromorphic extension to the whole plane defined recursively on half planes $\mathrm{Re}(s) > -k$ for any positive integer k by

$$\Gamma(s) = \frac{\Gamma(s+k)}{s(s+1)\cdots(s+k-1)} \tag{1.3}$$

thereby extending the following formula obtained by rThis meromorphic extension denoted by the same symbol Γ therefore has simple poles at negative integers. The following straightforward statements are useful for later applications.

Proposition 3 *1. The Gamma function is differentiable at any positive integer k and:*

$$\Gamma(k) = (k-1)! \quad \forall k \in \mathbb{N}, \quad \Gamma'(1) = -\gamma; \quad \Gamma'(k) = \Gamma(k)\left(\sum_{j=1}^{k-1}\frac{1}{j} - \gamma\right) \quad \forall k \in \mathbb{N} - \{1\},$$

where

$$\gamma := -\int_0^\infty \log x\, e^{-x}\,dx$$

is the Euler constant.

2. *The Gamma function has simple poles at any non positive integer k with residue:*

$$\mathrm{Res}_{-k}\Gamma = \frac{(-1)^k}{k!}.$$

Furthermore,

$$\Gamma^{\mathrm{reg}}(-k) := \mathrm{ev}_0^{\mathrm{reg}} \circ \Gamma(-k+\cdot) = \frac{(-1)^k}{k!}\left(\sum_{j=1}^{k}\frac{1}{j} - \gamma\right).$$

3. *The inverse Gamma function $\frac{1}{\Gamma(z)}$ defined on the half plane Re$(z) > 0$ extends to a holomorphic map at $z = 0$ and*

$$\frac{1}{\Gamma(z)} = z + \gamma z^2 + o(z^2).$$

In particular, $\left(\frac{1}{\Gamma}\right)'(0) = 1$.

Proof:

1. By integration by parts we have :

$$\Gamma(k) = (k-1)! \quad \forall k \in \mathbb{N}$$

and the derivative of Γ at 1 reads:

$$\Gamma'(1) = \partial_z \Gamma(1+z)|_{z=0} = \int_0^\infty \log x\, e^{-x}\,dx := -\gamma.$$

The derivative at $k \in \mathbb{N} - \{1\}$ reads:

$$\begin{aligned}
\Gamma'(k) &= \partial_z \left(\Gamma(z+k)\right)_{|z=0} \\
&= \partial_z \left((k+z-1)\cdots(z+1)\cdot\Gamma(z+1)\right)_{|z=0} \\
&= \partial_z \left((k+z-1)\cdots(z+1)\right)_{|z=0} \cdot \Gamma(1) + (k-1)!\,\Gamma'(1) \\
&= (k-1)!\left(\sum_{j=1}^{k-1}\frac{1}{j} - \gamma\right) = \Gamma(k)\left(\sum_{j=1}^{k-1}\frac{1}{j} - \gamma\right),
\end{aligned}$$

so that for $k \geq 2$

$$\frac{\Gamma'(k)}{\Gamma(k)} = \sum_{j=1}^{k-1}\frac{1}{j} - \gamma. \tag{1.4}$$

2. By (1.3), the Gamma function has a simple pole at any non positive integer k with residue:

$$\mathrm{Res}_{-k}\Gamma = \lim_{z\to 0}\Gamma(-k+z)\,z = \lim_{z\to 0}\frac{\Gamma(z+1)}{(-k+z)\cdots(-1+z)\,z}\,z = \frac{(-1)^k\,\Gamma(1)}{k!} = \frac{(-1)^k}{k!}.$$

Using (1.3) again we write

$$\begin{aligned}
\Gamma^{\mathrm{reg}}(-k) &:= \mathrm{ev}_0^{\mathrm{reg}} \circ \Gamma(-k+\cdot) \\
&= \lim_{z\to 0}\left(\Gamma(-k+z) - \frac{1}{z}\frac{(-1)^k}{k!}\right) \\
&= \lim_{z\to 0}\frac{1}{z}\left(\frac{\Gamma(z+1)}{(-k+z)\cdots(-1+z)} - \frac{(-1)^k}{k!}\right) \\
&= \partial_z\left(\frac{\Gamma(z+1)}{(-k+z)\cdots(-1+z)}\right)_{|z=0} \\
&= (-1)^k\frac{\Gamma'(1)}{k!} + \frac{(-1)^k}{k!}\sum_{j=1}^{k}\frac{1}{j} \\
&= \frac{(-1)^k}{k!}\left(\sum_{j=1}^{k}\frac{1}{j} - \gamma\right).
\end{aligned}$$

3. By (1.3) we have $\frac{1}{\Gamma(z)} = \frac{\Gamma(z+k)}{z(z+1)\cdots(z+k-1)}$ so that in particular

$$\frac{1}{\Gamma(z)} = \frac{z}{\Gamma(z+1)} = \frac{z}{\Gamma(1)+\Gamma'(1)z+o(z)} = z + \gamma z^2 + o(z^2).$$

□

It is useful for later purposes (related to dimensional regularisation) to express the volume of the unit sphere in terms of the Gamma function.

Lemma 1 *The volume of the unit sphere S^{d-1} in $\mathbb{R}^d$ is given by:*

$$\mathrm{Vol}(S^{d-1}) = \frac{2\pi^k}{(k-1)!} = \frac{2\,\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \tag{1.5}$$

if $d = 2k$ is even and

$$\mathrm{Vol}(S^{d-1}) = \frac{k!\pi^k 2^{2k+1}}{(2k)!} = \frac{\pi^{\frac{n}{2}-1}\,2^{n-1}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(d)}$$

if $d = 2k+1$ is odd.

Remark 2 *Since physicists usually work in dimension 4, they are mostly interested in the even dimensional case.*

Proof: Recall the well-known formula:

$$\int_{\mathbb{R}^d} e^{-|x|^2}\, dx = \pi^{\frac{n}{2}}. \tag{1.6}$$

Indeed, since

$$\int_{\mathbb{R}^d} e^{-|x|^2} dx = \prod_{i=1}^{n} \int_{\mathbb{R}} e^{-x_i^2} dx_i,$$

(1.6) follows from

$$\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta = \pi \int_0^{\infty} e^{-u} du = \pi.$$

On the other hand,

$$\int_{\mathbb{R}^d} e^{-|x|^2} dx = \mathrm{Vol}(S^{d-1}) \cdot \int_0^{\infty} e^{-r^2} r^{n-1} dr,$$

so we need to compute the integral $\int_0^\infty e^{-r^2} r^{d-1} dr$ to determine the volume of S^{n-1}.

- If $n = 2k$, iterated integrations by parts yield

$$\begin{aligned} \int_0^{\infty} e^{-r^2} r^{n-1} dr &= -\frac{1}{2} \int_0^{\infty} -2r e^{-r^2} r^{2(k-1)} dr \\ &= (k-1) \int_0^{\infty} e^{-r^2} r^{2(k-1)-1} dr \\ &= \cdots = (k-1)! \int_0^{\infty} e^{-r^2} r dr = \frac{(k-1)!}{2}. \end{aligned}$$

  Thus

$$\mathrm{Vol}(S^{2k-1}) = \frac{2\pi^k}{(k-1)!}.$$

- If $n = 2k+1$, similarly, we have

$$\begin{aligned} \int_0^{\infty} e^{-r^2} r^{n-1}\, dr &= -\frac{1}{2} \int_0^{\infty} -2r e^{-r^2} r^{2k-1} dr \\ &= \frac{1}{2}(2k-1) \int_0^{\infty} e^{-r^2} r^{2k-2} dr \\ &= \frac{2k-1}{2} \frac{2k-3}{2} \int_0^{\infty} e^{-r^2} r^{2k-4} dr \\ &= \frac{2k-1}{2} \frac{2k-3}{2} \cdots \frac{1}{2} \int_0^{\infty} e^{-r^2} dr \\ &= \sqrt{\pi} \cdot \frac{2k-1}{2} \frac{2k-3}{2} \cdots \frac{1}{2}. \end{aligned}$$

  Hence

$$\mathrm{Vol}(S^{2k}) = \frac{\pi^k 2^k}{(2k-1)(2k-3)\cdots 1} = \frac{\pi^k 2^k 2^k k!}{(2k)!} = \frac{\pi^k 2^{2k} k!}{(2k)!}.$$

1.3 The Mellin transform and regularised evaluators at zero

Let us define another evaluator at zero on another set of maps which we are about to introduce.

Definition 3 *Let b be a real number. Let $\mathcal{F}_0^{b,k}$ (resp.$\mathcal{F}^{b,k}$) denote the vector space generated by smooth functions on $]0,+\infty[$ with asymptotic behaviour at zero of the type*

$$f(\epsilon)\sim_0 \sum_{j=0}^\infty \alpha_j \epsilon^{\frac{j-b}{q}} + \sum_{l=0}^k \sum_{\frac{j-b}{q}\notin\mathbb{Z}} \beta_{j,l}\epsilon^{\frac{j-b}{q}}\log^l\epsilon + \sum_{l=0}^{k+1}\sum_{j=0}^\infty \gamma_{j,l}\epsilon^j\log^l\epsilon \tag{1.7}$$

for some positive q and some real numbers $b,\alpha_j,\beta_{j,l},\gamma_{j,l}$, $j\in\mathbb{N}, l=0,\cdots,k$ (depending on f) (resp. and such that for large enough ϵ,

$$|f(\epsilon)|\leq Ce^{-\epsilon\lambda}$$

for some $\lambda>0$, $C>0$.)
Let us set

$$\mathcal{F}_0^k := \bigcup_{b\in\mathbb{C}} \mathcal{F}_0^{b,k};\quad (\text{resp.}\quad \mathcal{F}^k := \bigcup_{b\in\mathbb{C}} \mathcal{F}^{b,k}).$$

For any non negative integer k, the linear space $\mathcal{F}_0^k$ (resp. $\mathcal{F}^k$) contains the space $C_0([0,+\infty[)$ of continuous functions on $[0,+\infty[$ (resp. the space $\mathcal{S}([0,+\infty[)$ of Schwartz functions on $[0,+\infty[)$ and the linear form

$$\begin{aligned} \mathrm{ev}_0^{\mathrm{reg}}: \mathcal{F}^{b,k} &\to \mathbb{C} \\ f &\mapsto \alpha_b+\gamma_{0,0} \quad \text{if} \quad b\in\mathbb{Z}_{\geq 0} \\ f &\mapsto \gamma_{0,0} \quad \text{if} \quad b\notin\mathbb{Z}_{\geq 0}, \end{aligned}$$

is a regularised evaluator on $\mathcal{F}_0^k$ (and hence on $\mathcal{F}^k$) which extends the ordinary evaluation at zero on $\mathcal{C}([0,+\infty[)$ (resp. $\mathcal{S}([0,+\infty[)$.
The Mellin transform which involves the Gamma function, carries a function in $\mathcal{F}^k$ to one in $\mathrm{Mer}_0(\mathbb{C})$.

Proposition 4 *The* **Mellin transform** *of a function $f\in\mathcal{F}^k$*

$$z\mapsto \mathcal{M}(f)(z) := \frac{1}{\Gamma(z)}\int_0^\infty \epsilon^{z-1}f(\epsilon)\,d\epsilon \tag{1.8}$$

defines a meromorphic map on the complex plane with poles of order $\leq k+1$ at 0. In particular, it is holomorphic [2] *at $z=0$ if f has no logarithmic divergence.*
With the notations of Definition 3, the finite part reads:

$$\mathrm{ev}_0^{\mathrm{reg}}\circ\mathcal{M}(f) = \mathrm{fp}_{\epsilon=0}f(\epsilon) + \sum_{l=1}^{k+1}\frac{(-1)^{l+1}\left(\Gamma^{-1}\right)^{(l+1)}(0)\,\beta_{j_0,l}}{l+1}.$$

and

$$\mathrm{Res}_0^{r+1}\left(\mathcal{M}(f)\right) = \sum_{l=r+1}^{k+1}\frac{(-1)^l\,l!}{(l-r)!}\left(\frac{1}{\Gamma}\right)^{(l-r)}(0)\,\beta_{0,l}.$$

In particular,

$$\mathrm{Res}_0^{k+1}\left(\mathcal{M}(f)(z)\right) = (-1)^{k+1}\,(k+1)!\,\gamma_{0,k+1}$$

and

$$\mathrm{ev}_0^{\mathrm{reg}}\circ\mathcal{M}(f) = \mathrm{fp}_{\epsilon=0}f(\epsilon) + \sum_{l=1}^{k+1}\frac{\left(\frac{1}{\Gamma}\right)^{(l+1)}(0)}{(l+1)!}\left(\mathrm{Res}_0^l\left(\mathcal{M}(f)\right) - \mathrm{Res}_0^{l+1}\left(\mathcal{M}(f)\right)\right).$$

When $k=0$ these formulae boil down to:

$$\mathrm{ev}_0^{\mathrm{reg}}\circ\mathcal{M}(f) = \mathrm{fp}_{\epsilon=0}f(\epsilon) + \gamma\,\mathrm{Res}_0\left(\mathcal{M}(f)\right) \tag{1.9}$$

since $\frac{\left(\Gamma^{-1}\right)^{(2)}(0)}{2}=\gamma$.

[2] The order of the poles at $z=0$ is given by the power of the logarithmic divergence of f at $\epsilon=0$

Proof of the proposition: For an integer N chosen large enough

$$\begin{aligned}
\mathcal{M}(f)(z) &= \frac{1}{\Gamma(z)}\int_0^\infty \epsilon^{z-1}\, f(\epsilon)\, d\epsilon \\
&= \frac{1}{\Gamma(z)}\int_0^1 \epsilon^{z-1}\, f(\epsilon)\, d\epsilon + \frac{1}{\Gamma(z)}\int_1^\infty \epsilon^{z-1}\, f(\epsilon)\, d\epsilon \\
&= \frac{1}{\Gamma(z)}\sum_{j=0}^{N}\alpha_j \int_0^1 \epsilon^{z-1}\epsilon^{\frac{j-b}{q}}\, d\epsilon + \frac{1}{\Gamma(z)}\sum_{l=0}^{k}\sum_{\frac{j-b}{q}\notin\mathbb{Z};j\leq N}\beta_{j,l}\int_0^1 \epsilon^{z-1}\epsilon^{\frac{j-b}{q}}\log^l\epsilon\, d\epsilon \\
&+ \frac{1}{\Gamma(z)}\sum_{l=0}^{k+1}\sum_{j=0}^{N}\gamma_{j,l}\int_0^1 \epsilon^{z-1}\epsilon^{j}\log^l\epsilon\, d\epsilon \\
&+ \frac{1}{\Gamma(z)}\int_1^\infty \epsilon^{z-1}\, f(\epsilon)\, d\epsilon + o(z) \\
&\left(\text{since}\quad \Gamma(z)\int_0^1 \epsilon^{z-1+a}\,\log^l\epsilon\, d\epsilon = o(z)\quad\text{for}\quad\text{large}\quad\mathrm{Re}(a)\right) \\
&= \frac{1}{\Gamma(z)}\sum_{j=0}^{\infty}\alpha_j\left[\frac{\epsilon^{z+\frac{j-b}{q}}}{z+\frac{j-b}{q}}\right]_0^1 \\
&+ \frac{1}{\Gamma(z)}\sum_{l=0}^{k}\sum_{i=0}^{l}\frac{l!\,(-1)^i}{(l-i)!}\sum_{j=0,\frac{j-b}{q}\notin\mathbb{Z}}^{\infty}\beta_{j,l}\left[\frac{\epsilon^{z+\frac{j-b}{q}}}{(z+\frac{j-b}{q})^{i+1}}\log^{l-i}\epsilon\right]_0^1 \\
&+ \frac{1}{\Gamma(z)}\sum_{l=0}^{k+1}\sum_{i=0}^{l}\frac{l!\,(-1)^i}{(l-i)!}\sum_{j=0}^{\infty}\gamma_{j,l}\left[\frac{\epsilon^{z+j}}{(z+j)^{i+1}}\log^{l-i}\epsilon\right]_0^1 \\
&+ \frac{1}{\Gamma(z)}\int_1^\infty \epsilon^{z-1}\, f(\epsilon)\, d\epsilon + o(z)
\end{aligned}$$

where we have used integration by parts repeatedly to write

$$\int_0^1 \epsilon^\alpha\,\log^l\epsilon\, d\epsilon = \sum_{i=0}^{l}\frac{(-1)^i\, l!}{(l-i)!}\left[\frac{\epsilon^{\alpha+1}}{(\alpha+1)^{i+1}}\,\log^{l-i}\epsilon\right]_0^1.$$

This shows the existence of a meromorphic extension on $\mathcal{M}(f)$ in a neighborhood of $z=0$. Writing $\frac{1}{\Gamma(z)}=\sum_{m=1}^{M}\frac{(\Gamma^{-1})^{(m)}(0)}{m!}\,z^m+o(z^M)$, we see that only $\frac{\alpha_j}{\Gamma(z)}\left[\frac{\epsilon^{z+\frac{j-b}{q}}}{z+\frac{j_0-b}{q}}\right]_0^1$ and $\sum_{l=0}^{k+1}\frac{l!\,(-1)^l}{\Gamma(z)}\sum_{j=0}^{\infty}\gamma_{j,l}\left[\frac{\epsilon^{z+j}}{(z+j)^{l+1}}\right]_0^1$ (which corresponds to the terms $l=i$) can actually contribute to the finite part. Since $\left(\frac{1}{\Gamma}\right)'(0)=1$, their contribution amounts to

$$\begin{aligned}
\mathrm{ev}_0^{\mathrm{reg}}\circ\mathcal{M}(f) &= \delta_{j-b}\left(\frac{1}{\Gamma(0)}\right)'(0)+\sum_{l=0}^{k}\frac{(-1)^l\,\left(\frac{1}{\Gamma}\right)^{(l+1)}(0)\,\gamma_{0,l}}{l+1} \\
&= \mathrm{fp}_{\epsilon=0}f(\epsilon)+\sum_{l=1}^{k}(-1)^l\,\frac{\left(\frac{1}{\Gamma}\right)^{(l+1)}(0)\,\gamma_{0,l}}{l+1}
\end{aligned}$$

since $\mathrm{fp}_{\epsilon=0}f(\epsilon)=\alpha_j\delta_{j-b}+\gamma_{0,0}$. Similarly, the only terms contributing to the complex residue come from $\sum_{l=0}^{k+1}\frac{l!\,(-1)^l}{\Gamma(z)}\sum_{j=0}^{\infty}\gamma_{j,l}\left[\frac{\epsilon^{z+j}}{(z+j)^{l+1}}\right]_0^1$ and we have

$$\mathrm{Res}_0^{r+1}\left(\mathcal{M}(f)(z)\right)=\sum_{l=r+1}^{k+1}\frac{(-1)^l\, l!}{(l-r)!}\left(\frac{1}{\Gamma}\right)^{(l-r)}(0)\,\gamma_{0,l}$$

using the fact that $\frac{1}{\Gamma}(0)=0$. In particular,

$$\mathrm{Res}_0^{k+1}\left(\mathcal{M}(f)(z)\right)=(-1)^{k+1}\,(k+1)!\,\gamma_{0,k+1}$$

and

$$\gamma_{0,l} = (-1)^l \frac{\mathrm{Res}_0^l\left(\mathcal{M}(f)\right) - \mathrm{Res}_0^{l+1}\left(\mathcal{M}(f)\right)}{l!}$$

so that

$$\mathrm{ev}_0^{\mathrm{reg}} \circ \mathcal{M}(f) = \mathrm{fp}_{\epsilon=0} f(\epsilon) + \sum_{l=1}^{k+1} \frac{\left(\frac{1}{\Gamma}\right)^{(l+1)}(0)}{(l+1)!} \left(\mathrm{Res}_0^l\left(\mathcal{M}(f)\right) - \mathrm{Res}_0^{l+1}\left(\mathcal{M}(f)\right)\right).$$

This ends the proof of the proposition. □

The regularised evaluator at zero on Mer_0 therefore relates to the regularised evaluator at zero on $\mathcal{F}^k$ as follows:

Corollary 1 *For any f in $\mathcal{F}^k$ and with the notations of Definition 3, we have:*

$$\mathrm{ev}_0^{\mathrm{reg}}\left(\mathcal{M}(f)\right) - \mathrm{ev}_0^{\mathrm{reg}}(f) = \sum_{l=1}^{k+1} \frac{\left(\frac{1}{\Gamma}\right)^{(l+1)}(0)}{(l+1)!} \left(\mathrm{Res}_0^l \mathcal{M}(f) - \mathrm{Res}_0^{l+1}\mathcal{M}(f)\right).$$

If $k=0$ then

$$\mathrm{ev}_0^{\mathrm{reg}} \circ \mathcal{M}(f) - \mathrm{ev}_0^{\mathrm{reg}}(f) = \gamma\, \mathrm{Res}_0 \mathcal{M}(f).$$

1.4 Discrepancies

Whereas evaluators are invariant under a change of variable, regularised evaluators are not in general. Indeed, given a holomorphic h function at zero such that $h(0)=0$, the ordinary evaluator on $\mathrm{Hol}_0(\mathbb{C})$ is invariant under the change of variable $z \mapsto h(z)$:

$$\mathrm{ev}_0(f \circ h) = \mathrm{ev}_0(f).$$

In contrast, such a change of variable produces from the regularised evaluator $\mathrm{ev}_0^{\mathrm{reg}}$ on $\mathrm{Mer}_0(\mathbb{C})$ another regularised evaluator:

$$h_* \mathrm{ev}_0^{\mathrm{reg}}(f) := \mathrm{ev}_0^{\mathrm{reg}}(f \circ h)$$

on $\mathrm{Mer}_0(\mathbb{C})$.

Proposition 5 *For any f in $\mathrm{Mer}_0^k(\mathbb{C})$, for any $h \in \mathrm{Hol}_0$ such that $h(0)=0$ and $h'(0) \neq 0$, the reparametrised regularised evaluator $h_*\mathrm{ev}_0^{\mathrm{reg}}$ differs from the ordinary regularised evaluator $\mathrm{ev}_0^{\mathrm{reg}}$ by a linear expression in the jets of h at zero up to order k:*

$$h_* \mathrm{ev}_0^{\mathrm{reg}}(f) - \mathrm{ev}_0^{\mathrm{reg}}(f) = \sum_{j=1}^{k} \frac{\partial^j\left(k^{-j}\right)(0)}{j!} \mathrm{Res}_0^j f,$$

where we have set $k(z) = \frac{h(z)}{z}$.

Remark 3 *A more explicit formula in terms of jets of h at zero requires Faà di Bruno's formula [FdB] which generalises the chain rule to higher derivatives.*

Proof: We write $f(z) = R_-(f) + R_+(f) = \sum_{j=1}^k a_j z^{-j} + R_+(f)(z)$ so that $f \circ h(z) = R_-(f) \circ h(z) + R_+(f) \circ h(z) = \sum_{j=1}^k a_j h(z)^{-j} + R_+(f) \circ h(z)$. Since

$$\mathrm{ev}_0^{\mathrm{reg}}\left(R_+(f) \circ h\right) = \mathrm{ev}_0\left(R_+(f) \circ h\right) = \mathrm{ev}_0\left(R_+(f)\right) = \mathrm{ev}_0^{\mathrm{reg}}(f)$$

we infer that

$$\mathrm{ev}_0^{\mathrm{reg}}(f \circ h) = \sum_{j=1}^{k} a_j \mathrm{ev}_0^{\mathrm{reg}}\left(z \mapsto h(z)^{-j}\right) + \mathrm{ev}_0^{\mathrm{reg}}(f).$$

We now compute each term $\mathrm{ev}_0^{\mathrm{reg}}\left(z \mapsto h(z)^{-j}\right)$. Since $h(z) = z\, k(z)$ with k holomorphic at zero such that $k(0) \neq 0$, we have:

$$\mathrm{ev}_0^{\mathrm{reg}}\left(z \mapsto h(z)^{-j}\right) = \mathrm{ev}_0^{\mathrm{reg}}\left(z \mapsto \frac{1}{z^j}\, k(z)^{-j}\right) = \frac{\partial^j\left(k^{-j}\right)(0)}{j!}$$

from which we infer the result of the proposition. □

Whereas the ordinary evaluation at zero is compatible with the product on holomorphic functions around zero, regularised evaluators do not a priori factorise on products.

Proposition 6 *For any $f \in \mathrm{Mer}_0^k(\mathbb{C})$ and $g \in \mathrm{Mer}_0^l(:\mathbb{C})$ for some non negative integers k and l,*

$$\mathrm{ev}_0^{\mathrm{reg}}(f\,g) - \mathrm{ev}_0^{\mathrm{reg}}(f)\,\mathrm{ev}_0^{\mathrm{reg}}(g) = \sum_{i=1}^{l};\frac{f^{(i)}(0)}{i!}\,\mathrm{Res}_0^i g(z) + \sum_{i=1}^{k}\frac{g^{(i)}(0)}{i!}\,\mathrm{Res}_0^i f(z).$$

Proof: By (1.1) we have

$$\begin{aligned}
\mathrm{ev}_0^{\mathrm{reg}}\,(f\,g) &= \mathrm{ev}_0\,(R_+(f\,g)) \\
&= \mathrm{ev}_0\,(R_+(f)\,R_+(g)) + \mathrm{ev}_0\,(R_+(f\,R_-(g))) + \mathrm{ev}_0\,(R_+(g\,R_-(f))) \\
&= \mathrm{ev}_0\,(R_+(f))\;\mathrm{ev}_0\,(R_+(g)) + \mathrm{ev}_0\,(R_+(f\,R_-(g))) + \mathrm{ev}_0\,(R_+(g\,R_-(f))) \\
&= \mathrm{ev}_0^{\mathrm{reg}}\,(f)\;\mathrm{ev}_0^{\mathrm{reg}}\,(g) + \sum_{i=1}^{l}\frac{f^{(i)}(0)}{i!}\,\mathrm{Res}_0^i g + \sum_{i=1}^{k}\frac{g^{(i)}(0)}{i!}\,\mathrm{Res}_0^i f,
\end{aligned}$$

from which the result of the proposition follows. □
Multiplication by a holomorphic function h around zero such that $h(0) \neq 0$ therefore yields another regularised evaluator $\mathrm{ev}_0^{\mathrm{reg}}$ on $\mathrm{Mer}_0(\mathbb{C})$:

$$f \mapsto \mathrm{ev}_0^{\mathrm{reg}}(h\,f).$$

Corollary 2 *For any f in $\mathrm{Mer}_0(\mathbb{C})$ with poles at zero of order k, the regularised evaluator*

$$f \mapsto \mathrm{ev}_0^{\mathrm{reg}}(h\,f)$$

differs from the regularised evaluator $f \mapsto \mathrm{ev}_0(h)\,\mathrm{ev}_0^{\mathrm{reg}}$ by a linear expression in the jets of h at zero up to order k:

$$\mathrm{ev}_0^{\mathrm{reg}}(h\,f) = \mathrm{ev}_0(h)\,\mathrm{ev}_0^{\mathrm{reg}}(f) + \sum_{i=1}^{k}\frac{h^{(i)}(0)}{i!}\,\mathrm{Res}_0^i f(z).$$

Proof: This follows from the above proposition applied to $f' = h$. □

2 A first characterisation of the noncommutative residue

We characterise the noncommutative residue as the unique (up to a multiplicaitve factor linear form on symbols which vanishes on smoothing symbols and on partial derivatives. later in these notes, we show that this chracterisation also holds dropping the assumption that it vanishes on smoothing symbols.

2.1 Classical symbols with constant coefficients on $\mathbb{R}^d$

We only give a few definitions and refer the reader to [Sh, Ta, Tr] for further details on classical pseudodifferential symbols.
For any complex number a, let us denote by $\mathcal{S}_{\rm c.c}^a(\mathbb{R}^d)$ the set of smooth functions on $\mathbb{R}^d$ called symbols with constant coefficients, such that for any multiindex $\beta \in \mathbb{N}^d$ there is a constant $C(\beta)$ satisfying the following requirement:

$$|\partial_\xi^\beta \sigma(\xi)| \leq C(\beta) |(1+|\xi|)^{{\rm Re}(a)-|\beta|} \tag{2.10}$$

where ${\rm Re}(a)$ stands for the real part of a, $|\xi|$ for the euclidean norm of ξ. We single out the subset $CS_{\rm c.c}^a(\mathbb{R}^d) \subset \mathcal{S}_{\rm c.c}^a(\mathbb{R}^d)$ of symbols σ, called classical symbols of order a with constant coefficients, such that

$$\sigma(\xi) = \sum_{j=0}^{N-1} \sigma_{a-j}(\xi) + \sigma_{(N)}(\xi) \quad \forall \xi \in \mathbb{R}^d, \quad \text{such} \quad \text{that} \quad |\xi| > 1 \tag{2.11}$$

where $\sigma_{(N)} \in \mathcal{S}_{\rm c.c}^{a-N}(\mathbb{R}^d)$ and $\sigma_{a-j}, , j \in \mathbb{N}_0$ are positively homogeneous of degree $a-j$.

Example 1 *Any polynomial of total degree M in the coordinates x_i of ξ defines a classical symbol of order M.*

Example 2 *The map $\xi \mapsto \frac{1}{|\xi|^2+1}$ has the following asymptotic behaviour*

$$\frac{1}{|\xi|^2+1} \sim_{|\xi|\to\infty} \sum_{k=0}^{\infty} (-1)^k \, |\xi|^{-2k-2} \tag{2.12}$$

and defines a classical symbol of order -2; only if $n = 2p$ is even does the expansion contain a homogeneous term of degree $-n$ with coefficient given by $(-1)^{p-1}$.

The ordinary product of functions sends $CS_{\rm c.c}^a(\mathbb{R}^d) \times CS_{\rm c.c}^b(\mathbb{R}^d)$ to $CS_{\rm c.c}^{a+b}(\mathbb{R}^d)$ provided $b - a \in \mathbb{Z}$; let

$$CS_{\rm c.c}(\mathbb{R}^d) = \langle \bigcup_{a\in\mathbb{C}} CS_{\rm c.c}^a(\mathbb{R}^d) \rangle \tag{2.13}$$

denote the algebra generated by all classical symbols with constant coefficients on $\mathbb{R}^d$. Let

$$CS_{\rm c.c}^{-\infty}(\mathbb{R}^d) = \bigcap_{a\in\mathbb{C}} CS_{\rm c.c}^a(\mathbb{R}^d)$$

be the algebra of smoothing symbols. It follows from (2.10) that

$$f \in CS^{-\infty}(\mathbb{R}^d) \Longrightarrow |f(\xi)| \leq C \, \langle \xi \rangle^a \quad \forall a \in \mathbb{R}$$

and hence that

$$CS^{-\infty}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{S}(\mathbb{R}^d)$ is the algebra of Schwartz functions on $\mathbb{R}^d$. We write $\sigma \sim \sigma'$ for two symbols σ, σ' which differ by a smoothing symbol.
We also denote by $CS_{\rm c.c}^{<p}(\mathbb{R}^d) := \bigcup_{{\rm Re}(a)<p} CS_{\rm c.c}^a(\mathbb{R}^d)$, the set of classical symbols of order with real part $< p$ and by

$$CS_{\rm c.c}^{\notin\mathbb{Z}}(\mathbb{R}^d) := \bigcup_{a\in\mathbb{C}-\mathbb{Z}} CS_{\rm c.c}^a(\mathbb{R}^d) \tag{2.14}$$

the set of non integer order symbols.

2.2 Closed linear forms on symbol valued forms

Symbol valued forms are defined in a similar manner to ordinary differential forms. Let $\mathcal{S}$ be a subset of $CS_{\rm c.c}(\mathbb{R}^d)$ and let

$$\Omega^k\mathcal{S} := \{\sum_{|I|=k}\alpha_I(\xi)\,d\xi_I, \quad \alpha_I\in\mathcal{S}\},$$

the set of $\mathcal{S}$-valued degree k forms on $\mathbb{R}^d$. Provided $\mathcal{A}$ is stable under derivatives, exterior differentiation on forms extends to $\mathcal{S}$- valued forms (see (5.14) in [LP]):

$$\begin{aligned} d:\Omega^k\mathcal{S} &\to \Omega^{k+1}\mathcal{S}\\ \alpha(\xi)\,d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_k} &\mapsto \sum_{i=1}^n\partial_i\alpha(\xi)\,d\xi_i\wedge d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_k}.\end{aligned}$$

We call a symbol valued form α closed if $d\alpha=0$ and exact if $\alpha=d\,\beta$ where β is a symbol valued form; this gives rise to the following cohomology groups

$$H^k\mathcal{A} := \{\alpha\in\Omega^k\mathcal{S}, \quad d\alpha=0\}\,/\,\{d\,\beta, \beta\in\Omega^{k-1}\mathcal{S}\}.$$

A linear form λ on $\mathcal{S}$ extends to $\mathcal{S}$-valued forms by:

$$\begin{aligned}\tilde\lambda:\Omega^\bullet\mathcal{S} &\to \mathbb{C}\\ \alpha(x)\,dx_{i_1}\wedge\cdots\wedge dx_{i_k} &\mapsto \delta_{k-d}\lambda(\alpha) \quad \text{if} \quad i_1<\cdots<i_k.\end{aligned}$$

It is closed if it vanishes on exact forms in which case it induces a linear form

$$\overline{\lambda}: H^\bullet\mathcal{S}\to\mathbb{C},$$

which therefore lies in the dual $\left(H^d(\mathcal{A})\right)'$.
Let us recall Stokes' theorem: If M is an oriented d-dimensional manifold with boundary ∂M equipped with the induced orientation, then for any $d-1$ form α on M with compact support,

$$\int_M d\alpha = \int_{\partial M}\alpha.$$

In particular, if M is boundaryless, $\int_M d\alpha=0$.
We shall use the following consequence of Stokes' theorem: Let M be a compact oriented d-dimensional Riemannian manifold with boundary ∂M equipped with the induced metric, with outward pointing unit normal vector field ν on ∂M, then for any smooth vector field X on M,

$$\int_M {\rm div}X\,d\,{\rm vol} = \int_{\partial M}\langle X,\nu\rangle\,d\,\sigma, \tag{2.15}$$

where $d$vol is the volume measure on M and $d\,\sigma$ the induced measure on the boundary.
If M is a submanifold of $\mathbb{R}^d$ equipped with the measure induced by the canonical measure on $\mathbb{R}^d$, we apply the above to the vector field $X=f\,e_i$ for some i in $\{1,\cdots,d\}$ where $\{e_1,\cdots,e_d\}$ is the orthonormal basis in $\mathbb{R}^d$, and any smooth function f on M. Equation (2.15) reads

$$\int_M \partial x_i f\,d\,{\rm vol} = \int_{\partial M} f\,\langle e_i,\nu\rangle\,d\,\sigma. \tag{2.16}$$

Before we give an exemple in the case of the sphere $S^{d-1}=\{x\in\mathbb{R}^d, \sum_{i=1}^d x_i^2=1\}$ seen as a submanifold of $\mathbb{R}^d$, let us recall the expression of the measure on S^{d-1} induced by the canonical volume form on $\mathbb{R}^d$.

Lemma 2 *[BG] The measure on S^{d-1} induced by the canonical volume form* d*vol* $:= dx_1\wedge\cdots\wedge dx_d$ *on $\mathbb{R}^d$ reads:*

$$d\mu_S(x) = \sum_{j=1}^d(-1)^{j-1}\,x_j\,dx_1\wedge\cdots\wedge d\hat{x}_j\wedge\cdots\wedge dx_d.$$

Moreover, for any transformation T in $GL_d(\mathbb{R})$ and any continuous function f on $\mathbb{R}^d$

$$\int_{S^{d-1}} f\circ T\,d\mu_S = \frac{1}{|{\rm det}T|}\int_{S^{d-1}} f\,d\mu_S. \tag{2.17}$$

Proof: Let us consider the Liouville (or radial) field

$$X(x)=\sum_{i=1}^d x_i\frac{\partial}{\partial x_i}$$

on $\mathbb{R}^d$, which transforms the canonical volume form $\omega(x)=dx_1\wedge\cdot\wedge dx_d$ on $\mathbb{R}^d$ to a $d-1$-form

$$i_X\omega(x)=\sum_{j=1}^d(-1)^{j-1}\,x_j\,dx_1\wedge\cdots\wedge d\hat{x}_j\wedge\cdots\wedge dx_d.$$

At any point $x\in S^{d-1}$, the vector $X(x)$ is normal to S^{d-1}. A basis $(\xi_1,\cdots,\xi_{d-1})$ of T_xS^{d-1} can be completed to a basis $(X(x),\xi_1,\cdots,\xi_{d-1})$ of $T_x\mathbb{R}^d$ such that

$$i_X\omega(x)(\xi_1,\cdots,\xi_{d-1})=\omega(x)(X(x),\xi_1,\cdots,\xi_{d-1})\neq 0$$

since $\omega(x)$ is a volume form on $\mathbb{R}^d$. It follows that $i_X\omega(x)$ defines a volume form on $S^{d(-1}$. Since $X(x)$ is the outgoing normal unitary vector field to S^{d-1}, the basis $(X(x),\xi_1,\cdots,\xi_{d-1})$ can be chosen orthonormal and positively oriented in which case $\omega(x)(X(x),\xi_1,\cdots,\xi_{d-1})=1$ so that $i_X\omega(x)$ is in fact the canonical volume form on the submanifold S^{d-1} of $\mathbb{R}^d$.
The covariance property (2.17) follows from that of the volume measure on $\mathbb{R}^d$ since

$$\left(T^{-1}\right)^*d\mu_S(T(x))=\left(T^{-1}\right)^*i_{T_*X}\omega(T(x))=i_X\left(\left(T^{-1}\right)^*\omega(T(x))\right)=\frac{1}{|\mathrm{det}T|}i_X\omega(x)=\frac{1}{|\mathrm{det}T|}\,d\mu_S(x).$$

□

Example 3 *Let us consider the boundaryless unit sphere $S^{d-1}=\{\xi\in\mathbb{R}^d,|\xi|=1\}$ then for any smooth function f on $\mathbb{R}^d$, we have*

$$\int_{S^{d-1}}\partial_i f\,d\mu_S=0,$$

so that the map $\lambda:\sigma\mapsto\int_{S^{d-1}}\partial_i f\,d\mu_S$ is closed.

Example 4 *Let us consider the unit ball $M=B(0,1):=\{\xi\in\mathbb{R}^d,\quad|\xi|\le 1\}$ with boundary the unit sphere $S^{d-1}=\{\xi\in\mathbb{R}^d,|\xi|=1\}$. Then*

$$\int_{B(0,1)}\partial_i f\,d\,\omega=\int_{S^{d-1}}f\,\langle e_i,\nu\rangle\,d\mu_S.\tag{2.18}$$

Since the subalgebra $CS^{-\infty}_{\mathrm{c.c}}(\mathbb{R}^d)$ of the algebra of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ is stable under derivatives, we infer the following straightforward statement.

Lemma 3 *Integration along $\mathbb{R}^d$ defines a linear form on $CS^{-\infty}(\mathbb{R}^d)$ which vanishes on partial derivatives.*

Proof: By Equation (2.18)

$$\begin{aligned}\int_{B(0,R)}\partial_i f\,\mathrm{dvol}&=R^d\int_{B(0,1)}(\partial_i f)(R\,\cdot)\,\mathrm{dvol}\\&=R^{d-1}\int_{B(0,1)}\partial_i\left(f(R\,\cdot)\right)\,\mathrm{dvol}\\&=R^{d-1}\int_{S^{d-1}}f(R\,\cdot)\langle e_i,\nu\rangle\,d\mu_S\end{aligned}$$

Since f is a Schwartz function on $\mathbb{R}^d$, there is a constant C such that

$$|f(x)|\le C|x|^{-d}$$

for large enough $|x|$ so that for large enough R we have

$$\begin{aligned}\left|\int_{B(0,R)} \partial_i f \, d\mathrm{vol}\right| &= R^{d-1}\left|\int_{S^{d-1}} f(R\,\cdot)\langle e_i, \nu\rangle \, d\mu_S\right| \\ &\leq C\, R^{-1}\, \mathrm{Vol}\left(S^{d-1}\right)\end{aligned}$$

and

$$\int_{\mathbb{R}^d} \partial_i f \, d\,\mathrm{vol} = \lim_{R\to\infty} \int_{B(0,R)} \partial_i f \, d\,\mathrm{vol} = 0.$$

□

Motivated by this example, we set the following definition.

Definition 4 *A linear form on a subspace $\mathcal{S}$ of $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ stable under derivatives fulfills Stokes' property (which by extension, we also call closed) if*

$$\lambda(\partial_i \sigma) = 0 \quad \forall \sigma \in \mathcal{S}.$$

Let us prove an easy but very useful result.

Lemma 4 *If $\mathcal{S}$ is stable under partial differentials and multiplication by polynomials then for any closed linear form $\lambda : \mathcal{S} \to \mathbb{C}$, we have*

$$|\beta| < |\alpha| \Longrightarrow \lambda(x^\beta \partial_x^\alpha \sigma(x)) = 0 \quad \forall \sigma \in \mathcal{S}. \tag{2.19}$$

Proof: We first observe that by asumption, $\sigma \in \mathcal{S} \Rightarrow (x \mapsto x^\beta \partial_x^\alpha \sigma(x)) \in \mathcal{S}$. The implcation (2.19) then follows by induction from repeatedly applying the property $\lambda(\partial_i \tau) = 0$ to $\tau(x) = x^\gamma \partial_x^\delta \sigma(x)$ with $\tau \in \mathcal{S}$ and appropriate multiindices γ and δ. □

Closed linear forms on $\Omega\mathcal{S}$ are in one to one correspondence with linear forms λ on $\mathcal{S}$ with Stokes' property since

$$\begin{aligned} & d\left(\sigma\, dx_1 \wedge \cdots \wedge d\hat{x}_i \wedge \cdots \wedge dx_d\right) = (-1)^i\, \partial_i \sigma\, dx_1 \wedge \cdots \wedge dx_d \\ \Longrightarrow\quad & \tilde{\lambda}\left(d\left(\sigma\, dx_1 \wedge \cdots \wedge d\hat{x}_i \wedge \cdots \wedge dx_d\right)\right) = (-1)^i \lambda(\partial_i \sigma).\end{aligned}$$

2.3 Closed linear forms which vanish on smoothing symbols

Let us first define the noncommutative residue originally introduced by Adler and Manin in the one dimensional case was later extended to all dimensions by Wodzicki in [Wo1] (see also [Wo2] and [Ka1] for a review) and independently by Guillemin [Gu1].

Definition 5 *The noncommutative residue is a linear form on $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ defined by*

$$\mathrm{res}(\sigma) := \frac{1}{(2\pi)^d} \int_{S^{n-1}} \sigma_{-d}(\xi)\, d\mu_S(\xi),$$

where as before

$$d\mu_S(\xi) := \sum_{j=1}^{d} (-1)^{j-1}\, \xi_j\, d\xi_1 \wedge \cdots \wedge d\hat{\xi}_j \wedge \cdots \wedge d\xi_d \tag{2.20}$$

denotes the volume measure on the unit sphere S^{d-1} in $\mathbb{R}^d$ induced by the canonical measure on that space.

The noncommutative residue clearly vanishes on smoothing symbols, which is why we first choose to work "modulo smoothing symbols".

Given a subset $\mathcal{S}$ of $CS_{\mathrm{c.c}}(\mathbb{R})$, we call a symbol valued form $\alpha \in \Omega\mathcal{S}$ closed "modulo smoothing symbols" if $d\alpha \sim 0$ and exact "modulo smoothing symbols" if $\alpha \sim d\,\beta$ where $\beta \in \Omega\mathcal{S}$ is a symbol valued form. Since $\alpha \sim d\beta \Rightarrow d\alpha \sim 0$, this gives rise to the following cohomology groups

$$H^k_\sim \mathcal{S} := \{\alpha \in \Omega^k \mathcal{S}, \quad d\alpha \sim 0\} \,/\, \{\alpha \sim d\,\beta, \beta \in \Omega^{k-1}\mathcal{A}\}.$$

Proposition 7 *The extended noncommutative residue vanishes on classical symbol valued forms which are exact up to smoothing symbols and therefore induces a linear form*

$$\overline{\mathrm{res}}: H^d_{\sim} CS_{\mathrm{c.c}}(\mathbb{R}^d) \to \mathbb{R}.$$

Proof: All we need to show is that the residue vanishes on partial derivatives. If $\sigma \sim \partial_i \tau$, then $\sigma_{-d} = \partial_i \tau_{-d+1}$ where now the index $-d$ (resp. $-d+1$) stands for the $-d$-th (resp. $-d+1$-th) positively homogeneous component of the symbol.
Thus, by (2.15) applied to $M = S^{d-1}$, $X = \tau_{-d+1}\, e_i$ we have:

$$\int_{S^{d-1}} \sigma_{-d}\, d\mu_S = \int_{S^{d-1}} \partial_i \tau_{-d+1}\, d\mu_S = \int_{\partial S^{d-1}} \tau_{-d+1}\, \langle e_i, \nu \rangle\, d\mu_S = 0$$

so that $\mathrm{res}(\partial_i \tau) = 0$. $\square$

The converse which is a result of [FGLS], follows from the following elementary lemma.

Lemma 5 *1. (Euler's theorem)*
For any positively homogeneous functions f of degree a on $\mathbb{R}^d - \{0\}$,

$$\sum_{i=1}^d x_i \partial_i f = a\, f.$$

2. Any positively homogeneous function f on $\mathbb{R}^d - \{0\}$ with vanishing residue is a finite sum of partial derivatives, i.e. there exist positively homogeneous functions $g_i, i = 1, \cdots, d$ such that

$$f = \sum_{i=1}^d \partial_i g_i. \tag{2.21}$$

Proof:

1.

$$\sum_{i=1}^d \partial_i (f(\xi))\, x_i = \frac{\partial}{\partial_t}_{|_{t=1}} f(t\,\xi) = \frac{\partial}{\partial_t}_{|_{t=1}} t^a f(\xi) = a\, f(\xi).$$

(a) If $a \neq -d$ it follows from the first part of the lemma that the positively homogeneous function $f_i(x) = \frac{x_i\, f(x)}{a+d}$ satisfies $\sum_{i=1}^d \partial_i f_i = f$.

(b) We now consider the case $a = -d$. In polar coordinates $(r, \omega) \in \mathbb{R}_0^+ \times S^{d-1}$ the Laplacian reads $\Delta = -\sum_{i=1}^d \partial_i^2 = -r^{1-d} \partial_r (r^{d-1} \partial_r) + r^{-2} \Delta_{S^{d-1}}$. Since $\Delta(g(\omega) r^{2-d}) = r^{-d}\, \Delta_{S^{d-1}} g(\omega)$ we have

$$\Delta(g(\omega) r^{2-d}) = f(r\omega) \Longleftrightarrow \Delta_{S^{d-1}} g = f_{|_{S^{d-1}}}.$$

Setting $F(r\,\omega) := g(\omega)\, r^{2-d}$ it follows that the equation $\Delta F = f$ has a solution if and only if $f \in \mathrm{Ker} \Delta^{\perp}_{S^{d-1}}$ i.e. if $\mathrm{res}(f) = 0$. In that case, $f = \sum_{i=1}^d \partial_i f_i$ where we have set $f_i := \partial_i F$.

$\square$

The following proposition is reminiscent of the characterisation of top degree exact forms by the vanishing of their integral over $\mathbb{R}^d$.

Proposition 8 *Any symbol $\sigma \in CS_{\mathrm{c.c}}(\mathbb{R}^d)$ with vanishing residue is up to some smoothing symbol, a finite sum of partial derivatives, i.e. there exist symbols $\tau_i \in CS_{\mathrm{c.c}}(\mathbb{R}^d), i = 1, \cdots, d$ such that*

$$\sigma \sim \sum_{i=1}^d \partial_i \tau_i. \tag{2.22}$$

Proof: We write $\sigma \sim \sum_{j=0}^\infty \chi\, \sigma_{a-j}$ with $\sigma_{a-j} \in C^\infty(\mathbb{R}^d - \{0\})$ positively homogeneous of degree $a-j$. Since $\mathrm{res}(\sigma_{a-j}) = 0$, by Lemma 5 there are homogeneous functions $\tau_{i,a-j+1}$ such that $\sum_{i=1}^d \partial_i \tau_{i,a-j+1} = \sigma_{a-j}$. Let $\tau_i \sim \sum_{j=1}^\infty \chi\, \tau_{i,a-j+1}$ then

$$\sigma \sim \sum_{i=0}^{d} \sum_{j=0}^{\infty} \chi\, \partial_i \tau_{i,a-j+1} \sim \sum_{i=1}^{d} \partial_i \tau_i. \tag{2.23}$$

Indeed, $\partial_i \chi$ has compact support so that the difference $\sigma - \sum_{i=1}^n \partial_i \tau_i$ is smoothing. Since the τ_i are by construction of order $a+1$, statement (2.22) of the proposition follows. □

Theorem 1 *The map* $\overline{\mathrm{res}} : H^d_\sim CS_{\mathrm{c.c}}(\mathbb{R}^d) \to \mathbb{R}$ *is an isomorphism so that* $H^d_\sim CS_{\mathrm{c.c}}(\mathbb{R}^d) \simeq \mathbb{R}$. *Equivalently, any closed singular linear form on* $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ *is proportional to the noncommutative residue.*

Proof: By Proposition 8, a form which lies in the kernel of $\widetilde{\mathrm{res}}$ is exact up to a smoothing symbol valued form, which yields the injectivity of the map $\overline{\mathrm{res}}$. It is clearly surjective; indeed let $\tau(\xi) := \chi(\xi)\, |\xi|^{-d}$ for some smooth function χ which is one outside the unit ball and vanishes in a neighborhood of 0, then $\mathrm{res}(\tau) \neq 0$ and if we set $\alpha(\xi) := \frac{\tau(\xi)}{\mathrm{res}(\tau)}$, then for any $\lambda \in \mathbb{R}$, $\sigma_\lambda := \lambda\, \alpha$ has residue λ. Thus $\overline{\mathrm{res}}$ is an isomorphism. □

3 A first characterisation of the canonical integral

3.1 Closed linear forms on smoothing symbols

The following straightforward lemma is useful to compute the de Rham cohomology groups $H^\bullet\mathcal{S}$ with values in $\mathcal{S} = CS^{-\infty}_{\rm c.c}(\mathbb{R}^d)$.

Lemma 6 *Let $\alpha \in CS^{-\infty}_{\rm c.c}(\mathbb{R}^{k+1})$, then*

$$\int_{\mathbb{R}} \alpha(\xi,t)\,dt = 0 \Longleftrightarrow \exists \beta \in CS^{-\infty}_{\rm c.c}(\mathbb{R}^{k+1}), \quad \text{such} \quad \text{that} \quad \alpha = \partial_t \beta.$$

More precisely, using a one form $e = e(t)\,dt \in \Omega^1 CS^{-\infty}_{\rm c.c}(\mathbb{R})$ chosen such that $\int_{\mathbb{R}} e(t)\,dt = 1$, we have:

$$\alpha(x,t)\,dt = \left(\alpha(x,t) - \int_{\mathbb{R}} \alpha(x,t)\,dt\right)\,dt + \left(\int_{\mathbb{R}} \alpha(x,t)\,dt\right)\,e(t)\,dt.$$

Proof: The implication from right to left is clear. Setting $\beta(t) := \int_{-\infty}^t \alpha(x,u)\,du$ yields the implication from left to right since $\alpha \in CS^{-\infty}_{\rm c.c}(\mathbb{R}^{k+1}) \Rightarrow \beta \in CS^{-\infty}_{\rm c.c}(\mathbb{R}^{k+1})$. $\square$

Proposition 9
1. *$H^k CS^{-\infty}_{\rm c.c}(\mathbb{R}^d) = 0$ if $k < d$ and*

$$H^d CS^{-\infty}_{\rm c.c}(\mathbb{R}^d) \simeq \mathbb{R}.$$

2. *Integration over $\mathbb{R}^d$ gives rise to an isomorphism*

$$\begin{array}{ccc} H^d CS^{-\infty}_{\rm c.c}(\mathbb{R}^d) & \to & \mathbb{R} \\ \alpha & \mapsto & \int_{\mathbb{R}^d} \alpha. \end{array}$$

3. *Any linear form on smoothing symbols which vanishes on partial derivative is proportional to the integration map over $\mathbb{R}^d$. In other words,*

$$\left(H^d CS^{-\infty}_{\rm c.c}(\mathbb{R}^d)\right)' \sim \mathbb{R}.$$

Proof:

1. That $H^d CS^{-\infty}_{\rm c.c}(\mathbb{R}^d) \simeq \mathbb{R}$ and $H^k CS^{-\infty}_{\rm c.c}(\mathbb{R}^d) = 0$ for any non negative integer $k < d$ can be shown by induction on d integrating along the fibres of the projection map

$$\begin{array}{cccc} \pi: & \mathbb{R}^{k+1} & \to & \mathbb{R}^k \\ & (x,t) & \mapsto & \xi \end{array}$$

to decrease the degree of the form and the dimension simultaneously. Let us make this more precise and set for $\alpha \in \Omega^{\bullet+1} CS^{-\infty}_{\rm c.c}(\mathbb{R}^{k+1})$

$$\pi_\star\left(\alpha(x,t)\,dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_j}\right) := \left(\int_{\mathbb{R}} \alpha(x,t)\,dt\right)\,dx_{i_1} \wedge \cdots \wedge dx_{i_j},$$

which lies in $\Omega^\bullet CS^{-\infty}_{\rm c.c}(\mathbb{R}^k)$ and

$$\pi_\star\left(\alpha(x,t)\,dx_{i_1} \wedge \cdots \wedge dx_{i_j}\right) := 0.$$

If for $\alpha \in \Omega^\bullet CS^{-\infty}_{\rm c.c}(\mathbb{R}^k)$ we furthermore set $e_\star(\alpha) := e \wedge \alpha$ which lies in $\Omega^{\bullet+1} CS^{-\infty}_{\rm c.c}(\mathbb{R}^{k+1})$. Since integration on smoothing symbols commutes with differentation:

$$d\,\pi_\star = \pi_\star\,d \quad \text{and} \quad d\,e_\star = e_\star\,d. \tag{3.24}$$

Thus π_* sends $H^{\bullet+1}CS^{-\infty}_{\rm c.c}(\mathbb{R}^{k+1})$ to $H^{\bullet}CS^{-\infty}_{\rm c.c}(\mathbb{R}^{k})$ and $e_\star$ sends $H^{\bullet}CS^{-\infty}_{\rm c.c}(\mathbb{R}^{k})$ to $H^{\bullet+1}CS^{-\infty}_{\rm c.c}(\mathbb{R}^{k+1})$. Moreover,

$$\pi_\star \circ e_\star = 1 \quad \text{and} \qquad e_\star \circ \pi_\star = 1 \quad \text{in} \quad \text{cohomology} \quad \text{by} \quad \text{Lemma} \quad 6. \tag{3.25}$$

Hence $\pi_\star$ and $e_\star$ are isomorphisms (inverse of each other) and $H^{\bullet+1}CS^{-\infty}_{\rm c.c}(\mathbb{R}^{k+1}) \simeq H^{\bullet}CS^{-\infty}_{\rm c.c}(\mathbb{R}^{k})$. But $H^1CS^{-\infty}_{\rm c.c}(\mathbb{R}) = \mathbb{R}$; indeed, we observe that

$$\alpha(t)\,dt = \alpha_0(t)\,dt + \left(\int_{\mathbb{R}} \alpha(t)\,dt\right)\, e(t)\,dt = d\,\beta(t) + \left(\int_{\mathbb{R}} \alpha(t)\,dt\right)\, e(t)\,dt,$$

which is a special instance of Lemma 6 with $k = 0$ and where we have set $\beta(t) := \int_{-\infty}^t \alpha_0(u)\,du$ which lies in $CS^{-\infty}_{\rm c.c}(\mathbb{R})$. Thus, up to an exact form, the form $\alpha(t)\,dt$ is entirely determined by its integral $\int_{\mathbb{R}} \alpha(t)\,dt$. On the other hand, $H^0CS^{-\infty}_{\rm c.c}(\mathbb{R}^k) = 0$ for any positive integer k since constant smoothing symbols vanish. Hence $H^kCS^{-\infty}_{\rm c.c}(\mathbb{R}^d) = 0$ for any $k < d$.

2. Since the integration map along $\mathbb{R}^d$ vanishes on exact forms in $\Omega CS^{-\infty}_{\rm c.c}(\mathbb{R}^d)$, it gives rise to a linear form

$$\begin{array}{rcl} H^dCS^{-\infty}_{\rm c.c}(\mathbb{R}^d) & \to & \mathbb{R} \\ \alpha & \mapsto & \displaystyle\int_{\mathbb{R}^d} \alpha. \end{array}$$

It is clearly onto since there exists $\beta \in \Omega^dCS^{-\infty}_{\rm c.c}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \beta = 1$. Since $H^dCS^{-\infty}_{\rm c.c}(\mathbb{R}^d)$ is one dimensional by the first part of the proof, it follows that the integration map is an isomorphism.

3. Let $\beta \in \Omega^dCS^{-\infty}_{\rm c.c}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \beta = 1$. Any smoothing symbol valued form $\alpha \in \Omega^dCS^{-\infty}_{\rm c.c}(\mathbb{R}^d)$ reads $\alpha = \alpha_0 + \left(\int_{\mathbb{R}^d} \alpha\right) \beta$ where we have set $\alpha_0 := \alpha - \left(\int_{\mathbb{R}^d} \alpha\right) \beta$. Since $\int_{\mathbb{R}^d} \alpha_0 = 0$ it follows from the isomorphism $\int_{\mathbb{R}^d} : H^dCS^{-\infty}_{\rm c.c}(\mathbb{R}^d) \to \mathbb{R}$ that α_0 is exact. Hence, any $\lambda \in \left(H^dCS^{-\infty}_{\rm c.c}(\mathbb{R}^d)\right)'$ acts on α by

$$\lambda(\alpha) = C \int_{\mathbb{R}^d} \alpha$$

where we have set $C = \lambda(\beta)$.

□

3.2 Closed linear forms on the kernel of the residue

By the previous paragraph, the ordinary integral does not extend to a linear form on the algebra of classical symbols which fulfills Stokes' property. If we insist on extending the ordinary integral,we need to restrict to the kernel of the noncommutative residue.

Definition 6 *We call a subset*

$$CS^{-\infty}_{\rm c.c}(\mathbb{R}^d) \subset \mathcal{S} \subset \mathrm{Ker(res)}$$

admissible if

1. *it is stable under partial differentiation*

$$\sigma \in \mathcal{S} \Longrightarrow \partial_i\sigma \in \mathcal{S} \quad \forall i \in \{1, \cdots, d\},$$

2. *for σ in $\mathcal{S}$, the symbols $\tau_i \sim \sum_{j=0}^\infty \tau_{i,a-j}\,\chi$ arising in the asymptotic expansion $\sigma \sim \sum_{i=1}^d \partial_i\tau_i$ (see (2.22)) can be chosen in $\mathcal{S}$.*

The kernel Ker(res) of the noncommutative residue, which is stable under partial differentiation, does not satisfy the second requirement, since the derivatives of a homogeneous function τ of degree $-d$ with non vanishing residue have vanishing residue.
Nevertheless, there are interesting subsets of Ker(res) with the above properties.

Example 5 *The non integrality of the order of the symbol is a property which is stable under partial differentiation and hence so is the set* $CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d)$ *of non integer order classical symbols with constant coefficients. Furthermore,* $CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d) \subset \mathrm{Ker(res)}$ *and the* $\tau_{i,a-j}$ *arising in (2.22) also have non integer order, so that* $CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d)$ *is an admissible subset of* Ker(res).

Example 6 *The odd-class property for a symbol* σ *in* $CS_{\rm c.c.}(\mathbb{R}^d)$ *of order* a *with homogeneous components, namely that it satisfies the following requirement*

$$\sigma_{a-j}(-\xi) = (-1)^{a-j}\sigma_{a-j}(\xi) \quad \forall j \in \mathbb{N}_0,$$

is a property stable under partial differentiation and hence so is the set

$$CS_{\rm c.c}^{\rm odd}(\mathbb{R}^d) := \{\sigma \in CS_{\rm c.c}^{\mathbb{Z}}(\mathbb{R}^d), \quad \sigma_{a-j}(-\xi) = (-1)^{a-j}\sigma_{a-j}(\xi) \quad \forall \xi \in \mathbb{R}^d \quad \text{if} \quad \mathrm{ord}(\sigma) = a\}$$

of odd-class classical symbols with constant coefficients. An easy computation further shows that $CS_{\rm c.c}^{\rm odd}(\mathbb{R}^d) \subset \mathrm{Ker(res)}$ *if* d *is odd. In that case, it is easy to check that the* $\tau_{i,a-j}$ *arising in (2.22) also lie in the odd-class. Thus,* $CS_{\rm c.c}^{\rm odd}(\mathbb{R}^d)$ *is an admissible subset of* Ker(res) *when* d *is odd.*

Example 7 *Similarly, the even-class property* $\sigma_{a-j}(-\xi) = (-1)^{a-j+1}\sigma_{a-j}(\xi)$ *is stable under partial differentiation and hence so is the set*

$$CS_{\rm c.c}^{\rm even}(\mathbb{R}^d) := \{\sigma \in CS_{\rm c.c}^{\mathbb{Z}}(\mathbb{R}^d), \quad \sigma_{a-j}(-\xi) = (-1)^{a-j+1}\sigma_{a-j}(\xi) \quad \forall \xi \in \mathbb{R}^d \quad \text{if} \quad \mathrm{ord}(\sigma) = a\}$$

of even-class classical symbols with constant coefficients. An easy computation further shows that $CS_{\rm c.c}^{\rm even}(\mathbb{R}^d) \subset \mathrm{Ker(res)}$ *if* d *is even. In that case, it is easy to check that the* $\tau_{i,a-j}$ *arising in (2.22) also lie in the even-class. Thus,* $CS_{\rm c.c}^{\rm odd}(\mathbb{R}^d)$ *is an admissible subset of* Ker(res) *when* d *is even.*

The following result plays an important part in the following.

Theorem 2 1. *Let*

$$CS_{\rm c.c}^{-\infty}(\mathbb{R}^d) \subset \mathcal{A} \subset \mathrm{Ker(res)}$$

be an admissible set.
The ordinary integration map $\int_{\mathbb{R}^d}$ *uniquely extends to a linear form on* $\mathcal{S}$ *with Stokes' property which we call* canonical integral *and denote by* $-\!\!\!\!\!\!\int_{\mathbb{R}^d}$.
Any other linear form on $\mathcal{S}$ *with Stokes' property is proportional to* $-\!\!\!\!\!\!\int_{\mathbb{R}^d}$.

2. *If moreover* $\mathcal{S}$ *is invariant under the action of* $GL_d(\mathbb{R})$ *i.e.:*

$$\sigma \in \mathcal{S} \Longrightarrow \sigma \circ C \in \mathcal{S} \quad \forall C \in GL_d(\mathbb{R}),$$

then $-\!\!\!\!\!\!\int_{\mathbb{R}^d}$ *is covariant i.e.:*

$$|\mathrm{det} C| -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma \circ C = -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma \quad \forall C \in GL_d(\mathbb{R}). \tag{3.26}$$

Proof:

1. **Uniqueness:** Let λ be a linear form on $\mathcal{A}$ with Stokes' property. When restricted to $CS_{\rm c.c.}^{-\infty}(\mathbb{R}^d)$, λ induces a linear form on the algebra of smoothing symbols with Stokes' property. By Proposition 9, this restriction is proportional to the ordinary integration map $\int_{\mathbb{R}^d}$:

$$\exists c \in \mathbb{R}, \quad \text{s.t.} \quad \lambda_{|_{CS_{\rm c.c.}^{-\infty}(\mathbb{R}^d)}} = c \int_{\mathbb{R}^d}.$$

But by Proposition 8, the linear form λ, is uniquely determined by its restriction to smoothing symbols. Indeed, by the assumptions on $\mathcal{S}$, given a symbol σ in $\mathcal{S}$ we have

$$\exists \tau_1, \cdots, \tau_d \in \mathcal{S} \quad \text{s.t.} \quad s_\sigma := \sigma - \sum_{i=1}^{d} \partial_i \tau_i \in CS^{-\infty}_{\text{c.c.}}(\mathbb{R}^d).$$

Since λ vanishes on partial derivatives we infer that

$$\lambda(\sigma) = \sum_{i=1}^{d} \lambda(\partial_i \tau_i) + \lambda(s_\sigma) = \lambda(s_\sigma) = c \int_{\mathbb{R}^d} s_\sigma$$

is uniquely determined by its value on the smoothing symbol s_σ.

2. **Existence:** We need to prove the existence of a linear form on $\mathcal{A}$ with Stokes' property which extends ordinary integration. We first observe that the integration map extends to a linear form on $CS^{<-d}_{\text{c.c.}}(\mathbb{R}^d) \cap \mathcal{S}$ with Stokes' property since symbols of order with real part $< -d$ lie in $L^1(\mathbb{R}^d)$.
   We now want to extend it to the whole set $\mathcal{S}$. By (2.11) a symbol $\sigma \in \mathcal{S}$ can be written:

$$\sigma(\xi) = \sum_{j=0}^{N-1} \sigma_{a-j}(\xi)\, \chi(\xi) + \sigma_{(N)}(\xi) \quad \forall \xi \in \mathbb{R}^d$$

   where χ is a smooth function which vanishes in a neighborhood of 0 and is identically one outside the unit ball and where $\sigma_{(N)}$ is a symbol of order $< -d$. By linearity, it therefore suffices to determine λ on a finite number of expressions of the type $\sigma_{a-j}\, \chi$ involving positively homogeneous components σ_{a-j}.
   We therefore need to define a linear extension λ of $\int_{\mathbb{R}^d}$ on expressions of the type $f\, \chi$ with f a positively homogeneous function in Ker(res). By Lemma 5

$$\text{res}(f) = 0 \Longrightarrow f = \sum_{i=1}^{d} \partial_i f_i$$

   for some homogeneous functions $f_1, \cdots, f_d$. By assumption, $\mathcal{S}$ being admissible, the f_i's can be chosen such that $f_i\, \chi$ lies in $\mathcal{S}$; we are therefore left to define $\lambda(\partial_i f_i\, \chi)$ for any homogeneous function f_i such that $f_i\, \chi$ lies in $\mathcal{S}$. Since it should satisfy Stokes' property, the linear form λ on $\partial_i f_i\, \chi$ reads:

$$\lambda(\partial_i f_i\, \chi) = -\lambda(f_i\, \partial_i \chi) = -\int_{\mathbb{R}^d} f_i\, \partial_i \chi. \tag{3.27}$$

   The second equality follows from the fact that $\partial_i \chi$ is smoothing and that λ coincides with ordinary integration on smoothing symbols. To make sure that equation (3.27) defines λ on $\mathcal{S} \cap \text{Ker(res)}$ consistently, we observe that this definition is independent of the choice of primitive $f_i + c$ of $g_i := \partial_i f_i$. Indeed, applying (2.15) to $X = \chi\, e_i$ we have

$$\int_{B(0,1)} \partial_i \chi\, d\xi = \int_{S^{d-1}} \chi\, \langle e_i, \nu \rangle\, d\mu_S = \int_{S^{d-1}} \langle e_i, \nu \rangle\, d\mu_S = 0,$$

   since the outward pointing normal vector ν to S^{d-1} points in opposite directions at diametrically situated points of the sphere.
   Applying these constructions to each homogeneous function σ_{a-j} with $\text{Re}(a) - j \geq -d$ defines λ on $\mathcal{S}$ by

$$\lambda(\sigma) := -\sum_{i=1}^{d} \sum_{j=0}^{N-1} \int_{\mathbb{R}^d} \tau_{a-j,i}\, \partial_i \chi + \int_{\mathbb{R}^d} \sigma_{(N)}.$$

   The fact that λ satisfies Stokes' property follows from its very construction.

3. **Covariance:** Due to the covariance property of the ordinary integration non L^1 functions and hence on $CS_{\rm c.c.}^{<-d}(\mathbb{R}^d)$, in order to check the covariance of λ, it suffices to check it on symbols $f\,\chi \in \mathcal{S}$ with f positively homogeneous. For any invertible matrix $C \in Gl_d(\mathbb{R})$ and for any homogeneous function $f = \sum_{i=1}^d \partial_i f_i$ such that $f\,\chi$ lies in $\mathcal{S}$, we have:

$$
\begin{aligned}
\lambda((f \circ C)\,(\chi \circ C)) &= \sum_{i=1}^d \lambda(((\partial_i f_i) \circ C)\,(\chi \circ C)) \\
&= \sum_{i=1}^d \sum_{j=1}^d b_{ji} \lambda(\partial_j (f_i \circ C)\,\chi \circ C) \\
&= -\sum_{i=1}^d \sum_{j=1}^d b_{ji} \lambda(f_i \circ C\,\partial_j(\chi \circ C)) \\
&= -\sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d c_{kj} b_{ji} \int_{\mathbb{R}^d} f_i \circ A\,(\partial_j \chi) \circ C \\
&= -|\mathrm{det} C|^{-1} \sum_{i=1}^d \sum_{j=1}^d \delta_{ij} \int_{\mathbb{R}^d} f_i\,(\partial_j \chi) \\
&= -|\mathrm{det} C|^{-1} \sum_{i=1}^d \int_{\mathbb{R}^d} f_i\,(\partial_i \chi) \\
&= |\mathrm{det} C|^{-1} \sum_{i=1}^d \lambda(\partial_i f_i\,\chi) \\
&= |\mathrm{det} C|^{-1}\,\lambda(f\,\chi)
\end{aligned}
$$

where we have set $C = (c_{ij})$ and $C^{-1} = (b_{ij})$ and used the covariance of the restriction of λ to smoothing symbols, which by assumption coincides with ordinary integration. Applying this to each homogeneous component σ_{a-j} of degree $\geq -d$ of a symbol $\sigma \in \mathcal{S}$ of order a yields the result.

□

Applying Theorem 2 to the examples exhibited at the beginning of the paragraph leads to the following statement.

Corollary 3 *Ordinary integration canonically extends to a linear form $⨍_{\mathbb{R}^d}$ on the subsets $CS_{\rm c.c.}^{\notin \mathbb{Z}}(\mathbb{R}^d)$, $CS_{\rm c.c.}^{\rm odd}(\mathbb{R}^d)$ if d is odd, $CS_{\rm c.c.}^{\rm even}(\mathbb{R}^d)$ if d is even, which satisfies Stokes' property.*

4 The cut-off (or Hadamard finite part) integral

The cut-off (or Hadamard finite part) integral provides a realisation of the canonical integral on non integer order symbols. The obstruction which prevents it from extending to a linear form with Stokes' property on the whole algebra of symbols is measured in terms of a noncommutative residue.

4.1 The cut-off integral

Let us first recall a useful technical lemma.

Lemma 7 *Given a symbol $\sigma \in CS^a_{\rm c.c}(\mathbb{R}^d)$, the map $R \mapsto \int_{B(0,R)} \sigma(\xi)\, d\xi$ has an asymptotic expansion as $R \to \infty$ of the form (with the notations of (2.11)):*

$$\int_{B(0,R)} \sigma(\xi)\, d\xi \sim_{R\to\infty} \alpha_0(\sigma) + \sum_{j=0, a-j+d\neq 0}^{\infty} \frac{R^{a-j+d}-1}{a-j+d} \int_{|\xi|=1} \sigma_{a-j}(\xi)\, d\xi + \log R \int_{S^{d-1}} \sigma_{-d} \quad (4.28)$$

for some scalar $\alpha_0(\sigma)$ corresponding to the finite part as $R \to \infty$.

Proof: By (2.11), we write $\sigma = \sum_{j=0}^{N-1} \sigma_{a-j}(\xi)\, \chi(\xi) + \sigma_{(N)}$ with the order of $\sigma_{(N)}$ decreasing with N, we have

$$\begin{aligned}\int_{B(0,R)} \sigma(\xi)\, d\xi &= \int_{B(0,1)} \sigma(\xi)\, \chi(\xi)\, d\xi + \int_{B(0,R)-B(0,1)} \sigma(\xi)\, d\xi \\ &= \int_{B(0,1)} \sigma(\xi)\, \chi(\xi)\, d\xi + \sum_{j=0}^{N-1} \int_{B(0,R)-B(0,1)} \sigma_{a-j}(\xi)\, d\xi + \int_{B(0,R)-B(0,1)} \sigma_{(N)}(\xi)\, d\xi.\end{aligned}$$

We only need to analyse the second integral since the other two converge as R tends to infinity provided N is chosen large enough. Each of the terms $\int_{B(0,R)-B(0,1)} \sigma_{a-j}(\xi)\, d\xi$ reads:

$$\begin{aligned}\int_{B(0,R)-B(0,1)} \sigma_{a-j}(\xi)\, d\xi &= \left(\int_1^R r^{a-j+d-1}\, dr\right) \left(\int_{|\xi|=1} \sigma_{a-j}(\xi)\, d\xi\right) \\ &= \frac{R^{a-j+d}-1}{a-j+d} \left(\int_{|\xi|=1} \sigma_{a-j}(\xi)\, d\xi\right) \quad \text{if} \quad a-j+d \neq 0 \\ &= \log R \left(\int_{S^{d-1}} \sigma_{a-j}(\xi)\, d\xi\right) \quad \text{if} \quad a-j+d = 0,\end{aligned}$$

giving rise to the asymptotic behaviour described in the lemma. □

The finite part $\sigma \mapsto \alpha_0(\sigma)$ defines a linear form which we call the cut-off integral of σ.

Definition 7 *We call cut-off regularised integral the linear form*

$$\begin{aligned}-\!\!\!\!\!\!\int_{\mathbb{R}^d} : \quad CS_{\rm c.c.}(\mathbb{R}^d) &\to \mathbb{R} \\ \sigma &\mapsto \mathrm{fp}_{R\to\infty} \int_{B(0,R)} \sigma(\xi)\, d\xi,\end{aligned}$$

where we have set:

$$\begin{aligned}\mathrm{fp}_{R\to\infty} \int_{B(0,R)} \sigma(\xi)\, d\xi &:= \int_{B(0,1)} \sigma(\xi)\, \chi(\xi)\, d\xi + \int_{\mathbb{R}^d - B(0,1)} \sigma_{(N)}(\xi)\, d\xi \\ &\quad - \sum_{a-j+d\neq 0,\, j=0}^{N-1} \frac{1}{a-j+d} \left(\int_{|\xi|=1} \sigma_{a-j}(\xi)\, d\xi\right), \quad (4.29)\end{aligned}$$

independently of N provided it is chosen large enough.
With these notations the asymptotic expansion (4.28) reads:

$$) \int_{B(0,R)} \sigma(\xi)\, đ\xi \sim_{R\to\infty} -\!\!\!\!\!\!\int_{\mathbb{R}^d} + \sum_{j=0, a-j+d\neq 0}^{\infty} \frac{R^{a-j+d}-1}{a-j+d} \int_{|\xi|=1} \sigma_{a-j}(\xi)\, đ\xi + \log R\, \mathrm{res}(\sigma). \tag{4.30}$$

The cut-off integral extends the ordinary integral since

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi)\, d\xi = \int_{\mathbb{R}^d} \sigma(\xi)\, d\xi \quad \forall \sigma \in CS^{<-d}_{\mathrm{c.c.}}(\mathbb{R}^d).$$

Example 8 *The cut-off regularised integral vanishes on polynomials; indeed for any polynomial $Q(\xi) = \sum_{|\alpha|\leq M} c_\alpha\, \xi^\alpha$ in d variables, we have*

$$\begin{aligned} \mathrm{fp}_{R\to\infty} \sum_{|\alpha|\leq M} c_\alpha \int_{B(0,R)} \xi^\alpha\, d\xi &= \sum_{|\alpha|\leq M} c_\alpha \left(\mathrm{fp}_{R\to\infty} \int_0^\infty r^{|\alpha|+d-1}\, dr\right) \int_{B(0,1)} \xi^\alpha\, d\xi \\ &= \sum_{|\alpha|\leq M} \left(\mathrm{fp}_{R\to\infty} \frac{R^{|\alpha|+d}}{|\alpha|+d}\right) c_\alpha \int_{B(0,1)} \xi^\alpha \\ &= 0. \end{aligned}$$

Example 9 *The cut-off regularised integral of the symbol $\xi \mapsto \frac{1}{|\xi|^2+1}$ on $\mathbb{R}^4$ vanishes:*

$$\begin{aligned} -\!\!\!\!\!\!\int_{\mathbb{R}^4} \frac{1}{|\xi|^2+1} &= \mathrm{fp}_{R\to\infty} \left(\int_0^R \frac{r^3}{r^2+1}\, dr\right) \mathrm{Vol}\left(S^3\right) \\ &= \frac{1}{2}\, \mathrm{fp}_{R\to\infty} \left(\int_0^R \frac{u}{u+1}\, du\right) \mathrm{Vol}\left(S^3\right) \\ &= \frac{1}{2} \left(\mathrm{fp}_{R\to\infty} \int_0^R \left(1 - \frac{1}{u+1}\right) du\right) \mathrm{Vol}\left(S^3\right) \\ &= \frac{1}{2}\, \mathrm{fp}_{R\to\infty} \left(R^2 - \log(1+R^2) + \log 1\right) \mathrm{Vol}\left(S^3\right) \\ &= 0. \end{aligned}$$

An alternative but rather lengthy computation using (4.29) and the asymptotic expansion (2.12) of the symbol as $|\xi|$ tends to infinity, yields the same result

$$\begin{aligned} -\!\!\!\!\!\!\int_{\mathbb{R}^4} \frac{1}{|\xi|^2+1}\, d\xi &= \int_{B_4(0,1)} \frac{1}{|\xi|^2+1}\, d\xi + \int_{\mathbb{R}^4 - B_4(0,1)} \left(\frac{1}{|\xi|^2+1} - |\xi|^{-2} + |\xi|^{-4}\right) d\xi - \frac{1}{2} \int_{S_4(0,1)} |\xi|^{-2}\, d\xi \\ &= \left(\int_0^1 \frac{r^3}{r^2+1}\, dr + \int_1^\infty \left(\frac{r^3}{r^2+1} - r + r^{-1}\right) dr - \frac{1}{2}\right) \mathrm{Vol}(S^3) \\ &= \left(\frac{1}{2} \int_0^1 \frac{u}{u+1}\, du + \int_1^\infty \frac{1}{r\,(r^2+1)}\, dr - \frac{1}{2}\right) \mathrm{Vol}(S^3) \\ &= \left(\frac{1}{2} \int_0^1 (1 - \frac{1}{u+1})\, du + \frac{1}{2} \int_1^\infty \frac{1}{u\,(u+1)}\, du - \frac{1}{2}\right) \mathrm{Vol}(S^3) \\ &= \left(\frac{1}{2} \int_0^1 (1 - \frac{1}{u+1})\, du + \frac{1}{2} \int_1^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right) du - \frac{1}{2}\right) \mathrm{Vol}(S^3) \\ &= \left(\frac{1-\log 2}{2} + \frac{\log 2 - 1}{2}\right) \mathrm{Vol}(S^3) \\ &= 0. \end{aligned}$$

Example 10 *For $k \in \mathbb{N} - \{0\}$*

$$\begin{aligned}
-\!\!\!\!\!\!\int_{\mathbb{R}^4} \frac{1}{(|\xi|^2+1)^{2+k}}\,d\xi &= \frac{1}{2}\,\mathrm{fp}_{R\to\infty}\left(\int_0^R \frac{u}{(u+1)^{2+k}}\,du\right)\mathrm{Vol}\left(S^3\right)\\
&= \frac{1}{2}\,\mathrm{fp}_{R\to\infty}\left(\int_0^R \frac{1}{(u+1)^{k+1}}\,du - \int_0^R \frac{1}{(u+1)^{k+2}}\,du\right)\mathrm{Vol}\left(S^3\right)\\
&= \frac{1}{2}\,\mathrm{fp}_{R\to\infty}\left[\frac{(u+1)^{-k}}{-k} - \frac{(u+1)^{-k-1}}{-k-1}\right]_0^R \mathrm{Vol}\left(S^3\right)\\
&= \frac{1}{2}\,\mathrm{fp}_{R\to\infty}\left[\frac{(R+1)^{-k}}{-k} + \frac{(R+1)^{-k-1}}{k+1} + \frac{1}{k(k+1)}\right]\mathrm{Vol}\left(S^3\right)\\
&= \frac{1}{2\,k(k+1)}\,\mathrm{Vol}\left(S^3\right) = \frac{\pi}{2\,k(k+1)}
\end{aligned}$$

by (1.5) whereas when the order of the symbol is minus the dimension (this corresponds to a logarithmic divergence in the physics terminology),we have

$$\begin{aligned}
-\!\!\!\!\!\!\int_{\mathbb{R}^4} \frac{1}{(|\xi|^2+1)^{2}}\,d\xi &= \frac{1}{2}\,\mathrm{fp}_{R\to\infty}\left(\int_0^R \frac{u}{(u+1)^{a}}\,du\right)\mathrm{Vol}\left(S^3\right)\\
&= \frac{1}{2}\,\mathrm{fp}_{R\to\infty}\left(\int_0^R \frac{1}{(u+1)}\,du - \int_0^R \frac{1}{(u+1)^{2}}\,du\right)\mathrm{Vol}\left(S^3\right)\\
&= \frac{1}{2}\,\mathrm{fp}_{R\to\infty}\left(\log(1+R) + \frac{1}{R+1} - 1\right)\mathrm{Vol}\left(S^3\right)\\
&= -\,\mathrm{Vol}\left(S^3\right) = -2\pi^2.
\end{aligned}$$

4.2 An explicit realisation of the canonical integral on non integer order symbols

Theorem 3 *Given a symbol σ in $CS_{\mathrm{c.c.}}(\mathbb{R}^d)$ we have*

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d} \partial_i\sigma(\xi)\,đ\xi = \mathrm{res}(\xi \mapsto \chi(\xi)\,\sigma(\xi)\,\xi_i\,|\xi|^{-2}), \tag{4.31}$$

independently of the choice of smooth cut-off function χ which vanishes in a neighborhood of zero and is one outside the unit ball.
In particular, the cut-off regularised integral verifies Stokes' property on $CS_{\mathrm{c.c.}}^{\notin\mathbb{Z}}(\mathbb{R}^d)$, where it coincides with the canonical integral.

Proof: We first observe that $CS_{\mathrm{c.c}}^{\notin\mathbb{Z}}(\mathbb{R}^d)$ is an admissible set. By Theorem 2, since the cut-off integral coincides with the usual integral on smoothing symbols, it suffices to show that it vanishes on partial derivatives in order to identify it with the canonical integral. By (2.15) for any index i in $\{1, \cdots, d\}$ and any symbol $\sigma \in CS_{\mathrm{c.c}}^{a}(\mathbb{R}^d)$ we have

$$\begin{aligned}
-\!\!\!\!\!\!\int_{\mathbb{R}^d} \partial_i\sigma(\xi)\,đ\xi &= \mathrm{fp}_{R\to\infty}\int_{B(0,R)} \partial_i\sigma(\xi)\,đ\xi\\
&= \mathrm{fp}_{R\to\infty} R^d \int_{B(0,1)} \partial_i\sigma(R\,\xi)\,đ\xi\\
&= \mathrm{fp}_{R\to\infty} R^{d-1} \int_{B(0,1)} \partial_i(\sigma(R\,\xi))\,đ\xi\\
&= \mathrm{fp}_{R\to\infty} R^{d-1} \int_{S^{d-1}} \sigma(R\,\xi)\,\langle e_i, \nu\rangle\,đ_S\xi
\end{aligned}$$

Hence

$$\begin{aligned}
-\!\!\!\!\!\!\int_{\mathbb{R}^d} \partial_i\sigma(\xi)\, đ\xi &= \sum_{j=0}^{N-1} \mathrm{fp}_{R\to\infty} R^{d-1} \int_{S^{d-1}} \sigma_{a-j}(R\,\xi)\,\langle e_i,\nu\rangle\, đ_S\xi + \mathrm{fp}_{R\to\infty} R^{d-1} \int_{S^{d-1}} \sigma_{(N)}(R\,\xi)\,\langle e_i,\nu\rangle\, đ_S\xi \\
&= \sum_{j=0}^{N-1} \mathrm{fp}_{R\to\infty} R^{a-j+d-1} \int_{S^{d-1}} \sigma_{a-j}(\xi)\,\langle e_i,\nu\rangle\, đ_S\xi \\
&= \delta_{a-j+d-1} \int_{S^{d-1}} \sigma_{a-j}(\xi)\,\langle e_i,\nu\rangle\, đ_S\xi \\
&= \int_{S^{d-1}} \sigma_{-d+1}(\xi)\ \xi_i\, đ_S\xi \\
&= \mathrm{res}(\xi\mapsto \chi(\xi)\,\sigma(\xi)\,\xi_i\,|\xi|^{-2}),
\end{aligned}$$

independently of the choice of smooth cut-off funciton χ which vanishes in a neighborhood of zero and is one outside the unit ball.
Here as before $đ_S\xi = \frac{1}{(2\pi)^d} d_S\xi$ and $d_S\xi = \sum_{i=1}^d (-1)^{d-1}\,\xi_i\, d\xi_1\wedge\cdots\wedge d\xi_{i-1}\, d\xi_{i+1}\cdots d\xi_d$ is the induced measure on the unit sphere S^{d-1} and $\eta = (\frac{xi_1}{|\xi|},\cdots,\frac{xi_d}{|\xi|})$ the outward pointing normal vector at ξ. In particular, $-\!\!\!\!\!\int_{\mathbb{R}^d} \partial_i\sigma(\xi)\, d\xi = 0$ if the order a of σ is non integer. $\square$

Corollary 4 *The cut-off integral does not obey Stokes' property on $CS_{\mathrm{c.c}}(\mathbb{R}^d)$.*

Proof: Take $\sigma^i(\xi) = \chi(\xi)\,\frac{\xi_i}{|\xi|^d}$ where as before χ is a smooth cut-off function which is 1 outside the unit ball and vanishes in a neighborhood of zero. Then

$$\sum_{i=1}^d \int_{S^{d-1}} \sigma^i_{-d+1}(\xi)\ \xi_i\, đ_S\xi = d\int_{S^{d-1}} |\xi|^{-d+2}\, đ_S\xi = \frac{d\,\mathrm{Vol}(S^{d-1})}{(2\pi)^d} \neq 0$$

so that $\sum_{i=1}^d -\!\!\!\!\!\int_{\mathbb{R}^d} \partial_i\sigma^i \neq 0$ which shows that $-\!\!\!\!\!\int_{\mathbb{R}^d}$ does not fulfill Stokes' property on $CS_{\mathrm{c.c}}(\mathbb{R}^d)$. $\square$

5 Translation invariant linear forms on symbols

We first show that the canonical integral on non integer order symbols is translation invariant. We then prove that the canonical integral is the unique translation invariant linear extension of the ordinary integration map to admissible sets of classical symbols in Ker(res). Similarly, we construct a unique $\mathbb{Z}^d$-translation invariant linear extension to admissible sets of classical symbols in Ker(res) of the ordinary discrete summation on L^1 symbols.

5.1 The action of the translation group on the cut-off integral

The following lemma shows that $\mathbb{R}^d$ acts via translations on the algebra $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ of classical symbols.

Lemma 8 *Given a symbol σ in $CS^a_{\mathrm{c.c}}(\mathbb{R}^d)$, for any $\eta \in \mathbb{R}^d$, the translated symbol $t^*_\eta\sigma := \sigma(\cdot + \eta)$ lies in $CS^a_{\mathrm{c.c}}(\mathbb{R}^d)$.*

Proof:

1. We first show that $t^*_\eta\sigma$ is a symbol of order a i.e. that for any η in $\mathbb{R}^d$ and for any multiindex α, there is a constant $C_\alpha(\eta)$ such that

$$|\partial_x^\alpha \sigma(\xi+\eta)| \leq C_\alpha(\eta)\langle\xi\rangle^{\mathrm{Re}(a)-|\alpha|} \quad \forall \xi \in \mathbb{R}^d. \tag{5.32}$$

Since σ is a symbol of order a, we know there exists a constant C_α such that

$$\left|\partial_\xi^\alpha \sigma(\xi+\eta)\right| \leq C_\alpha\langle\xi+\eta\rangle^{\mathrm{Re}(a)-|\alpha|} \quad \forall \xi \in \mathbb{R}^d.$$

Since $\lim_{|\xi|\to\infty}\frac{\langle\xi+\eta\rangle}{\langle\xi\rangle} = 1$, there are constants $C'(\eta)$ and $C''(\eta)$ such that

$$C'(\eta)\langle\xi\rangle \leq \langle\xi+\eta\rangle \leq C''(\eta)\langle\xi\rangle,$$

hence the existence of a constant $\partial_\alpha(\eta)$ such that

$$\langle\xi+\eta\rangle^{\mathrm{Re}(a)-|\alpha|} \leq \partial_\alpha(\eta)\,\langle\xi\rangle^{\mathrm{Re}(a)-|\alpha|}.$$

The constant $C_\alpha(\eta) := \partial_\alpha(\eta)\,C_\alpha$ thus satisfies (5.32).

2. We now show that $t^*_\eta\sigma$ is classical. Since σ is classical, we have

$$\sigma(\xi+\eta) \sim \sum_{j=0}^\infty \sigma_{a-j}(\xi+\eta)\,\chi(\xi+\eta),$$

where as before χ is a smooth function which vanishes in a neighborhood of zero and is identically one outside the unit ball. Since $\xi \mapsto \chi(\chi+\eta) - \chi(\xi)$ has compact support, it follows that

$$\sigma(\xi+\eta) \sim \sum_{j=0}^\infty \sigma_{a-j}(\xi+\eta)\,\chi(\xi).$$

For any non negative integer j the following Taylor expansion at $\eta = 0$:

$$t^*_\eta\sigma_{a-j}(\xi) = \sum_{|\beta|\leq N-1} \partial_\xi^\beta\sigma_{a-j}(\xi)\frac{\eta^\beta}{\beta!} + N\sum_{|\beta|=N}\left(\int_0^1 (1-u)^{N-1}\partial_\xi^\beta\sigma_{a-j}((1-u)\xi+u\eta)du\right)\frac{\eta^\beta}{\beta!}$$

shows that

$$t^*_\eta\sigma(\xi) \sim \sum_{i=0}^\infty (t^*_\eta\sigma)_{a-i}(\xi)\,\chi(\xi), \quad \text{where} \quad (t^*_\eta\sigma)_{a-i}(\xi) := \sum_{j+|\beta|=i}\partial_\xi^\beta\sigma_{a-j}(\xi)\frac{\eta^\beta}{\beta!},$$

which is positively homogeneous of degree $a-i$.

□

The following proposition shows that the cut-off integral is not invariant under the action of the translation group.

Proposition 10 *For any σ in $CS_{\rm c.c.}(\mathbb{R}^d)$, the difference*

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d} t_\eta^*\sigma - -\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma = \sum_{|\beta|\leq[{\rm Re}(a)]+d}\left(-\!\!\!\!\!\!\int_{\mathbb{R}^d}\partial_\xi^\beta\sigma(\xi)\,d\xi\right)\frac{\eta^\beta}{\beta!}$$

is a polynomial in η. If σ lies in $CS_{\rm c.c.}^{\notin\mathbb{Z}}(\mathbb{R}^d)$, then

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d} t_\eta^*\sigma = -\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma.$$

Proof: A Taylor expansion at $\eta=0$ yields:

$$t_\eta^*\sigma(\xi) = \sum_{|\beta|\leq N-1}\partial_\xi^\beta\sigma(\xi)\frac{\eta^\beta}{\beta!} + N\sum_{|\beta|=N}\left(\int_0^1(1-u)^{N-1}\partial_\xi^\beta\sigma((1-u)\xi+u\eta)du\right)\frac{\eta^\beta}{\beta!}. \tag{5.33}$$

If σ has order a, the map $\xi\mapsto\sigma((1-u)\xi+u\eta)$ defines a family of symbols of order $a-j$ parametrised by u in $[0,1[$ which becomes a constant function in ξ for $u=1$. The map $\xi\mapsto\int_0^1(1-u)^{N-1}\sigma((1-u)\xi+u\eta)\,du$ therefore defines a symbol of same order a. For large enough $|\beta|$, the symbol $\xi\mapsto\partial_\xi^\beta\sigma((1-u)\xi+u\eta)$ lies in $CS_{\rm c.c.}^{<-d}(\mathbb{R}^d)$ and hence so does $\xi\mapsto\int_0^1(1-u)^{N-1}\partial_\xi^\beta\sigma((1-u)\xi+u\eta)\,du$. On $CS_{\rm c.c.}^{<-d}(\mathbb{R}^d)$ the canonical integral coincides with the ordinary integral so that for $|\beta|>{\rm Re}(a)+d$ we have

$$\int_{\mathbb{R}^d}\left(\int_0^1(1-u)^{N-1}\partial_\xi^\beta\sigma((1-u)\xi+u\eta)\,du\right)d\xi = \int_0^1(1-u)^{N-1}\left(\int_{\mathbb{R}^d}\partial_\xi^\beta\sigma((1-u)\xi+u\eta)\,d\xi\right)du.$$

Hence by linearity of the canonical integral and for $N=[{\rm Re}(a)]+d+1$,

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}t_\eta^*\sigma(\xi)\,d\xi = \sum_{|\beta|\leq N-1}\left(-\!\!\!\!\!\!\int_{\mathbb{R}^d}\partial_\xi^\beta\sigma(\xi)d\xi\right)\frac{\eta^\beta}{\beta!}+N\sum_{|\beta|=N}\left(\int_0^1(1-u)^{N-1}\left(\int_{\mathbb{R}^d}\partial_\xi^\beta\sigma((1-u)\xi+u\eta)d\xi\right)du\right)\frac{\eta^\beta}{\beta!}.$$

Since the ordinary integral on $CS_{\rm c.c.}^{<-n}(\mathbb{R}^d)$ vanishes on partial derivatives, it follows that

$$\begin{aligned}
&-\!\!\!\!\!\!\int_{\mathbb{R}^d}t_\eta^*\sigma(\xi)\,d\xi - -\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma(\xi)\,d\xi\\
=\;& \sum_{0<|\beta|\leq N-1}\left(-\!\!\!\!\!\!\int_{\mathbb{R}^d}\partial_\xi^\beta\sigma(\xi)d\xi\right)\frac{\eta^\beta}{\beta!}+-\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma(\xi)\,d\xi + N\sum_{|\beta|=N}\left(\int_0^1(1-u)^{N-1}\left(\int_{\mathbb{R}^d}\partial_\xi^\beta(t_{u\eta}^*\sigma)((1-u)\xi)d\xi\right)du\right)\frac{\eta^\beta}{\beta!}\\
=\;& \sum_{0<|\beta|\leq N-1}\left(-\!\!\!\!\!\!\int_{\mathbb{R}^d}\partial_\xi^\beta\sigma(\xi)d\xi\right)\frac{\eta^\beta}{\beta!}.
\end{aligned}$$

If σ has non integer order, then the cut-off integral which coincides with the canonical integral on non integer order vanishes on the derivatives $\partial_\xi^\beta\sigma$ so that

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}t_\eta^*\sigma(\xi)\,d\xi = -\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma(\xi)\,d\xi.$$

□

5.2 Translation invariance versus closedness

Let $\mathcal{S}\subset CS_{\rm c.c}(\mathbb{R}^d)$ be a subset stable under $\mathbb{R}^d$- (resp. $\mathbb{Z}$-) translations i.e:

$$\sigma\in\mathcal{S}\Longrightarrow t_\eta^\star\sigma\in\mathcal{S}\quad\forall\xi\in\mathbb{R}^d\quad({\rm resp}\mathbb{Z}^d).$$

Definition 8 *A linear form* $\lambda : \mathcal{S} \to \mathbb{C}$ *is said to be* $\mathbb{R}^d$ *(resp.* $\mathbb{Z}^d$*) -translation invariant whenever it is invariant under the action of the translation group:*

$$t_\eta^* \lambda = \lambda \quad \forall \eta \in \mathbb{R}^d \quad (\text{resp.} \quad \forall \eta \in \mathbb{Z}^d),$$

where

$$t_\eta^\star \lambda(\sigma) := \lambda(t_\eta^\star \sigma) \quad \forall \sigma \in CS_{\rm c.c}(\mathbb{R}^d).$$

The following proposition relates $\mathbb{R}^d$- translation invariance and Stokes' property.

Proposition 11 *With the notations of the above definition, let* $\lambda : \mathcal{S} \to \mathbb{C}$ *be a linear form which restricts on* $\mathcal{S} \cap CS^{<-d}(\mathbb{R}^d)$ *to a translation invariant linear form.*

1. *The linear form* λ *is* $\mathbb{R}^d$*-translation invariant if and only if it vanishes on partial derivatives.*
2. *If the linear form* λ *is* $\mathbb{Z}^d$*-translation invariant, then its value on derivatives* $\lambda(\partial^\beta \sigma), \quad \beta \neq 0, \sigma \in CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d)$ *is uniquely determined by its restriction to* $\mathcal{S} \cap CS^{<-d}(\mathbb{R}^d)$.

Proof: Let ρ denote the restriction of λ to $\mathcal{S} \cap CS^{<-d}(\mathbb{R}^d)$, which by assumption is translation invariant.

1. Using the Taylor expansion (5.33) , we have

$$t_\eta^* \lambda(\sigma) = \lambda(\sigma) \Longleftrightarrow \sum_{0<|\beta|\leq N} \lambda(\partial^\beta \sigma) \frac{\eta^\beta}{\beta!} + N \sum_{|\beta|=N+1} \frac{\eta^\beta}{\beta!} \int_0^1 (1-t)^N \rho\left(\partial^\beta \sigma(\cdot + t\eta)\right) dt = 0.$$

   Differentiating this identity with respect to the coordinates of η at $\eta = 0$ yields the first part of the assertion.

2. Let λ_1 and λ_2 be two $\mathbb{Z}^d$-translation invariant linear forms on $CS^{\notin \mathbb{Z}}(\mathbb{R}^d)$ satisfying the assumptions of the theorem with the same restriction ρ. The Taylor formula (5.33) applied to $\sigma \in \mathcal{S}$ yields by linearity of λ_i and for N chosen large enough:

$$\lambda_i\left(t_\eta^* \sigma\right) = \sum_{|\beta|\leq N} \lambda_i\left(\partial^\beta \sigma\right) \frac{\eta^\beta}{\beta!} + N \sum_{|\beta|=N+1} \frac{\eta^\beta}{\beta!} \int_0^1 (1-t)^N \rho\left(\partial^\beta \sigma(\cdot + t\eta)\right) dt,$$

   so that $t_\eta^* \lambda_1(\sigma) - \sum_{|\beta|\leq N} \lambda_1(\partial^\beta \sigma) \frac{\eta^\beta}{\beta!} = t_\eta^* \lambda_2(\sigma) - \sum_{|\beta|\leq N} \lambda_2(\partial^\beta \sigma) \frac{\eta^\beta}{\beta!} \quad \forall \eta \in \mathbb{Z}^d$ since λ_1 and λ_2 both coincide with ρ on $\mathcal{S} \cap CS_{\rm c.c}^{<-d}(\mathbb{R}^d)$. Since $t_\eta^* \lambda_i = \lambda_i$ for any $\eta \in \mathbb{Z}^d$, this implies that the polynomial expressions $\sum_{0<|\beta|\leq N} \lambda_1(\partial^\beta \sigma) \frac{\eta^\beta}{\beta!}$ and $\sum_{0<|\beta|\leq N} \lambda_2(\partial^\beta \sigma) \frac{\eta^\beta}{\beta!}$ in the coordinates of η coincide for all $\eta \in \mathbb{Z}^d$ and hence that their coefficients coincide $\lambda_1(\partial^\beta \sigma) = \lambda_2(\partial^\beta \sigma)$ when $0 < |\beta| < N$. Since this holds for any large enough N, we conclude that $\lambda_1(\partial^\beta \sigma) = \lambda_2(\partial^\beta \sigma)$ when $\beta \neq 0$. It follows that the value of λ on derivatives $\lambda(\partial^\beta \sigma), \quad \beta \neq 0, \quad \sigma \in \mathcal{A}$ is uniquely determined by the restriction ρ to $CS_{\rm c.c}^{<-d}(\mathbb{R}^d)$.

□

The following theorem then directly follows from Theorem 1.

Theorem 4 *Any translation invariant linear form on* $CS_{\rm c.c}(\mathbb{R}^d)$ *which vanishes on smoothing symbols and fulfills Stokes' property is proportional to the noncommutative residue.*

The following theorem then directly follows from Theorem 2.

Theorem 5 *Let*

$$CS_{\rm c.c}^{-\infty}(\mathbb{R}^d) \subset \mathcal{A} \subset \text{Ker(res)}$$

be an admissible set.
The ordinary integration map $\int_{\mathbb{R}^d}$ *uniquely extends to a translation invariant linear form on* $\mathcal{S}$*, which coincides with the canonical integral* $⨍_{\mathbb{R}^d}$ *introduced in Theorem 2. Equivalently, any translation invariant linear form on* $\mathcal{A}$ *which fulfills Stokes' property is proportional to the canonical integral.*

A similar statement gives the uniqueness (but not the existence) of $\mathbb{Z}^d$-translation invariant linear extensions of the discrete summation on L^1-symbols to admissible sets.

Theorem 6 *Let*

$$CS_{\rm c.c}^{-\infty}(\mathbb{R}^d) \subset \mathcal{A} \subset {\rm Ker(res)}$$

be an admissible set.
Whenever the ordinary discrete summation map $\sum_{\mathbb{Z}^d}$ on L^1-classical symbols extends to a $\mathbb{Z}^d$-translation invariant linear form on $\mathcal{S}$, this extension is unique. We call it the canonical sum and denote it by $\sum_{\mathbb{Z}^d}$.

Proof: Let λ be a $\mathbb{Z}^d$- translation invariant linear form on $\mathcal{S}$ which extends the ordinary discrete summation on L^1-symbols. By Proposition 8, a symbol σ in Ker(res), can be written

$$\sigma = \sum_{i=1}^{d} \partial_i \tau_i + \tau$$

as a finite sum of derivatives of symbols τ_i in $\mathcal{S}$ and a smoothing symbol τ. By assumption, λ coincides with the ordinary summation map on smoothing symbols, so that

$$\lambda(\sigma) = \sum_{i=1}^{d} \lambda(\partial_i \tau_i) + \sum_{\mathbb{Z}^d} \tau.$$

By Proposition 11, λ being $\mathbb{Z}^d$-translation invariant, its value on partial derivatives is entirely determined by its restriction to L^1-symbols also given by the ordinary summation map, so that $\lambda(\sigma)$ is entirely determined by its restriction $\sum_{\mathbb{Z}^d}$ to L^1-symbols. This determines λ uniquely. $\square$

A large part of the next section is dedicated to the construction of the cut-off discrete sum on $CS_{\rm c.c.}^{\notin \mathbb{Z}}(\mathbb{R}^d)$.

6 The cut-off discrete (or Hadamard finite part) sum on non integer order symbols

We provide an explicit realisatton of the canonical sum on non integer order symbols, namely of the unique linear extension of the ordinary discrete sum on L^1 symbols which fulfills Stokes' property.

6.1 The classical Euler-MacLaurin formula

The classical Euler-MacLaurin formula which relates a sum to an integral, involves the Bernoulli numbers defined by the following Taylor expansion at $t = 0$:

$$\frac{t}{e^t - 1} := \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{6.34}$$

Since $\frac{t}{e^t-1} + \frac{t}{2} = \frac{t}{2}\frac{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}$ is an even function, $B_1 = -\frac{1}{2}$ and $B_{2k+1} = 0$ for any positive integer k.

Remark 4 *In view of a generalisation to higher dimensions, it is useful to observe that* $\frac{t}{e^t-1} = \mathrm{Td}(-t)$ *where* $\mathrm{Td}(t) := \frac{t}{1-e^{-t}}$ *is the Todd function so that*

$$\mathrm{Td}(t) = \sum_{n=0}^{\infty}(-1)^n B_n \frac{t^n}{n!} = \frac{t}{2} + \sum_{k=0}^{\infty} B_{2k} \frac{t^{2k}}{(2k)!}. \tag{6.35}$$

Here are some values of the Bernoulli numbers see e.g. [Ca]: $B_0 = 1, \quad B_1 = -\frac{1}{2}; \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}; \quad B_6 = \frac{1}{42}; \quad B_8 = -\frac{1}{30}; B_{10} = \frac{5}{66}$.
Bernoulli polynomials are defined similarly by:

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t\, e^{t\,x}}{e^t - 1}, \tag{6.36}$$

so that in particular, $B_n(0) = B_n$. This initial condition combined with the differential equations obtained from differentiating (6.36) with respect to x

$$\partial_x B_n(x) = n\, B_{n-1}(x), \tag{6.37}$$

completely determine the Bernoulli polynomials. Indeed, we have:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, x^k, \tag{6.38}$$

so that for example $B_1(x) = -\frac{1}{2} + x$.
Furthermore, $\sum_{n=0}^{\infty} (B_n(1) - B_n(0)) \frac{t^n}{n!} = \frac{te^t - t}{e^t-1} = t$ so that

$$B_1(1) = B_1(0) + 1 \quad \text{and} \quad B_n(1) = B_n(0) \quad \forall n \geq 2.$$

It is useful to observe that since $B_n(1) = B_n$ for any $n \geq 2$, setting $x = 1$ we have

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k \quad \forall n \geq 2. \tag{6.39}$$

Let us recall the Euler-MacLaurin formula (see e.g. [Ha]).

Proposition 12 *For any function f in $C^\infty(\mathbb{R})$ and any two integers $M < N$*

$$\begin{aligned}\sum_{n=M}^{N} f(n) \quad &= \quad \frac{f(M)+f(N)}{2} + \int_1^N f(x)\,dx + \sum_{k=2}^{K}(-1)^k \frac{B_k}{k!}\left(f^{(k-1)}(N) - f^{(k-1)}(M)\right) \\ &+ \quad \frac{(-1)^{K-1}}{K!}\int_M^N \overline{B_K}(x)\, f^{(K)}(x)\,dx \end{aligned} \tag{6.40}$$

with $\overline{B_k}(x) = B_k\,(x - [x])$ and where K is any positive integer larger than 1.

Remark 5 • *As it will become clear from the proof, the lower bound* 1 *in the discrete sum on the l.h.s. can be replaced by any other integer, in which case the lower bound of the integral on the r.h.s. should also be replaced by this integer.*

- $\overline{B_k}(x) = B_k(x-j)$ *is smooth on any interval* $[j, j+1[, j \in \mathbb{Z}$.
- *The index* K *can be chosen arbitrarily large.*

Proof of the Proposition: Let us set for convenience:

$$S_K(f) := \sum_{k=2}^{K} (-1)^k \frac{B_k}{k!} \left(f^{(k-1)}(N) - f^{(k-1)}(M) \right); \quad I_K(f) := \frac{(-1)^{K-1}}{K!} \int_M^N \overline{B_K}(x)\, f^{(K)}(x)\, dx.$$

We first observe that

$$S_K(f) + I_K(f) = S_{K+1}(f) + I_{K+1}(f)$$

which shows that K can be chosen arbitrarily large. Indeed, using (6.37) as well as $B_{k+1}(1) = B_{k+1}(0)$ we have by integration by parts

$$\begin{aligned}
S_{K+1}(f) + I_{K+1}(f) &= \sum_{k=2}^{K+1} (-1)^k \frac{B_k}{k!} \left(f^{(k-1)}(N) - f^{(k-1)}(M) \right) + \frac{(-1)^K}{(K+1)!} \int_M^N \overline{B_{K+1}}(x)\, f^{(K+1)}(x)\, dx \\
&= \sum_{k=2}^{K+1} (-1)^k \frac{B_k}{k!} \left(f^{(k-1)}(N) - f^{(k-1)}(1) \right) + \frac{(-1)^K}{(K+1)!} \sum_{j=M}^{N-1} \int_j^{j+1} B_{K+1}(x-j)\, f^{(K+1)}(x)\, dx \\
&= \sum_{k=2}^{K+1} (-1)^k \frac{B_k}{k!} \left(f^{(k-1)}(N) - f^{(k-1)}(M) \right) + \frac{(-1)^{K-1}}{K!} \sum_{j=M}^{N-1} \int_j^{j+1} B_K(x-j)\, f^{(K)}(x)\, dx \\
&+ \frac{(-1)^K}{(K+1)!} \sum_{j=M}^{N-1} \left[B_{K+1}(1)\, f^{(K)}(j+1) - B_{K+1}(0) f^{(K)}(j) \right] \\
&= \sum_{k=2}^{K} (-1)^k \frac{B_k}{k!} \left(f^{(k-1)}(N) - f^{(k-1)}(M) \right) + \frac{(-1)^{K-1}}{K!} \int_M^N \overline{B_K}(x)\, f^{(K)}(x)\, dx \\
&= S_K(f) + I_K(f).
\end{aligned}$$

Iterating this as K decreases and using (6.37) we find:

$$\begin{aligned}
S_K(f) + I_K(f) &= S_2(f) + I_2(f) \\
&= \frac{B_2}{2} \left(f'(N) - f'(M) \right) - \frac{1}{2} \int_1^N \overline{B_2}(x)\, f^{(2)}(x)\, dx \\
&= \frac{B_2}{2} \left(f'(N) - f'(M) \right) + \frac{1}{2} \int_M^N \overline{B_2}'(x)\, f'(x)\, dx - \frac{1}{2} \left[\overline{B_2}(x)\, f'(x) \right]_M^N \\
&= \frac{B_2}{2} \left(f'(N) - f'(M) \right) + \int_1^N \overline{B_1}(x)\, f'(x)\, dx - \frac{1}{2} \left(\overline{B_2}(N)\, f'(N) - \overline{B_2}(1)\, f'(1) \right) \\
&= \frac{B_2}{2} \left(f'(N) - f'(1) \right) + \int_M^N \overline{B_1}(x)\, f'(x)\, dx - \frac{B_2}{2} \left(f'(N) - f'(M) \right) \\
&= \int_M^N \overline{B_1}(x)\, f'(x)\, dx
\end{aligned}$$

and hence

$$
\begin{aligned}
S_K(f)+I_K(f) &= -\frac{1}{2}\left(f(N)-f(1)\right)+\sum_{j=M}^{N-1}\int_j^{j+1}(x-j)\,f'(x)\,dx\\
&= -\frac{1}{2}\left(f(N)-f(M)\right)-\sum_{j=M}^{N-1}\int_j^{j+1}f(x)\,dx+\sum_{j=M}^{N-1}\left[(x-j)\,f(x)\right]_j^{j+1}\\
&= -\frac{1}{2}\left(f(N)-f(M)\right)-\int_M^N f(x)\,dx+\sum_{j=1}^{N-1}f(j+1)\\
&= -\frac{1}{2}\left(f(N)-f(M)\right)-\int_M^N f(x)\,dx+\sum_{j=1}^{N}f(j)-f(1)\\
&= -\frac{1}{2}\left(f(N)+f(M)\right)-\int_M^N f(x)\,dx+\sum_{j=M}^{N}f(j)
\end{aligned}
$$

which proves the required formula. □

When f is polynomial of degree D, the Euler-MacLaurin formula reduces to

$$
\sum_{n=M}^{N}f(n)=\frac{f(M)+f(N)}{2}+\int_M^N f(x)\,dx+\sum_{k=2}^{D+1}(-1)^k\frac{B_k}{k!}\left(f^{(k-1)}(N)-f^{(k-1)}(M)\right). \tag{6.41}
$$

6.2 The Euler-MacLaurin formula extended to symbols

Let $f\in\mathbb{C}^\infty(\mathbb{R}^d)$, for any positive integer N we now want to compare the discrete sum

$$
P_N(f):=\sum_{\eta\in\mathbb{Z}^d\cap C(0,N)}f(\eta)
$$

on integer points of the hypercube $C(0,N):=[-N,N]^d$ with the integral

$$
\tilde{P}_N(f):=\int_{C(0,N)}f(\xi)\,dx
$$

over the hypercube. The Euler MacLaurin formula did the job when $d=1$; we need a generalised Euler MacLaurin formula to higher dimensions. We only quote the results in higher dimensions without proofs, indicating some references where proofs can be found.
When $f(\xi)=\sum_{|\alpha|\leq D}c_\alpha\,\xi_1^{\alpha_1}\cdots\xi_k^{\alpha_k}$ is polynomial of degree D in the x_i's, then by the Euler-MacLaurin formula for polynomials (6.41) we have:

$$
\begin{aligned}
P_N(f) &= \sum_{|\alpha|\leq D}c_\alpha\prod_{i=1}^{d}\sum_{n=-N}^{N}n_i^{\alpha_i}\\
&= \sum_{|\alpha|\leq D}c_\alpha\prod_{i=1}^{d}\left(\frac{(-N)^{\alpha_i}+N^{\alpha_i}}{2}+\int_{-N}^{N}x^{\alpha_i}\,dx\right.\\
&+\left.\sum_{k_i=2}^{D+1}(-1)^{k_i}\frac{B_{k_i}}{k_i!}\left(\alpha_i\,(\alpha_i-1)\cdots(\alpha_i-k)\,(N^{\alpha_i-k_i+1}-(-N)^{\alpha_i-k_i+1})\right)\right).
\end{aligned} \tag{6.42}
$$

This can be written in a more compact form at the cost of introducing further notations borrowed from [GSW] which generalise the Taylor expansion (6.35) at zero to higher dimensions:

$$
\mathrm{Todd}\,(\xi):=\prod_{i=1}^{m}\frac{\xi_i}{1-e^{\xi_i}}=\sum_\alpha\frac{B_\alpha}{\alpha!}x^\alpha,\quad\forall\xi\in\mathbb{R}^d
$$

for some constants B_α and where we have set $\alpha! := \alpha_1! \cdots \alpha_d!$.
The Khovanskii-Pukhlikov formula [KP] relates the discrete sum with the integral by:

$$P_N(f) - \tilde{P}_N(f) = \Big((\mathrm{Todd}(\partial_h) - Id)\, \tilde{P}_N(f)(h) \Big)\,|_{h=0},$$

where we have set

$$\tilde{P}_N(f)(h) := \int_{\prod_{i=1}^d [-N-h_i, N+h_i]\cap \mathbb{Z}} \sigma(\xi)\, d\xi.$$

The Khovanskii-Pukhlikov formula was generalised to classical symbols in [GSW] (formula (15), see also [AW], [KSW1] and [KSW2] for previous results along these lines) in which case the formula is not exact anymore but only holds asymptotically.

Proposition 13 *[GSW] Given a symbol $\sigma \in CS^a_{\rm c.c}(\mathbb{R}^d)$ with complex order a,*

$$P_N(\sigma) - \tilde{P}_N(\sigma) \sim_{N\to\infty} (\mathrm{Todd}(\partial_h) - Id)\, \tilde{P}_N(\sigma)(h)|_{h=0} + C(\sigma).$$

More precisely, there are polynomials $M^{[j]}, j \in \mathbb{N}$ on $\mathbb{R}^d$ such that

$$P_N(\sigma) - \tilde{P}_N(\sigma) = \Big(\Big(M^{[j]}(\partial_h) - Id \Big)\, \tilde{P}_N(\sigma)(h) \Big)_{|_{h=0}} + R^j(\sigma)(N) \tag{6.43}$$

where

$$R^j(\sigma)(N) := \sum_p (-1)^p \int_{C_{p,N}} \sum_{|\alpha|=j}^{|\alpha|=d\,j} \phi^p_{\alpha,j}(\xi)\, \partial^\alpha \sigma(\xi)\, d\xi$$

tends to $C(\sigma)$ as $N \to \infty$. The $C_{p,N}$ are convex polytopes growing with N and $\phi^p_{\alpha,j}$ bounded piecewise smooth periodic functions as described in [KSW2] and [AW].

6.3 The cut-off discrete sum on $\mathbb{Z}^d$

Combining the cut-off regularised integral built in the previous chapter with the Euler-MacLaurin-Khovanskii-Puklikov formula for classical symbols, we build a cut-off regularised discrete sum on $CS_{\rm c.c}(\mathbb{R}^d)$ which provides a realisation of the canonical discrete sum on $CS^{\notin Z}_{\rm c.c}(\mathbb{R}^d)$, which by Theorem 6 is uniquely determined whenever it exists.

Proposition 14 *Let $\sigma \in CS^a_{\rm c.c}(\mathbb{R}^d)$ for some complex number a. The map $R \mapsto \int_{[-R,R]^d} \sigma(\xi)\, d\xi$ has the same type of asymptotic expansion as the map $R \mapsto \int_{B(0R)} \sigma(\xi)\, d\xi$ as R tends to ∞. The constant term $\mathrm{fp}_{R\to\infty} \int_{[-R,R]^d} \sigma(\xi)\, d\xi$ in its asymptotic expansion relates to the cut-off regularised integral finite part $-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi)\, d\xi = \mathrm{fp}_{R\to\infty} -\!\!\!\!\!\!\int_{B(0,R)} \sigma(\xi)\, d\xi$ as follows*

$$\mathrm{fp}_{R\to\infty} \int_{[-R,R]^d} \sigma(\xi)\, d\xi = -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi)\, d\xi + \int_{C(0,1)\ominus B(0,1)} \sigma_{-d}(\xi)\, d\xi \quad \forall \sigma \in CS_{\rm c.c}(\mathbb{R}^d), \tag{6.44}$$

where we have set $C(0,R) = [-R,R]^d$ and $A \ominus B := (A - (A\cap B)) \cup (B - (A \cap B))$ the symmetric difference. Consequently,

$$\mathrm{fp}_{R\to\infty} \int_{[-R,R]^d} \sigma(\xi)\, d\xi = -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi)\, d\xi \quad \forall \sigma \in CS^{\notin \mathbb{Z}}(\mathbb{R}^d).$$

Remark 6 *A similar statement holds [Pa6] replacing $[-R,R]^d$ by any d-dimensional expanded convex polytope $R\,\Delta$; here we have $\Delta = [-1,1]^d$. Changing the polytope a priori changes the value of the finite part.*

Proof: Writing $\sigma = \sum_{j=0}^{N-1} \sigma_{a-j} + \sigma_{(N)}$ as in (2.11), for R chosen large enough, we can estimate the difference:

$$\begin{aligned} & \int_{C(0,R)} \sigma(\xi)\, d\xi - \int_{B(0,R)} \sigma(\xi)\, d\xi \\ = & \sum_{j=0}^{N-1} \int_{C(0,R)\ominus B(0,R)} \sigma_{a-j}(\xi)\, d\xi + \int_{C(0,R)\ominus B(0,R)} \sigma_{(N)}(\xi)\, d\xi \\ = & \sum_{j=0}^{N-1} \int_{C(0,R)\ominus B(0,R)} \sigma_{a-j}(\xi)\, d\xi + \int_{C(0,R)\ominus B(0,R)} \sigma_{(N)}(\xi)\, d\xi. \end{aligned} \tag{6.45}$$

Since $\sigma_{(N)}$ is a symbol of order $a - N$ we have:

$$|\sigma_{(N)}(\xi)| \leq C(1 + |\xi|^2)^{\frac{\mathrm{Re}(a)-N}{2}}$$

for some constant C. Hence, for large enough N

$$\begin{aligned} |\int_{C(0,R)-(C(0,R)\cap B(0,R))} \sigma_{(N)}(\xi)\, d\xi| & \leq C\,(1+R^2)^{\frac{\mathrm{Re}(a)-N}{2}} \mathrm{Vol}\,(C(0,R) - (C(0,R)\cap B(0,R))) \\ & \leq C\, R^d (1+R^2)^{\frac{\mathrm{Re}(a)-N}{2}} \mathrm{Vol}\,(C(0,1)) \\ & \leq C\,(1+R^2)^{\frac{\mathrm{Re}(a)+d-N}{2}} \mathrm{Vol}\,(C(0,1))\,. \end{aligned}$$

Using once more the fact that $\sigma_{(N)}$ is a symbol of order $a - N$ combined with the equivalence of the supremum and the Euclidean norms we also have:

$$|\sigma_{(N)}(\xi)| \leq C'\,(1 + |\xi|_{\mathrm{sup}})^{\mathrm{Re}(a)-N}$$

for some constant C' and

$$\begin{aligned} |\int_{B(0,R)-(C(0,R)\cap B(0,R))} \sigma_{(N)}(\xi)\, d\xi| & \leq C'\,(1+d\,R)^{\mathrm{Re}(a)-N} \mathrm{Vol}\,(B(0,R) - (C(0,R)\cap B(0,R))) \\ & \leq C'\, R^d (1+R)^{\mathrm{Re}(a)-N} \mathrm{Vol}\,(B(0,1)) \\ & \leq C'\,(1+R)^{\mathrm{Re}(a)+d-N} \mathrm{Vol}\,(B(0,1))\,. \end{aligned}$$

Consequently, we can choose N sufficiently large so that

$$\int_{C(0,R)\ominus B(0,R)} \sigma_{(N)}(\xi)\, d\xi = O((1+R)^{\mathrm{Re}(a)+d-N}).$$

This settles the case of integrals involving the remainder term $\sigma_{(N)}$. As for integrals of homogeneous symbols $\int_{C(0,R)\ominus B(0,R)} \sigma_{a-j}(\xi)\, d\xi$, we have

$$\int_{C(0,R)\ominus B(0,R)} \sigma_{a-j}(\xi)\, d\xi = \int_{C(0,1)\ominus B(0,1)} \sigma_{a-j}(R\,\eta)\, R^d\, d\eta = R^{a-j+d} \int_{C(0,1)\ominus B(0,1)} \sigma_{a-j}(\eta)\, d\eta,$$

which shows they are homogeneous of degree $a - j + d$.
Combining these results shows that $R \mapsto \int_{C(0,R)\ominus B(0,R)} \sigma(\xi)\, d\xi$ defines a classical symbol of order $a+d$ with constant term given by

$$\mathrm{fp}_{R\to\infty} \int_{C(0,R)\ominus B(0,R)} \sigma(\xi)\, d\xi = \int_{C(0,1)\ominus B(0,1)} \sigma_{-d}(\eta)\, d\eta.$$

Thus the map $R \mapsto \int_{C(0,R)} \sigma(\xi)\, d\xi$ by (6.45) has the same type of asymptotic expansion as $R \to \infty$ as $\int_{B(0,R)} \sigma(\xi)\, d\xi$ and its finite part differs from $\mathrm{fp}_{R\to\infty} \int_{B(0,R)} \sigma(\xi)\, d\xi$ by $\mathrm{fp}_{R\to\infty} \int_{C(0,R)\ominus B(0,R)} \sigma(\xi)\, d\xi = \int_{C(0,1)\ominus B(0,1)} \sigma_{-d}(\eta)\, d\eta$ which vanishes when σ has non integer order. $\square$

6.4 A realisation of the canonical discrete sum on non integer order symbols

On the grounds of the above proposition, given a symbol $\sigma \in CS_{\rm c.c}(\mathbb{R}^d)$, the map

$$N \mapsto \sum_{\vec{n}\in\mathbb{Z}^d\cap C(0,N)} \sigma(\vec{n})$$

has an asymptotic expansion as $N \to \infty$ of type (4.28). The constant term ${\rm fp}_{N\to\infty} \sum_{\vec{n}\in\mathbb{Z}^d\cap[-N,N]^d} \sigma(\vec{n})$ in the expansion gives rise to a linear form which extends the ordinary discrete summation map $\sum_{\mathbb{Z}^d}$ on $CS_{\rm c.c}^{<-d}(\mathbb{R}^d)$ as a result of the Euler-MacLaurin/ Khovanskii-Pukhlikov formula for symbols and which is defined as follows.

Definition 9 *We call the linear form defined by*

$$\begin{array}{rccl} \sum\limits_{\vec{n}\in\mathbb{Z}^d} : & CS_{\rm c.c}(\mathbb{R}^d) & \to & \mathbb{C} \\ & \sigma & \mapsto & {\rm fp}_{N\to\infty} \sum\limits_{\vec{n}\in\mathbb{Z}^d\cap[-N,N]^d} \sigma(\vec{n}) \end{array}$$

the cut-off regularised sum of σ. By the Kohvanskii-Pukhlikov formula, for any σ in $CS_{\rm c.c}(\mathbb{R}^d)$ we have:

$$\sum_{\mathbb{Z}^d} \sigma - {\int\!\!\!\!\!\!-}_{\mathbb{R}^d} \sigma = {\rm fp}_{N\to\infty} \left(\left(M^{[j]}(\partial_h) - Id\right) \int_{-N-h_1\leq x_1\leq N+h_1} \cdots \int_{-N-h_d\leq x_d\leq N+h_d} \sigma\right)_{|h=0} + C(\sigma). \tag{6.46}$$

Proposition 15 *For a polynomial function P in d-variables,*

$$\sum_{\mathbb{Z}^d} P = P(0).$$

The cut-off regularised sum is not translation invariant on $CS_{\rm c.c}(\mathbb{R}^d)$.

Proof: By (6.42) applied to $f = P$ we have

$$\sum_{\mathbb{Z}^d} f = {\rm fp}_{N\to\infty} P_N(f) = c_0 = f(0)$$

since only the constant terms survive in the finite part procedure as $N \to \infty$. Thus $\sum_{\mathbb{Z}^d} P = P(0)$, from which it follows that $\sum_{\mathbb{Z}^d}$ is not translation invariant. Indeed, for any $\eta \in \mathbb{Z}^d$, the function $t_\eta^* P = P(\cdot + \eta)$ is again a polynomial in d variables so that $\sum_{\mathbb{Z}^d} P(\cdot+\eta) = P(\eta)$ which in general does not coincide with $P(0) = \sum_{\mathbb{Z}^d} P$. □

The cut-off regularised sum actually provides an explicit construction of the canonical discrete sum $\sum_{\mathbb{Z}^d}$ on non integer order classical symbols introduced in Theorem 6.

Theorem 7 *The cut-off regularised sum extends the ordinary discrete sum to a $\mathbb{Z}^d$-translation invariant linear form on $CS_{\rm c.c}^{\notin\mathbb{Z}}(\mathbb{R}^d)$ and we have:*

$$\sum_{\mathbb{Z}^d} \sigma := {\rm fp}_{N\to\infty} \sum_{\vec{n}\in\mathbb{Z}^d\cap C(0,N)} \sigma(\vec{n}) = {\int\!\!\!\!\!\!-}_{\mathbb{R}^d} \sigma + C(\sigma) \quad \forall \sigma \in CS_{\rm c.c}^{\notin\mathbb{Z}}(\mathbb{R}^d). \tag{6.47}$$

The cut-off regularised sum therefore provides a realisation of the canonical discrete sum on non integer order symbols. Moreover, the map $\sigma \mapsto C(\sigma)$ is $\mathbb{Z}^d$-translation invariant on $CS_{\rm c.c}^{\notin\mathbb{Z}}(\mathbb{R}^d)$ i.e.

$$C(t_\eta^*\sigma) = C(\sigma) \quad \forall \eta \in \mathbb{Z}^d, \quad \forall \sigma \in CS_{\rm c.c}^{\notin\mathbb{Z}}(\mathbb{R}^d).$$

Proof: By Proposition 13, the asymptotic behaviour of the map $N \mapsto P_N(\sigma) = \sum_{\vec{n} \in \mathbb{Z}^d \cap C(0,N)} \sigma(\vec{n})$ follows from that of the map $N \mapsto \tilde{P}_N(\sigma) = \int_{C(0,N)} \sigma(\xi)\, d\xi$ since the difference, which involves derivatives of integrals $\int_{-N-h_1 \leq x_1 \leq N+h_1} \cdots \int_{-N-h_d \leq x_d \leq N+h_d} \sigma(\xi)\, d\xi$ in the h_i's, can be shown to be of that type (with no logarithmic terms) [Pa6]. Taking finite parts as $N \to \infty$ on either side of (6.43) yields (6.46).
Since $\sum_{\mathbb{Z}^d}$ coincides with the ordinary summation map $\sigma \mapsto \sum_{\mathbb{Z}^d} \sigma(\vec{n})$ on $CS_{\rm c.c}^{<-d}(\mathbb{R}^d)$ which is admissible, it suffices to show that it is $\mathbb{Z}^d$-translation invariant on $CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d)$ in order to view it as the unique linear form on $CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d)$ with these properties which we called the canonical sum on $CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d)$. We therefore need to check that

$$\mathrm{fp}_{N\to\infty} \sum_{C(0,N)\cap \mathbb{Z}^d} t_\eta^* \sigma(\vec{n}) = \mathrm{fp}_{N\to\infty} \sum_{C(0,N)\cap \mathbb{Z}^d} \sigma(\vec{n}) \quad \forall \sigma \in CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d) \quad \forall \eta \in \mathbb{Z}^d.$$

To prove translation invariance, we observe that $\sum_{C(0,N)\cap \mathbb{Z}^d} t_\eta^* \sigma(\vec{n}) = \sum_{t_{-\eta}^* C(0,N) \cap \mathbb{Z}^d} \sigma(\vec{n})$ is the sum over integer points of a polytope corresponding to the translated hypercube $t_{-\eta}^* C(0,N)$. Using the Euler-MacLaurin/ Khovanskii-Pukhlikov applied to the polytope $t_{-\eta}^* C(0,N)$, we can relate this sum with the integral $\int_{t_{-\eta}^* C(0,N)} \sigma(\xi)\, d\xi$, which for non integer order symbols, can be shown to have the same finite part as $N \to \infty$ as $\int_{C(0,N)} \sigma(\xi)\, d\xi$. On the other hand, using again the fact that σ has non integer order, we check that

$$\mathrm{fp}_{N\to\infty} \partial_{h_i}^{\gamma_i} \left(\tilde{P}_{(N)}(t_\eta^* \sigma)(h) \right)_{|h=0} = \mathrm{fp}_{N\to\infty} \partial_{h_i}^{\gamma_i} \left(\tilde{P}_{(N)}(\sigma)(h) \right)_{|h=0} = 0.$$

hence,

$$\begin{aligned}
\sum_{\mathbb{Z}^d} t_{-\eta}^* \sigma &:= \mathrm{fp}_{N\to\infty} \sum_{\mathbb{Z}^d \cap t_{-\eta}^* C(0,N)} \sigma \\
&= \mathrm{fp}_{N\to\infty} \int_{t_{-\eta}^* C(0,N)} \sigma(\xi)\, d\xi + C(\sigma) \\
&= \mathrm{fp}_{N\to\infty} \int_{C(0,N)} \sigma(\xi)\, d\xi + C(\sigma) \\
&= -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi)\, d\xi + C(\sigma) \\
&= \sum_{\mathbb{Z}^d} \sigma.
\end{aligned}$$

Since both $\sum_{\mathbb{Z}^d}$ and $-\!\!\!\!\!\!\int_{\mathbb{R}^d}$ are invariant under translation by $\eta \in \mathbb{Z}^d$ on non integer order symbols, so is the map $\sigma \mapsto C(\sigma)$, consequently

$$C\left(t_\eta^* \sigma\right) = C(\sigma) \quad \forall \sigma \in CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}) \quad \forall \eta \in \mathbb{Z}^d$$

which shows (6.47). □

Remark 7 *The canonical sum on non integer order symbols can also be derived from finite parts* $\mathrm{fp}_{N\to\infty} \sum_{N\,\Delta \cap \mathbb{Z}^d} \sigma$ *of sums over integer points in expanded polytopes* $N\,\Delta$ *independently of the choice of the convex polytope* Δ. *Whereas for a general symbol* $\sigma \in CS_{\rm c.c}(\mathbb{R}^d)$ *the expression* $\mathrm{fp}_{N\to\infty} \sum_{N\,\Delta \cap \mathbb{Z}^d} \sigma(\vec{n})$ *depends on the choice of the polytope* Δ, *it does not when* σ *is of non integer order.*

Example 11 *Cut-off regularised sums vanish on polynomials. Indeed, let* Q *be a polynomial in d variables, then (6.46) applied to polynomials reads*

$$\sum_{\mathbb{Z}^d} Q - -\!\!\!\!\!\!\int_{\mathbb{R}^d} Q = \mathrm{fp}_{N\to\infty} \left(\left(M^{[j]}(\partial_h) - Id \right) \int_{-N-h_1 \leq x_1 \leq N+h_1} \cdots \int_{-N-h_d \leq x_d \leq N+h_d} Q \right)_{|h=0},$$

since $C(\sigma) = 0$. *The derivatives which arise from* $\left(\left(M^{[j]}(\partial_h) - Id \right) \int_{-N-h_1 \leq x_1 \leq N+h_1} \cdots \int_{-N-h_d \leq x_d \leq N+h_d} Q \right)_{|h=0}$ *give rise to non constant monomials in* N *and hence do not contribute to the finite part. It follows that* $\sum_{\mathbb{Z}^d} Q = -\!\!\!\!\!\!\int_{\mathbb{R}^d} Q$, *vanishes since the cut-off integral vanishes on polynomials.*

7 An alternative characterisation of the noncommutative residue

Using the canonical integral on non integer order symbols, we classify linear forms on the algebra of symbols which fulfill Sotkes' property on non integer order symbols. This leads to a characterisation of the noncommutative residue as the unique (up to a multiplicative factor) linear form on the algebra of classical symbols, which fulfills Stokes' property.

7.1 Canonical integrals of holomorphic symbols

The notion of holomorphic family of classical pseudodifferential symbols and operators was first introduced by Guillemin in [Gu1] and extensively used by Kontsevich and Vishik in [KV]. The idea is to embed a symbol σ in a family $z \mapsto \sigma(z)$ depending holomorphically on a complex parameter z.
A family $\{f(z)\}_{z\in W}$ in a topological vector space $\mathcal{S}$ which is parametrised by a complex domain Ω, is holomorphic at $z_0 \in \Omega$ if the corresponding function $f : \Omega \to \mathcal{S}$ admits a Taylor expansion in a neighborhood N_{z_0} of z_0

$$f(z) = \sum_{k=0}^{\infty} f^{(k)}(z_0)\, \frac{(z-z_0)^k}{k!} \tag{7.48}$$

which is convergent, uniformly on compact subsets of N_{z_0}, with respect to the topology on $\mathcal{S}$. The vector space of functions we consider here is $C^\infty(U \times \mathbb{R}^d) \otimes \mathrm{End}(V)$ equipped with the uniform convergence of all derivatives on compact subsets.
A family $\sigma(z)$ of classical symbols on $\mathbb{R}^d$ parametrised by a domain Ω is holomorphic at point $z_0 \in \Omega$ if:

1. $\sigma(z)$ is holomorphic at z_0 as a function of z with values in $C^\infty(\mathbb{R}^d)$ and

$$\sigma(z) \sim \sum_{j\geq 0} \sigma(z)_{\alpha(z)-j} \ \in CS_{\rm c.c}^{\alpha(z)}(\mathbb{R}^d), \tag{7.49}$$

   where the function $\alpha : \Omega \to \mathbb{C}$ is holomorphic at z_0;

2. for any integer $N \geq 1$ the remainder

$$\sigma_{(N)}(z) := \sigma(z) - \sum_{j=0}^{N-1} \sigma_{\alpha(z)-j}(z)$$

   is holomorphic at z_0 as a function of z with values in $C^\infty(\mathbb{R}^d)$ with $k^{\rm th}$ z-derivative

$$\sigma_{(N)}^{(k)}(z) := \partial_z^k(\sigma_{(N)}(z)) \tag{7.50}$$

   a symbol on $\mathbb{R}^d$ of order $\alpha(z) - N + \epsilon$ for any $\epsilon > 0$ locally uniformly in z, i.e the k-th derivative $\partial_z^k \sigma_{(N)}(z)$ satisfies a uniform estimate (2.10) in z on compact subsets in Ω.

If $\sigma(z)$ is holomorphic at every point $z_0 \in \Omega$, it is called a holomorphic family of symbols parametised by Ω. We shall also use finite sums of holomorphic families of symbols, in which case the notion of holomorphic order does not make sense any longer; we then need to handle each holomorphic symbol in the sum separately. We call a family $\sigma(z)$ of classical symbols parametrised by Ω meromorphic if there are a finite number of complex numbers $z_1, \cdots, z_k$ in Ω corresponding to the poles of σ and holomorphic families $\tau_1(z), \cdots, \tau_k(z)$ of classical symbols parametrised by Ω such that $\sigma(z) = \sum_{i=1}^k \tau_i(z)\,(z-z_i)^{-m_i}$, for some $m_1, \cdots, m_k \in \mathbb{N}$.
The following technical result is useful to recover a result by Kontsevich and Vishik to holomorphic families of classical symbols.

Lemma 9 *Let $z \mapsto \tau(z)$ be a holomorphic family of classical symbols in $CS_{\rm c.c}(\mathbb{R}^d)$ parametrized by Ω with holomorphic order $\alpha(z)$. Then the map*

$$z \mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d} \tau(z)(\xi)\, đ\xi$$

is meromorphic with at most a simple pole at points in $\Omega \cap \alpha^{-1}(\mathbb{Z} \cap [-d, +\infty[)$ *and at any such point* z_0 *we have:*

$$\mathrm{ev}_{z_0}^{\mathrm{reg}} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \tau(z)(\xi)\, đ\xi - -\!\!\!\!\!\!\int_{\mathbb{R}^d} \tau(z_0)(\xi)\, đ\xi = -\mathrm{ev}_{z_0}^{\mathrm{reg}} \left(\frac{1}{\alpha(z) - \alpha(z_0)} \int_{S^{d-1}} \tau_{\alpha(z)-j_0}(z)(\omega)\, đ\omega_S \right),$$

$$\mathrm{Res}_{z_0} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \tau(z)(\xi)\, đ\xi = -\mathrm{Res}_{z_0} \left(\frac{1}{\alpha(z) - \alpha(z_0)} \int_{S^{d-1}} \tau_{\alpha(z)-j_0}(z)(\omega)\, đ_S\omega \right),$$

where j_0 *is chosen such that* $\alpha(z_0) - j_0 = -n$ *and where we have set* $\mathrm{ev}_{z_0}^{\mathrm{reg}}(f) := \mathrm{ev}_0^{\mathrm{reg}} f(z_0 + \cdot)$ *with* $\mathrm{ev}_0^{\mathrm{reg}}$ *the regularised evaluator at zero.*

These formulae still hold if $z \mapsto \tau(z)$ *is a meromorphic family of classical symbols which is holomorphic at the point* z_0.

Proof: We write $\tau(\xi) = \sum_{j=0}^{N-1} \chi(\xi)\, \tau_{a-j}(\xi) + \tau_{(N)}(\xi)$ with N chosen large enough so that $\tau_{(N)}$ has order $< -d$. An explicit derivation of the finite part yields

$$\begin{aligned} & -\!\!\!\!\!\!\int_{\mathbb{R}^d} \tau(\xi)\, đ\xi \\ = \; & \int_{\mathbb{R}^d} \tau_{(N)}(\xi)\, đ\xi + \sum_{j=0}^{N-1} \int_{B(0,1)} \chi(\xi)\, \tau_{a-j}(\xi)\, đ\xi - \sum_{j \neq a-n}^{N-1} \frac{1}{a - j + n} \int_{S^{d-1}} \tau_{a-j}(\omega)\, đ_S\omega, \end{aligned} \tag{7.51}$$

where $B(0,1)$ stands for the unit ball. In particular, for the holomorphic family $\tau(z)$ of order $\alpha(z)$ this yields the following identity of meromorphic maps:

$$\begin{aligned} & -\!\!\!\!\!\!\int_{\mathbb{R}^d} \tau(z)(\xi)\, đ\xi \\ = \; & \int_{\mathbb{R}^d} \tau_{(N)}(z)(\xi)\, đ\xi + \sum_{j=0}^{N-1} \int_{B(0,1)} \chi(\xi)\, \tau_{\alpha(z)-j}(z)(\xi)\, đ\xi - \sum_{j=0}^{N-1} \frac{1}{\alpha(z) - j + n} \int_{S^{d-1}} \tau_{\alpha(z)-j}(z)(\omega)\, đ_S\omega \\ = \; & \int_{\mathbb{R}^d} \tau_{(N)}(z)(\xi)\, đ\xi + \sum_{j=0}^{N-1} \int_{B(0,1)} \chi(\xi)\, \tau_{\alpha(z)-j}(z)(\xi)\, đ\xi - \sum_{j=0, j\neq j_0}^{N-1} \frac{1}{\alpha(z) - j + n} \int_{S^{d-1}} \tau_{\alpha(z)-j}(z)(\omega)\, đ_S\omega \\ - \; & \frac{1}{\alpha(z) - \alpha(z_0)} \int_{S^{d-1}} \tau_{\alpha(z)-j_0}(z)(\omega)\, đ_S\omega, \end{aligned} \tag{7.52}$$

where j_0 is such that $\alpha(z_0) - j_0 = -d$.

Since the family $\tau(z)$ is holomorphic at z_0, the expression

$$\int_{\mathbb{R}^d} \tau_{(N)}(z)(\xi)\, đ\xi + \sum_{j=0}^{N-1} \int_{B(0,1)} \chi(\xi)\, \tau_{\alpha(z)-j}(z)(\xi)\, đ\xi - \sum_{j=0, j\neq j_0}^{N-1} \frac{1}{\alpha(z) - j + d} \int_{S^{d-1}} \tau_{\alpha(z)-j}(z)(\omega)\, đ_S\omega$$

which involves integrals on compact sets $B(0,1)$ and S^{d-1} of homogeneous components $\tau_{\alpha(z)-j}$ is holomorphic at z_0 and the integral over $\mathbb{R}^d$ of a remainder term $\tau_{(N)}(z)$, converges to its value at z_0. This holomorphic part converges to

$$\begin{aligned} & -\!\!\!\!\!\!\int_{\mathbb{R}^d} \tau(z_0)(\xi)\, đ\xi \\ = \; & \int_{\mathbb{R}^d} \tau_{(N)}(z_0)(\xi)\, đ\xi + \sum_{j=0}^{N-1} \int_{B(0,1)} \chi(\xi)\, \tau_{\alpha(z_0)-j}(z_0)(\xi)\, đ\xi - \sum_{j=0, j\neq j_0}^{K_N} \frac{1}{\alpha(z_0) - j + n} \int_{S^{d-1}} \tau_{\alpha(z_0)-j}(z_0)(\omega)\, đ_S\omega, \end{aligned}$$

where we have used (7.51). The only poles of $-\!\!\!\!\!\int_{\mathbb{R}^d} \tau(z)(\xi)\, đ\xi$ which come from the remaining part $\frac{1}{\alpha(z)-\alpha(z_0)} \int_{S^{d-1}} \tau_{\alpha(z)-j_0}(z)(\omega)\, đ_S\omega$ arise at points z_0 for which $\alpha(z_0) = j_0$ lies in $[-d + \infty[\cap \mathbb{Z}$.

Combining this with (7.52) shows that the map $z \mapsto -\!\!\!\!\!\int_{\mathbb{R}^d} \tau(z)(\xi)\, đ\xi$ is meromorphic with simple poles in $\alpha^{-1}(\mathbb{Z} \cap [-d, +\infty[) \cap \Omega$ with finite part at a pole $z = z_0$ given by

$$\mathrm{ev}_{z_0}^{\mathrm{reg}} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \tau(z)(\xi)\, đ\xi = -\!\!\!\!\!\!\int_{\mathbb{R}^d} \tau(z_0)(\xi)\, đ\xi - \mathrm{ev}_{z_0}^{\mathrm{reg}} \left(\frac{1}{\alpha(z) - \alpha(z_0)} \int_{S^{d-1}} \tau_{\alpha(z)-j_0}(z)(\omega)\, đ_S\omega \right),$$

and a simple pole at $z = z_0$ given by:

$$\mathrm{Res}_{z_0} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \tau(z)(\xi)\, đ\xi = -\mathrm{Res}_{z_0} \left(\frac{1}{\alpha(z)-\alpha(z_0)} \int_{S^{d-1}} \tau_{\alpha(z)-j_0}(z)(\omega)\, đ_S\omega \right).$$

Since the last two formulae were derived from local computations at point z_0, they still hold if $z \mapsto \tau(z)$ is a meromorphic family of classical symbols which is only holomorphic at the point z_0. □

7.2 The noncommutative residue as a complex residue

The following theorem recalls a result of Kontsevich and Vishik [KV] which relates the complex residue of the cut-off integral of a holomorphic family of symbols to its noncomutative residue and yields back a result derived in [PS] which describes the finite part of the cut-off integral of a holomorphic family of symbols.

Theorem 8 *Let $z \mapsto \sigma(z)$ be a holomorphic family of classical symbols in $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ of holomorphic order $\alpha(z)$ parametrized by Ω.*

1. *The map $-\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi) đ\xi$ is meromorphic with simple poles in $\Omega \cap \alpha^{-1}(\mathbb{Z} \cap [-d, +\infty[)$.*
2. *[KV] Provided $\alpha'(z_0) \neq 0$, then the complex residue at $z_0 \in \Omega \cap \alpha^{-1}(\mathbb{Z} \cap [-d, +\infty[)$ is given by:*
$$\mathrm{Res}_{z_0} \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi) đ\xi \right) = -\frac{1}{\alpha'(z_0)} \mathrm{res}(\sigma(z_0)). \quad (7.53)$$
3. *[PS] Provided $\alpha'(z_0) \neq 0$, the finite part at $z_0 \in \Omega$ differs from the cut-off regularised integral $-\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z_0)(\xi) đ\xi$ by*
$$\mathrm{ev}_{z_0}^{\mathrm{reg}} \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi) đ\xi \right) - -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z_0)(\xi) đ\xi = -\frac{1}{\alpha'(z_0)} \mathrm{res}(\sigma'(z_0)) + \frac{\alpha''(z_0)}{2\,(\alpha'(z_0))^2} \mathrm{res}(\sigma(z_0)), \quad (7.54)$$
where we havse set $\mathrm{ev}_{z_0}^{\mathrm{reg}}(f) = \mathrm{ev}_0^{\mathrm{reg}} f(z_0 + \cdot)$ as before and extended the noncomutative residue to the possibly non classical symbol $\tau(z_0) = \sigma'(z_0)$[3] by implementing the same formula
$$\mathrm{res}(\tau(z_0)) := \int_{S^{d-1}} \tau_{-n}(z_0)(\xi)\, đ\xi.$$

Remark 8 *Formula (7.54) formally follows from (7.53) applied to the family $\tau(z) = \frac{\sigma(z)-\sigma(z_0)}{z-z_0}$ since $\tau(z_0) = \sigma'(z_0)$. However, the proof is not quite so straightforward since $\tau(z)$ is not a holomorphic family of classical symbols (outside z_0) but only a linear combination of such a holomorphic family $\sigma(z)$ and a constant symbol $\sigma(z_0)$.*

Proof:

1. By Lemma 9 the complex residue reads

$$\begin{aligned}
\mathrm{Res}_{z_0} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi) đ\xi &= -\mathrm{Res}_{z=0} \left(\frac{1}{\alpha(z)-\alpha(z_0)} \int_{S^{d-1}} \sigma_{\alpha(z)-j_0}(z)(\omega)\, đ_S\omega \right) \\
&= -\mathrm{Res}_{z_0} \left(\frac{1}{\alpha'(z_0)(z-z_0)} \int_{S^{d-1}} \sigma_{\alpha(z)-j_0}(z)(\omega)\, đ_S\omega \right) \\
&= -\frac{1}{\alpha'(z_0)} \int_{S^{d-1}} \sigma_{-n}(z)(\omega)\, đ_S\omega \\
&= -\frac{1}{\alpha'(z_0)} \mathrm{res}_0(\sigma(z_0)),
\end{aligned}$$

where we used the Taylor expansion at order zero:

$$(\sigma(z))_{\alpha(z)-j_0} = (\sigma(z_0))_{-n} + O(z-z_0).$$

[3] The asymptotic expansion of $\tau(z_0)(\xi)$ as $|\xi| \to, \infty$ might present logarithmic terms in $|\xi$, which vanish on the unit sphere and therefore do not explicitely arise in the following definition.

2. Let us first observe that

$$\begin{aligned}\frac{1}{\alpha(z)-\alpha(z_0)} &= \frac{1}{\alpha'(z_0)(z-z_0)(1+\frac{\alpha''(z_0)}{2\,\alpha'(z_0)}(z-z_0)+o(z-z_0))}\\ &= \frac{1}{\alpha'(z_0)(z-z_0)}\left(1-\frac{\alpha''(z_0)}{2\alpha'(z_0)}(z-z_0)+o(z-z_0)\right)\end{aligned}$$

so that

$$\mathrm{fp}_{z_0}\left(\frac{1}{\alpha(z)-\alpha(z_0)}\right) = -\frac{\alpha''(z_0)}{2\left(\alpha'(z_0)\right)^2}.$$

Note that this vanishes if α is affine.
By Lemma 9 the finite part $\mathrm{fp}_{z_0} -\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma(z)(\xi)$ differs from $-\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma(z_0)(\xi)\,đ\xi$ by

$$\begin{aligned}&\mathrm{fp}_{z_0} -\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma(z)(\xi)\,đ\xi - -\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma(z_0)(\xi)\,đ\xi\\ =& -\mathrm{fp}_{z_0}\left(\frac{1}{\alpha(z)-\alpha(z_0)}\int_{S^{d-1}}\sigma_{\alpha(z)-j_0}(z)(\omega)\,đ_S\omega\right)\\ =& -\mathrm{fp}_{z_0}\left(\frac{1}{\alpha(z)-\alpha(z_0)}\int_{S^{d-1}}\left(\sigma(z)-\sigma(z_0)\right)_{\alpha(z)-j_0}(\omega)\,đ_S\omega\right)\\ -& \ \mathrm{fp}_{z_0}\left(\frac{1}{\alpha(z)-\alpha(z_0)}\right)\int_{S^{d-1}}\left(\sigma(z_0)\right)_{\alpha(z_0)-j_0}(\omega)\,đ_S\omega.\end{aligned}$$

Setting $\tau(z)=\frac{\sigma(z)-\sigma(z_0)}{z-z_0}$ and using the Taylor formula at order zero around z_0

$$\left(\tau(z)\right)_{\alpha(z)-j_0} = \left(\sigma'(z_0)\right)_{-n}+O(z-z_0),$$

this yields

$$\begin{aligned}&\mathrm{fp}_{z_0} -\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma(z)(\xi)\,đ\xi - -\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma(z_0)(\xi)\,đ\xi\\ =& -\mathrm{Res}_{z_0}\left(\frac{1}{\alpha(z)-\alpha(z_0)}\int_{S^{d-1}}\left(\tau(z)\right)_{\alpha(z)-j_0}(\omega)\,đ_S\omega\right)\\ +& \ \frac{\alpha''(z_0)}{2\left(\alpha'(z_0)\right)^2}\int_{S^{d-1}}\left(\sigma(z_0)\right)_{\alpha(z_0)-j_0}\\ =& -\left(\frac{1}{\alpha'(z_0)}\int_{S^{d-1}}\left(\tau(z)\right)_{-n}(\omega)\,đ_S\omega\right)+\frac{\alpha''(z_0)}{2\left(\alpha'(z_0)\right)^2}\int_{S^{d-1}}\left(\sigma(z_0)\right)_{-n}\\ =& -\frac{1}{\alpha'(z_0)}\mathrm{res}_0(\tau(z_0))+\frac{\alpha''(z_0)}{2\left(\alpha'(z_0)\right)^2}\mathrm{res}_x(\sigma(z_0))\\ =& -\frac{1}{\alpha'(z_0)}\mathrm{res}_0(\sigma'(z_0))+\frac{\alpha''(z_0)}{2\left(\alpha'(z_0)\right)^2}\mathrm{res}_x(\sigma(z_0)).\end{aligned}$$

□

7.3 An alternative characterisation of the noncommutative residue

The following proposition characterises linear extensions to the whole algebra $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ of the canonical integral defined on non integer order symbols.

Proposition 16 *Any linear form on $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ which satisfies Stokes' property on $CS_{\mathrm{c.c}}^{\notin\mathbb{Z}}(\mathbb{R}^d)$ is of the form*

$$c\cdot -\!\!\!\!\!\!\int_{\mathbb{R}^d} + \quad \mu\,\mathrm{res},\quad (c,\mu)\in\mathbb{C}^2.$$

Proof: Let λ be a linear form on $CS_{\rm c.c}\,\mathbb{R}^d)$ which restricts to a linear form on $CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d)$ which fulfills Stokes' property. By Theorem 2 applied to $\mathcal{S} = CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d)$, the restriction is proportional to the canonical integral $-\!\!\!\!\!\!\int_{\mathbb{R}^d}$:

$$\exists c \in \mathbb{C}, \quad \lambda_{|_{CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d)}} = c -\!\!\!\!\!\!\int_{\mathbb{R}^d}.$$

We want to describe all possible linear extensions λ of $-\!\!\!\!\!\!\int_{\mathbb{R}^d}$ to classical symbols with integer order. Given a symbol $\sigma \in CS^{\mathbb{Z}}(\mathbb{R}^d)$ with integer order a, we build a holomorphic family

$$\sigma(z)(\xi) = (1 - \chi(\xi))\,\sigma(\xi) + \chi(\xi)\,\sigma(\xi)\,|\xi|^{-z}$$

whose order $a - z$ avoids integers in a small neighborhood of 0. Thus, in a small neighborhood of zero

$$\lambda(\sigma(z)) = -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z).$$

The remaining degree of freedom left to define $\lambda(\sigma)$ is the choice of a regularised evaluator at $z = 0$. But by Proposition 2 (here $k = 1$), regularised evaluators at zero are of the form ${\rm ev}_0^{\rm reg} + \nu\,{\rm Res}_0$, with ν a complex number. Hence,

$$\begin{aligned} \lambda(\sigma) &= c\,{\rm ev}_0^{\rm reg} \circ -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma) + \mu\,{\rm Res}_{z=0} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z) \\ &= c -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm reg} \sigma + \mu\,{\rm res}(\sigma), \end{aligned} \tag{7.55}$$

where we have set $-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm reg} \sigma := {\rm ev}_0^{\rm reg} \circ -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma)$ and used the fact that $\alpha(z) = -z + \alpha(0)$.
This holds in particular for the cut-off regularised integral $\lambda = -\!\!\!\!\!\!\int_{\mathbb{R}^d}$, which by (3) coincides with the canonical integral on $CS^{\notin \mathbb{Z}}(\mathbb{R}^d)$. Thus there are constants c' and μ' such that

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d} = c' -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm reg} + \mu'\,{\rm res}.$$

Since $-\!\!\!\!\!\!\int_{\mathbb{R}^d}$ and $-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm reg}$ coincide with the ordinary integral on L^1-symbols on which the residue vanishes, $c' = 1$ from which we infer that

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm reg} = -\!\!\!\!\!\!\int_{\mathbb{R}^d} - \mu'\,{\rm res}.$$

Inserting this back in (25.268) yields the existence of constants c and μ such that

$$\lambda = c -\!\!\!\!\!\!\int_{\mathbb{R}^d} + \mu\,{\rm res}$$

as announced. □

The lack of translation invariance of the cut-off regularised integral on $CS_{\rm c.c}(\mathbb{R}^d)$ observed in Corollory 4 combined with Proposition 16 leads to an alternative characterisation of the noncommutative residue.

Theorem 9 *Any linear form λ on $CS_{\rm c.c}(\mathbb{R}^d)$ which is translation invariant, or equivalently which satisfies Stokes' property, is proportional to the noncommutative residue*

$$\lambda = \mu\,{\rm res}, \quad \mu \in \mathbb{C}.$$

Proof: We first recall from Proposition 11 that translation invariance is equivalent to satisfying Stokes' property.
By the above theorem linear forms on $CS_{\rm c.c}(\mathbb{R}^d)$ with Stokes' property, which by definition restrict to linear forms on $CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d)$ with Stokes' property, are linear combinations of the cut-off integration map and the noncommutative residue. But by Corollary 4, in contrast to the noncommutative residue, the cut-off integral does not satisfy Stokes' property on $CS_{\rm c.c}(\mathbb{R}^d)$. It follows that linear forms on $CS_{\rm c.c}(\mathbb{R}^d)$ with Stokes' property are proportional to the noncommutative residue.

8 Holomorphic regularisation schemes

We compare different regularisation schemes and describe the discrepancies arising from regularised integrals such as the lack of covariance and translation invariance.

8.1 Regularised integrals of symbols

Definition 10 *A holomorphic* regularisation *scheme on* $CS_{\rm c.c}(\mathbb{R}^d)$ *is a linear map*

$$\mathcal{R}: \sigma \mapsto (z \mapsto \sigma(z))$$

which sends a symbol σ *to a holomorphic family of symbols* $\sigma(z)$ *parametrised by* $z \in \mathbb{C}$ *such that* $\sigma(0) = \sigma$ *of order* $z \mapsto \alpha(z)$ *with non vanishing derivative at zero* $\alpha'(0) \neq 0$.

Example 12 *Riesz regularisations*

$$\mathcal{R}(\sigma)(z)(x) = \sigma(\xi)\,|\xi|^{-z} \quad \text{if} \quad |\xi| \geq 1 \tag{8.56}$$

and the slightly more general regularisations (which as we shall see below include dimensional regularisation)

$$\mathcal{R}(\sigma)(z)(x) = H(z)\,\sigma(\xi)\,|\xi|^{-z} \quad \text{if} \quad |\xi| \geq 1 \tag{8.57}$$

with H *holomorphic such that* $H(0) = 1$, *are holomorphic regularisation schemes.*

Remark 9 *In the above examples*

1. $\sigma \mapsto \sigma(z)$ *is not an algebra morphism since* $\sigma(z)\,\tau(z) \neq (\sigma\,\tau)\,(z)$.
2. *the order* $\alpha(z) = \alpha(0) - z$ *of* $\sigma(z)$ *is affine in* z.

Definition 11 *By Theorem 8, to a holomorphic regularisation* $\mathcal{R}: \sigma \mapsto \sigma(z)$ *we assign a meromorphic map* $z \mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma)(z)$ *with a simple pole at* $z = 0$. *Combining it with the regularised evaluator* ${\rm ev}_0^{\rm reg}$ *at zero defined in (1.2) which amounts to taking the finite part* ${\rm fp}z = 0$, *we build a linear form:*

$$\begin{array}{rccc} -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}}: & CS_{\rm c.c}(\mathbb{R}^d) & \to & \mathbb{C} \\ & \sigma & \mapsto & {\rm ev}_0^{\rm reg} \circ \int_{\mathbb{R}^d} \mathcal{R}(\sigma) \end{array}$$

called the $\mathcal{R}$*-regularised integral of* σ.

Let us introduce a regularisation scheme which we call dimensional regularisation in reference to a similar regularisation scheme used in the physics litterature.

Definition 12 *Let* $H(z) := \frac{{\rm Vol}(S^{d-z-1})}{{\rm Vol}(S^{d-1})}$ *where using (1.5) we have set*

$${\rm Vol}(S^{d-z-1}) := \frac{2\pi^{\frac{d-z}{2}}}{\Gamma\left(\frac{d-z}{2}\right)}.$$

Let $\mathcal{R}$ *be a regularisation of the type described in Example (8.57),*

$$\mathcal{R}_H(\sigma)(z)(\xi) = (1 - \chi(\xi))\sigma(\xi) + \chi(\xi)\,H(z)\,\sigma(\xi)\,|\xi|^{-z}$$

where χ *is any smooth cut-off function which is identically one outside the unit ball and vanishes in a neighborhood of* 0.
For any symbol $\sigma \in CS_{\rm c.c.}(\mathbb{R}^d)$ *we call*

$$-\!\!\!\!\!\!\int_{\rm dim.reg} \sigma(\xi)\, d\!\!\!{}^-\xi := -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi)\, d\!\!\!{}^-\xi$$

the dimensional regularised integral of σ.

The terminology "dimensional regularisation" is justified by the following proposition which shows how on radial symbols, dimensional regularisation amounts to "complexifying" the dimension $d \to d - z$.

Proposition 17 *For any radial symbol $\sigma(\xi) = f(|\xi|) \in CS_{\rm c.c}(\mathbb{R}^d)$ we have*

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm dim.reg} \sigma(\xi)\, d\!\!{}^-\xi = -\!\!\!\!\!\!\int_{|\xi|\leq 1} \sigma(\xi)\, d\!\!{}^-\xi + {\rm ev}_0^{\rm reg}\left(\frac{{\rm Vol}(S^{d-z-1})}{(2\pi)^d} -\!\!\!\!\!\!\int_1^\infty f(r)\, r^{d-z}\, dr\right). \tag{8.58}$$

Proof: By definition of the dimensional regularised integral we have

$$\begin{aligned} -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm dim.reg} \sigma(\xi)\, d\!\!{}^-\xi &= -\!\!\!\!\!\!\int_{|\xi|\leq 1} \sigma(\xi)\, d\!\!{}^-\xi + {\rm ev}_0^{\rm reg}\left(H(z) -\!\!\!\!\!\!\int_{\mathbb{R}^d - B(0,1)} \sigma(\xi)\, |\xi|^{-z}\, d\!\!{}^-\xi\right) \\ &= -\!\!\!\!\!\!\int_{|\xi|\leq 1} \sigma(\xi)\, d\!\!{}^-\xi + \frac{{\rm Vol}(S^{d-1})}{(2\pi)^d}\, {\rm ev}_0^{\rm reg}\left(H(z) -\!\!\!\!\!\!\int_1^\infty f(r)\, r^{d-z}\, dr\right) \\ &= -\!\!\!\!\!\!\int_{|\xi|\leq 1} \sigma(\xi)\, d\!\!{}^-\xi + {\rm ev}_0^{\rm reg}\left(\frac{{\rm Vol}(S^{d-z-1})}{(2\pi)^d} -\!\!\!\!\!\!\int_1^\infty f(r)\, r^{d-z}\, dr\right). \end{aligned}$$

□

8.2 Dimensional versus cut-off regularised integrals

The following proposition compares $\mathcal{R}$-regularised integrals with the cut-off regularised integral.

Proposition 18 1. *If the holomorphic regularisation $\mathcal{R}$ sends a symbol of order a to a symbol of affine order $\alpha(z) = a - q\,z$ with $q \neq 0$, then*

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi) d\!\!{}^-\xi - -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi) d\!\!{}^-\xi = \frac{1}{q}{\rm res}(\sigma'(0)).$$

2. *In particular, if $\mathcal{R}(\sigma)(z)(\xi) = H(z)\,\sigma(\xi)\,|\xi|^{-z}$ if $|\xi| \geq 1$ is of type (8.57), then*

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi) d\!\!{}^-\xi - -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi) d\!\!{}^-\xi = H'(0)\, {\rm res}(\sigma).$$

3. *Riesz regularised integrals coincide with cut-off regularised integrals.*

4. *In even dimensions $d = 2k$, the dimensional regularised integral defined by (8.58), of a radial symbol $\sigma(\xi) = f(|\xi|)$ relates to its cut-off regularised integral by*

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm dim.reg} \sigma(\xi) d\!\!{}^-\xi - -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi) d\!\!{}^-\xi = -\left(\log \pi + \gamma - \sum_{j=1}^{k-1} \frac{1}{j}\right) {\rm res}(\sigma).$$

Proof: Let us set $\sigma(z) := \mathcal{R}(\sigma)(z)$.

1. By (7.54) we have at $z_0 = 0$

$$\begin{aligned} & -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi) d\!\!{}^-\xi - -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi) d\!\!{}^-\xi \\ = \;& {\rm ev}_0^{\rm reg} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi) d\!\!{}^-\xi - -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi) d\!\!{}^-\xi \\ = \;& -\frac{1}{\alpha'(0)}{\rm res}(\sigma'(0)) + \frac{\alpha''(0)}{2\,(\alpha'(0))^2}{\rm res}(\sigma(0)) \\ = \;& \frac{1}{q}{\rm res}_0(\sigma'(0)) \end{aligned} \tag{8.59}$$

since $\alpha(z) = a - qz$.

2. If $\mathcal{R}(\sigma)(z)(\xi) = H(z)\,\sigma(\xi)\,|\xi|^{-z} \quad \text{if} \quad vert\xi| \geq 1$ is of type (8.57), then $q = 1$ and

$$\sigma'(z)(\xi) = H'(z)\,\sigma(\xi)\,|\xi|^{-z} - z\,H(z)\,\sigma(\xi)\,\log|\xi|\,|\xi|^{-z} \quad \text{if} \quad vert\xi| \geq 1$$

so that $\mathrm{res}(\sigma'(0)) = H'(0)\,\mathrm{res}(\sigma)$ and $-\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi) đ\xi - -\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi) đ\xi = H'(0)\,\mathrm{res}(\sigma)$.

3. Setting $H \equiv 1$ yields the result for Riesz regularisation.

4. Setting $H(z) := \frac{\frac{2\pi^{\frac{d-z}{2}}}{\Gamma\left(\frac{d-z}{2}\right)}}{\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}}$ yields the result for dimensional regularisation since

$$H'(0) = -\log\pi + \frac{\Gamma'(k)}{\Gamma(k)} = -\log\pi - \gamma + \sum_{j=1}^{k-1}\frac{1}{j},$$

where we have used (1.4).

□

Example 13 *Let $d = 2k = 4$ and $\sigma(\xi) = \frac{1}{|\xi|^2+1}$. By the asymptotic expansion $\sigma(\xi) \sim |\xi|^{-2} - |\xi|^{-4}\cdots$ combined with (1.5), we compute its residue*

$$\mathrm{res}(\sigma) = -\frac{1}{(2\pi)^4}\,\mathrm{Vol}(S^3) = -\frac{1}{8\pi^2}$$

so that

$$-\!\!\!\!\!\!\int_{\mathbb{R}^4}^{\mathrm{dim.reg}} \frac{1}{|\xi|^2+1} = -\!\!\!\!\!\!\int_{\mathbb{R}^4} \frac{1}{|\xi|^2+1} + \frac{\log\pi + \gamma - \sum_{j=1}^{k-1}\frac{1}{j}}{8\pi^2} = \frac{\log\pi + \gamma - 1}{8\pi^2},$$

since $-\!\!\!\!\!\int_{\mathbb{R}^4} \frac{1}{|\xi|^2+1}$ vanishes by Example 9.

8.3 Discrepancies

Regularised integrals present discrepancies which can be measured in terms of the noncommutative residue. Unlike ordinary integrals on Schwartz functions, they are not covariant and do not vanish on derivatives.

Proposition 19 *Let $\mathcal{R}$ be a holomorphic regularisation on $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ which sends a symbol σ in $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ to a symbol of order $\alpha(z)$.*

1. *For any $C \in \mathrm{GL}_d(\mathbb{R})$*

$$|\mathrm{det}C| -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(C\,\xi)\,đ\xi - -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi)\,đ\xi = -\frac{|\mathrm{det}C|}{\alpha'(0)}\,\mathrm{res}\left((\mathcal{R}'(\sigma\circ C)(0) - \mathcal{R}'(\sigma)(0)\circ C\right).$$

2. *Setting $C = \lambda I$ yields for $\mathcal{R}$ of type (8.57)*

$$|\lambda|^d -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\lambda\,\xi)\,đ\xi - -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi)\,đ\xi = \mathrm{res}\,(\sigma)\,. \tag{8.60}$$

3. *For any $i \in \{1, \cdots, d\}$,*

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \partial_i\sigma(\xi)\,đ\xi = -\frac{1}{\alpha'(0)}\,\mathrm{res}\left(\mathcal{R}'\left(\partial_i\sigma\right)(0) - \partial_i\left(\mathcal{R}'\left(\sigma\right)(0)\right)\right).$$

In particular, if $\mathcal{R}$ is of type (8.57) then,

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \partial_i\sigma(\xi)\,đ\xi = \mathrm{res}(\xi \mapsto \sigma(\xi)\,\xi_i\,|\xi|^{-2}). \tag{8.61}$$

4. *For any* $\eta \in \mathbb{R}^d$

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} t_\eta^* \sigma - -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma = -\frac{1}{\alpha'(0)} \operatorname{res}\left(\mathcal{R}'\left(t_\eta^*\sigma\right)(0) - t_\eta^*\left(\mathcal{R}'(\sigma)(0)\right)\right).$$

which applied to $\mathcal{R}$ *is of type (8.57), reads:*

$$-\!\!\!\!\!\!\int^{\mathcal{R}} t_\eta^* \sigma - -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma = \operatorname{res}\left(\sigma(\xi)\ \left(\log|\xi| - \log(|t_{-\eta}^*\xi|)\right)\right). \tag{8.62}$$

Remark 10 *When* $\mathcal{R}$ *is Riesz regularisation which corresponds to a particular case of regularisation of type (8.57), then* $-\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \partial_i \sigma = -\!\!\!\!\!\int_{\mathbb{R}^d} \partial_i \sigma$ *computes the obstruction already derived in (4.31) preventing the cut-off integral from fulfilling Stokes' property.*

Proof:

1. We first observe that for large $\operatorname{Re}(z)$ the cut-off integral $-\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma)(z)(\xi)\, d\!\!{}^-\!\xi$ which coincides with an ordinary integral, is covariant. By analytic continuation we infer the following identity of meromorphic maps,

$$|\det C| -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma)(z)(C\,\xi)\, d\!\!{}^-\!\xi = -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma)(z)(\xi)\, d\!\!{}^-\!\xi.$$

Hence, by (7.54) applied to the holomorphic symbol $\mathcal{R}(\sigma\circ C)(z)(\xi) - \mathcal{R}(\sigma)(z)(C\xi)$ of order $\alpha(z)$, we write

$$\begin{aligned}
& |\det C| -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(C\,\xi)\, d\!\!{}^-\!\xi - -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi)\, d\!\!{}^-\!\xi \\
= \;& |\det C|\, \mathrm{ev}_0^{\mathrm{reg}} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma\circ C)(z)(\xi) d\!\!{}^-\!\xi - \mathrm{ev}_0^{\mathrm{reg}} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma)(z)(\xi) d\!\!{}^-\!\xi \\
= \;& |\det C|\, \mathrm{ev}_0^{\mathrm{reg}} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma\circ C)(z)(\xi) d\!\!{}^-\!\xi - |\det C|\, \mathrm{ev}_0^{\mathrm{reg}} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma)(z)(C\xi) d\!\!{}^-\!\xi \\
= \;& |\det C|\, \mathrm{ev}_0^{\mathrm{reg}} \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \left(\mathcal{R}(\sigma\circ C)(z)(\xi) - \mathcal{R}(\sigma)(z)\circ C(\xi)\right) d\!\!{}^-\!\xi \right) \\
= \;& |\det C|\, \mathrm{Res}_0 \frac{-\!\!\!\!\!\int_{\mathbb{R}^d} \left(\mathcal{R}(\sigma\circ C)(z)(\xi) - \mathcal{R}(\sigma)(z)\circ C(\xi)\right) d\!\!{}^-\!\xi}{z} \\
= \;& -\frac{|\det C|}{\alpha'(0)} \operatorname{res}\left(\mathcal{R}'(\sigma\circ C)(0) - \mathcal{R}'(\sigma)(0)\circ C\right).
\end{aligned}$$

2. For $C = \lambda I$ and a holomorphic regularisation

$$\mathcal{R}(\sigma)(z)(\xi) = H(z)\,\sigma(\xi)\,|\xi|^{-z} \quad \text{if} \quad |\xi| \geq 1$$

we have

$$\mathcal{R}(\sigma\circ C)'(0)(\xi) - \mathcal{R}(\sigma)'(0)\circ C((\xi) = -\sigma(\lambda\,\xi)\, \log|\xi| + \sigma(\lambda\xi)\, \log|\lambda\xi| \quad \forall |\xi| \geq 1,$$

so that in view of the positive homogeneity of σ_{-d} arising in the non commutative residue we find:

$$\operatorname{res}\left(\mathcal{R}'(\sigma\circ C)(0) - \mathcal{R}'(\sigma)(0)\circ C\right) = |\lambda|^{-d}\, \log|\lambda| \operatorname{res}(\sigma)$$

from which the result follows.

3. We first observe that for large $\operatorname{Re}(z)$ the cut-off integral $-\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma)(z)(\xi)\, d\xi$ which coincides with an ordinary integral, satisfies Stokes' property, i.e. $-\!\!\!\!\!\int_{\mathbb{R}^d} \partial_i \mathcal{R}(\sigma)(z)(\xi)\, d\xi = 0$ for any

$i \in \{1, \cdots, d\}$. By analytic continuation, this holds on the whole complex plane as an equality of meromorphic functions. Hence,

$$\begin{aligned}
-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \partial_i \sigma(\xi)\, d\!\!\!{}^-\xi &= \mathrm{ev}_0^{\mathrm{reg}} \circ -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}\left(\partial_i \sigma\right)(\xi)\, d\!\!\!{}^-\xi \\
&= \mathrm{ev}_0^{\mathrm{reg}} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \partial_i \left(\mathcal{R}\left(\sigma\right)\right)(\xi)\ d\!\!\!{}^-\xi + \mathrm{ev}_0^{\mathrm{reg}} \circ -\!\!\!\!\!\!\int_{\mathbb{R}^d} \left[\mathcal{R}\left(\partial_i \sigma\right) - \partial_i \left(\mathcal{R}\left(\sigma\right)\right)\right]\ (\xi)\, d\!\!\!{}^-\xi \\
&= \mathrm{ev}_0^{\mathrm{reg}} \circ -\!\!\!\!\!\!\int_{\mathbb{R}^d} \left[\mathcal{R}\left(\partial_i \sigma\right) - \partial_i \left(\mathcal{R}\left(\sigma\right)\right)\right]\ (\xi)\, d\!\!\!{}^-\xi \\
&= \mathrm{Res}_{z=0} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \left[\mathcal{R}\left(\partial_i \sigma\right)(z) - \partial_i \left(\mathcal{R}\left(\sigma\right)(z)\right)\right]\ (\xi)\, d\!\!\!{}^-\xi \\
&= -\frac{1}{\alpha'(0)}\, \mathrm{res}\left(\mathcal{R}'\left(\partial_i \sigma\right)(0) - \partial_i \left(\mathcal{R}'\left(\sigma\right)(0)\right)\right).
\end{aligned}$$

For a holomorphic regularisation

$$\mathcal{R}(\sigma)(z)(\xi) = H(z)\, \sigma(\xi)\, |\xi|^{-z} \quad \text{if} \quad |\xi| \geq 1$$

we have for $|\xi| \geq 1$

$$\mathcal{R}'\left(\partial_i \sigma\right)(0)(\xi) - \partial_i \left(\mathcal{R}'\left(\sigma\right)(0)\right)(\xi) = \sigma(\xi)\, \partial_i \log|\xi| = \sigma(\xi)\, \xi_i\, |\xi|^{-2}$$

from which the result in this particular case follows.

4.

$$\begin{aligned}
-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} t_\eta^* \sigma - -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma &= \mathrm{ev}_0^{\mathrm{reg}} \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \left(\mathcal{R}\left(t_\eta^* \sigma\right)(z) - \mathcal{R}(\sigma)(z)\right)\right) \\
&= \mathrm{ev}_0^{\mathrm{reg}} \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \left(\mathcal{R}\left(t_\eta^* \sigma\right)(z) - t_\eta^*\left(\mathcal{R}(\sigma)(z)\right)\right)\right) \\
&= -\frac{1}{\alpha'(0)}\, \mathrm{res}\left(\mathcal{R}'\left(t_\eta^* \sigma\right)(0) - t_\eta^*\left(\mathcal{R}'(\sigma)(0)\right)\right).
\end{aligned}$$

For a regularisation of type (8.57), we have for $|\xi| \geq 1$

$$\mathcal{R}'\left(t_\eta^* \sigma\right)(0)(\xi) - t_\eta^*\left(\mathcal{R}'\left(\sigma\right)(0)\right)(\xi) = -\sigma(t_\eta^* \xi)\, \left(\log|\xi| - \log|t_\eta^* \xi|\right),$$

which is a classical symbol. Using the translation invariance of the noncommutative residue on classical symbols, we infer that

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} t_\eta^* \sigma - -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma = \mathrm{res}\left(\sigma(t_\eta^* \xi)\, \left(\log|t_\eta^* \xi| - \log|\xi|\right)\right) = \mathrm{res}\left(\sigma(\xi)\, \left(\log|\xi| - \log(|t_{-\eta^*} \xi|\right)\right).$$

□

Example 14 *For $m \neq 0$, we compute $-\!\!\!\!\!\int_{\mathbb{R}^4}^{\mathrm{dim.reg}} \frac{1}{m^2 + |\xi|^2}\, d\!\!\!{}^-\xi$ By (8.60) we have*

$$\begin{aligned}
-\!\!\!\!\!\!\int_{\mathbb{R}^4}^{\mathrm{dim.reg}} \frac{1}{m^2 + |\xi|^2}\, d\!\!\!{}^-\xi &= m^{-2} -\!\!\!\!\!\!\int_{\mathbb{R}^4}^{\mathrm{dim.reg}} \frac{1}{1 + |\xi/m|^2}\, d\!\!\!{}^-\xi \\
&= m^2 -\!\!\!\!\!\!\int_{\mathbb{R}^4}^{\mathrm{dim.reg}} \frac{1}{1 + |\xi|^2}\, d\xi + m^2\, \log m\, \mathrm{res}\left(\frac{1}{1 + |\xi|^2}\right) \\
&= m^2 -\!\!\!\!\!\!\int_{\mathbb{R}^4}^{\mathrm{dim.reg}} \frac{1}{1 + |\xi|^2}\, d\xi - m^2\, \log m.
\end{aligned}$$

9 Regularised discrete sums on symbols

We build linear extensions to the whole algebra $CS_{\rm c.c}(\mathbb{R}^d)$ of symbols, of the canonical sum $\sum\!\!\!\!\!\!-_{\mathbb{Z}^d}$ on non integer order symbols. We show that any $\mathbb{Z}^d$-translation linear form on $CS_{\rm c.c}(\mathbb{R}^d)$ is proportional to the noncommutative residue.

9.1 Regularised discrete sums on the algebra of symbols

Theorem 10 *Let $\mathcal{R}(\sigma) : z \mapsto \mathcal{R}(\sigma)(z) := \sigma(z)$ be a holomorphic regularisation of $\sigma \in CS_{\rm c.c.}(\mathbb{R}^d)$ with order $\alpha(z)$. The map*

$$z \mapsto \sum_{\mathbb{Z}^d} \sigma(z)(\vec{n})$$

is meromorphic with a discrete of simple poles in $\alpha^{-1}([-d, \infty[\cap \mathbb{Z})$ and complex residue at $z = 0$ given by :

$$\mathrm{Res}_0 \left(\sum_{\mathbb{Z}^d} \sigma(z)(\vec{n}) \right) = -\frac{1}{\alpha'(0)} \mathrm{res}(\sigma(0)). \tag{9.63}$$

The constant term in the Laurent series at $z = 0$

$$\sum_{\mathbb{Z}^d}^{\mathcal{R}} \sigma(\vec{n}) := \mathrm{ev}_0^{\rm reg} \left(\sum_{\mathbb{Z}^d} \sigma(z)(\vec{n}) \right),$$

called the $\mathcal{R}$-regularised discrete sum of σ, reads

$$\sum_{\mathbb{Z}^d}^{\mathcal{R}} \sigma(\vec{n}) = -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi)\, d\xi + C(\sigma), \tag{9.64}$$

where we have set $-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi)\, d\xi = \mathrm{ev}_0 \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi)\, d\xi \right)$, with $C(\sigma) = \lim_{z\to 0} C(\sigma(z))$.
Whenever the order of σ has real part $< -d$ (resp. is non integer), the map $z \mapsto \sum_{\mathbb{Z}^d} \sigma(z)(\vec{n})$ is holomorphic at $z = 0$ and converges to the ordinary sum $\sum_{\mathbb{Z}^d} \sigma(\vec{n})$ (resp. cut-off regularised sum $\sum\!\!\!\!\!\!-_{\mathbb{Z}^d} \sigma(\vec{n})$) as $z \to 0$ so that in that case

$$\sum_{\mathbb{Z}^d}^{\mathcal{R}} \sigma(\vec{n}) = \sum_{\mathbb{Z}^d} \sigma(\vec{n}), \quad \left(\text{resp.} \quad \sum_{\mathbb{Z}^d}^{\mathcal{R}} \sigma(\vec{n}) = \sum\!\!\!\!\!\!-_{\mathbb{Z}^d} \sigma(\vec{n}) \right)$$

Remark 11 *The term $C(\sigma)$, which is independent of $\mathcal{R}$, arises here as a difference of regularised integrals, thus confirming a result of [GSW].*

Proof: By Theorem 7, outside the set $\alpha^{-1}([-d, \infty[\cap \mathbb{Z})$ we have:

$$\sum_{\mathbb{Z}^d} \sigma(z) = -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi)\, d\xi + C(\sigma(z)). \tag{9.65}$$

On the one hand, by results Theorem 8 we know that the map $z \mapsto \int_{\mathbb{R}^d} \sigma(z)(\xi)\, d\xi$ is meromorphic with a discrete set of simple poles in $\alpha^{-1}([-d, \infty[\cap \mathbb{Z})$ and that at zero

$$\mathrm{Res}_{z=0} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi)\, d\xi = -\frac{1}{(2\pi)^d\, \alpha'(0)} \mathrm{res}(\sigma). \tag{9.66}$$

On the other hand, we know from [GSW] that $z \mapsto C(\sigma(z))$ is holomorphic[4]. It therefore follows from (9.65) that the map $z \mapsto \sum_{\mathbb{Z}^d} \sigma(z)(\vec{n})$ is meromorphic with a discrete of simples poles in $\alpha^{-1}([-d, \infty[\cap \mathbb{Z})$ and complex residue at $z = 0$ given by

$$\mathrm{Res}_{z=0} \sum_{\mathbb{Z}^d} \sigma(z)(\vec{n}) = -\frac{1}{(2\pi)^d\, \alpha'(0)} \mathrm{res}(\sigma(0)).$$

[4]Their proof can easily be generalised to our more general setup of holomorphic families with any non constant affine order.

Taking finite parts at $z=0$ in (9.65) yields (9.64) since $\lim_{z\to 0} C(\sigma(z)) = C(\sigma)$.
When the order of σ has real part $< -d$ (resp. is non integer), the map $z \mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi)\, d\xi$ is holomorphic at 0 since σ has vanishing residue. Its limit at $z=0$ coincides with the ordinary integral $\int_{\mathbb{R}^d} \sigma(\xi)\, d\xi$ (resp. the cut-off regularised integral $-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi)\, d\xi$.). By (9.65) and since $z \mapsto C(\sigma(z))$ is known to be holomorphic ([GSW]), the map $z \mapsto \sum_{\mathbb{Z}^d} \sigma(z)(\vec{n}) = \sum_{\mathbb{Z}^d} \sigma(z)(\vec{n})$ is also holomorphic at $z=0$ and its limit reads:

$$\sum_{\mathbb{Z}^d}^{\mathcal{R}} \sigma(\vec{n}) = \lim_{z\to 0} \sum_{\mathbb{Z}^d} \sigma(z)(\vec{n}) = \int_{\mathbb{R}^d} \sigma(\xi)\, d\xi + C(\sigma) = \sum_{\mathbb{Z}^d} \sigma(\vec{n}),$$

resp.

$$\sum_{\mathbb{Z}^d}^{\mathcal{R}} \sigma(\vec{n}) = \lim_{z\to 0} \sum_{\mathbb{Z}^d} \sigma(z)(\vec{n}) = -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi)\, d\xi + C(\sigma) = \sum_{\mathbb{Z}^d} \sigma(\vec{n})$$

where the last sum is an ordinary sum (resp. a cut-off regularised sum). □

9.2 $\mathbb{Z}^d$-translation invariant linear forms on symbols

The following theorem provides a classification of regularised discrete sums which are $\mathbb{Z}^d$-translation invariant on non integer order symbols.

Theorem 11 *Linear forms on the algebra $CS_{\rm c.c}(\mathbb{R}^d)$ which are translation invariant on $CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d)$ are of the form:*

$$c \sum_{\mathbb{Z}^d} \quad + \quad \mu\, \mathrm{res}, \quad (c,\mu) \in \mathbb{C}^2.$$

Proof: Let λ be such a linear form; its restriction to $CS^{\notin \mathbb{Z}}(\mathbb{R}^d)$ being translation invariant, by Theorem 6, it is proportional to $\sum_{\mathbb{Z}^d}$.
Let $\sigma \in CS^{\mathbb{Z}}(\mathbb{R}^d)$. A holomorphic regularisation $\mathcal{R}$ modifies the order of σ from a to $\alpha(z)$ which avoids integers in a small neighborhood of 0 since $\alpha'(0) \neq 0$. Thus, for small enough non zero $|z|$,

$$\lambda(\sigma(z)) = \sum_{\mathbb{R}^d} \mathcal{R}(\sigma)(z).$$

The only degree of freedom left to define $\lambda(\sigma)$ is the choice of a regularised evaluator at $z=0$. But by Proposition 2 (here $k=1$), regularised evaluators at zero are of the form $\mathrm{ev}_0^{\rm reg} + \mu\, \mathrm{Res}_0$, with μ a real number. Hence,

$$\begin{aligned} \lambda(\sigma) &= c\, \mathrm{ev}_0^{\rm reg} \circ \sum_{\mathbb{Z}^d} \mathcal{R}(\sigma) + \mu\, \mathrm{Res}_0 \left(\sum_{\mathbb{Z}^d} \mathrm{calR}(\sigma)(z) \right) \\ &= c \sum_{\mathbb{Z}^d}^{\mathcal{R}} \sigma + \mu\, \mathrm{res}(\sigma). \end{aligned}$$

This applies to the cut-off summation map $\lambda = \sum_{\mathbb{Z}^d}$, which by Theorem 6 restricts to a $\mathbb{Z}^d$-translation invariant form on $CS_{\rm c.c}(\mathbb{R}^d)$ so that there are constants c' and μ' such that

$$\sum_{\mathbb{Z}^d} = c' \sum_{\mathbb{Z}^d}^{\mathcal{R}} + \mu'\, \mathrm{res}.$$

Since $\sum_{\mathbb{Z}^d}$ and $\sum_{\mathbb{Z}^d}^{\mathcal{R}}$ both coincide with the ordinary discrete summation map on L^1-symbols, we have $c' = 1$.It follows that $\sum_{\mathbb{Z}^d}^{\mathcal{R}} = \sum_{\mathbb{Z}^d} - \mu'\, \mathrm{res}$ from which we infer the existence of constants c and μ such that

$$\lambda = c \sum_{\mathbb{Z}^d}^{\mathcal{R}} + \mu\, \mathrm{res}$$

as anounced. □

Corollary 5 *Any $\mathbb{Z}^d$-translation invariant linear form on $CS^{\notin \mathbb{Z}}(\mathbb{R}^d)$ is proportional to the noncommutative residue.*

Proof: By the above theorem $\mathbb{Z}^d$-translation invariant linear forms on $CS_{\rm c.c}(\mathbb{R}^d)$, which by definition restrict to $\mathbb{Z}^d$-translation invariant linear forms on $CS_{\rm c.c}^{\notin \mathbb{Z}}(\mathbb{R}^d)$, are linear combinations of the cut-off discrete summation map and the noncommutative residue. Since the noncommutative residue vanishes on partial derivatives, it is invariant under $\mathbb{R}^d$-translations and is therefore $\mathbb{Z}^d$-translation invariant by Proposition 11. In contrast, we know from Proposition 15, that $\sum_{\mathbb{Z}^d}^{\mathcal{R}}$ is not $\mathbb{Z}^d$-translation invariant.Hence $c = 0$ and the result follows. $\square$

10 The zeta function

We introduce the **zeta function** and somes generalisations such as the Hurwitz and Epstein zeta functions and derive some of its properties using techniques previously described.

10.1 Zeta and Hurwitz zeta functions

Applying Theorem 8 in the one dimensional case to $\sigma_s(z)(\xi) = (|\xi|+v)^{-s-z}\,\chi(\xi)$ for some real number v and some complex number s and any smooth cut-off funciton χ which vanishes in a small neighborhood of 0 and is one outside the unite interval, leads to the following statement.

Proposition 20 *For any real number v, and any complex number s, the map $z \mapsto \sum_{n=1}^\infty (n+v)^{-s+z}$ is meromorphic with a simple pole at $z=0$ for $s=1$ given by 1 and with finite part*

$$\zeta(s;v) := \mathrm{ev}_0^{\mathrm{reg}}\left(\sum_{n=1}^\infty (n+v)^{-s+z}\right).$$

When $v=0$, $\zeta(s;v)$ is called the zeta function at argument s. For positive q, $\zeta(s,q) := \zeta(s;q-1)$ is called the Hurwitz function at argument s with parameter q.

The values at negative integers are rational provided the parameter v is rational. The result is well-known, we follow the proof of [MP2].

Proposition 21 *Let $z \mapsto \gamma(z) = z + \mu z^2 + O(z^3)$ be a holomorphic function with $\mu = \frac{\gamma''(0)}{2} \in \mathbb{R}$. Let $\lambda \in \mathbb{C} - \{0\}$. The map $z \mapsto \sum_{n=1}^\infty (n+v)^{a-\lambda\gamma(z)}$ is holomorphic at zero for any $a \neq -1$. For any $a \in \mathbb{N}$ and any non negative rational v its limit at zero:*

$$\mathrm{ev}_0^{\mathrm{reg}}\left(\sum_{n=1}^\infty (n+v)^{a-\lambda\gamma(z)}\right) = \lim_{z\to 0}\sum_{n=1}^\infty (n+v)^{a-\lambda\gamma(z)} = -\frac{B_{a+1}(1+v)}{a+1} = \zeta(-a, 1+v), \tag{10.67}$$

is a rational number. When $v=0$ this yields $\zeta(-a) = -\frac{B_{a+1}}{a+1}$ (see e.g. [Ca]). When $a=-1$ the residue at 0 reads $\frac{1}{\lambda}$.

Proof: Applied to $f(x) = (x+v)^a$ with $a \in \mathbb{C}$, the classical Euler-MacLaurin formula (6.40) gives:

$$\begin{aligned}
&\sum_{0<n\leq N} (n+v)^a = \frac{(N+v)^a + (1+v)^a}{2} + \int_1^N (x+v)^a\,dx \\
&+ \sum_{k=2}^{2K} B_k \frac{[a]_{k-1}}{k!}\left((N+v)^{a-k+1} - (1+v)^{a-k+1}\right) + \frac{[a]_{2K+1}}{(2K+1)!}\int_1^N \overline{B_{2K+1}}(x)\,(x+v)^{a-2K-1}\,dx \\
&= (1-\delta_{a+1})\frac{(N+v)^{a+1}}{a+1} - (1-\delta_{a+1})\frac{(1+v)^{a+1}}{a+1} + \frac{(N+v)^a + (1+v)^a}{2} + \delta_{a+1}\left(\log(N+v) - \log(1+v)\right) \\
&+ \sum_{k=2}^{2K} B_k \frac{[a]_{k-1}}{k!}\left((N+v)^{a-k+1} - (1+v)^{a-k+1}\right) + \frac{[a]_{2K+1}}{(2K+1)!}\int_1^N \overline{B_{2K+1}}(x)\,(x+v)^{a-2K-1}\,dx.
\end{aligned} \tag{10.68}$$

Let us set

$$R_K(a) := \frac{[a]_{2K+1}}{(2K+1)!}\int_1^N \overline{B_{2K+1}}(x)\,(x+v)^{a-2K-1}\,dx; \quad S_K(a) := \sum_{k=2}^{2K} B_k \frac{[a]_{k-1}}{k!}(1+v)^{a-k+1}.$$

Replacing a in (10.68) by $a - \lambda\gamma(z)$ and taking finite parts as $N \to \infty$ we have:

$$\sum_1^\infty (n+v)^{a-\lambda\gamma(z)} = -\frac{(1+v)^{a-\lambda\gamma(z)+1}}{a-\lambda\gamma(z)+1} + \frac{(1+v)^{a-\lambda\gamma(z)}}{2} - S_K\big(a-\lambda\gamma(z)\big) + R_K\big(a-\lambda\gamma(z)\big). \tag{10.69}$$

Hence $\text{Res}_{z=0}\sum_1^\infty (n+v)^{a-\lambda\gamma(z)} = \delta_{a+1}\,\frac{1}{\lambda}$. Taking the finite part at 0 then yields:

$$\text{fp}_{z=0}\sum_1^\infty (n+v)^{a-\lambda\gamma(z)} = -(1-\delta_{a+1})\frac{(1+v)^{a+1}}{a+1} - \delta_{a+1}\,\log(1+v) + \frac{(1+v)^a}{2} - S_K(a) + R_K(a),$$

which for a non negative integer a gives:

$$\lim_{z\to 0}\sum_0^\infty (n+v)^{a-\lambda\gamma(z)} = -\sum_{k=0}^{a+1}\binom{a+1}{k} B_k\,(1+v)^{a+1} = -\frac{B_{a+1}(1+v)}{a+1}.$$

□

Remark 12 *Whereas $\zeta(-a)$ is rational for non positive integers a, derivatives $\zeta^{(b)}(-a)$ are not expected to be rational for non positive integers a. For example, $\zeta'(0) = -\frac{1}{2}\log 2\pi$ [Ca]. As we saw previously, such derivatives involve taking sums of logarithmic expressions.*

□

10.2 Zeta functions associated with quadratic forms

Cut-off regularised sums $\sum_{\mathbb{Z}^d}$ are useful to build meromorphic extensions of ordinary sums of holomorphic families of symbols; we recover this way the existence of meromorphic extensions of zeta functions associated with quadratic forms.
To a positive definite quadratic form $q(x_1, \cdots, x_d)$ and a smooth cut-off function χ which vanishes in a small neighborhood of 0 and is identically one outside the unit euclidean ball, we assign the classical symbol

$$\xi \mapsto \sigma_{q,s}(\xi) := \chi(\xi)\, q(\xi)^{-s} \in CS_{\text{c.c}}(\mathbb{R}^d). \tag{10.70}$$

Theorem 12 *Given any complex number s the map*

$$z \mapsto \sum_{\vec{n}\in\mathbb{Z}^d-\{0\}} \sigma_{q,s+z} = \sum_{\vec{n}\in\mathbb{Z}^d-\{0\}} q(\vec{n})^{-(s+z)}$$

which is holomorphic on the half plane $\text{Re}(s+z) > d/2$ extends to a meromorphic map

$$z \mapsto -\!\!\!\!\!\!\sum_{\vec{n}\in\mathbb{Z}^d-\{0\}} \sigma_{q,s+z} = -\!\!\!\!\!\!\sum_{\vec{n}\in\mathbb{Z}^d-\{0\}} q(\vec{n})^{-(s+z)}$$

with simple pole at $z=0$ given by:

$$\text{Res}_{z=0} -\!\!\!\!\!\!\sum_{\vec{n}\in\mathbb{Z}^d-\{0\}} q(\vec{n})^{-(s+z)} = \delta_{2s=d}\int_{|\omega|=1} q(\omega)^{-d/2}\, d\mu_S(\omega)$$

and constant term at $z=0$:

$$Z_q(s) := \text{ev}_0^{\text{reg}} -\!\!\!\!\!\!\sum_{\vec{n}\in\mathbb{Z}^d-\{0\}} q(\vec{n})^{-(s+z)}. \tag{10.71}$$

Moreover,

$$Z_q(s) = -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma_{q,s} + C\left(\sigma_{q,s}\right).$$

Proof: Up to the pole which we compute separately, the result follows from Theorem 10 applied to $\sigma_{q,s}$ and Riesz regularisation $\mathcal{R}:\sigma \mapsto \sigma(\xi)\,|\xi|^{-z}$ combined with the fact that Riesz regularised integrals coincide with ordinary cut-off regularised integrals:

$$\text{ev}_0^{\text{reg}} -\!\!\!\!\!\!\int_{\mathbb{R}} \sigma(\xi)\,|\xi|^{-z}\, d\xi = -\!\!\!\!\!\!\int_{\mathbb{R}} \sigma(\xi)\, d\xi \quad \forall \sigma \in CS_{\text{c.c}}(\mathbb{R}^d).$$

Now, by (9.65) the pole at $z = 0$ is given by the pole of $⨍_{\mathbb{R}^d} \sigma_{q,s}(\xi) \, |\xi|^{-z} \, d\xi$. We write

$$\begin{aligned}
& \mathrm{Res}_{z=0} ⨍_{\mathbb{R}^d} \chi(\xi) \, q(\xi)^{-s} \, |\xi|^{-z} \, d\xi \\
= \; & \mathrm{Res}_{z=0} \int_{0 \leq |\xi| \leq 1} \chi(\xi) \, q(\xi)^{-s} \, |\xi|^{-z} \, d\xi + \mathrm{Res}_{z=0} \left(\mathrm{fp}_{R \to \infty} \int_{1 \leq |\xi| \leq R} q(\xi)^{-s} \, |\xi|^{-z} \, d\xi \right) \\
= \; & \mathrm{Res}_{z=0} \left(\mathrm{fp}_{R \to \infty} \int_{|\omega|=1} \int_1^R q(r\omega)^{-s} \, r^{-z+d-1} \, d\mu_S(\omega) \right) \\
= \; & \mathrm{Res}_{z=0} \left(\left(\mathrm{fp}_{R \to \infty} \int_1^R r^{-(2s+z)+d-1} \, dr \right) \left(\int_{|\omega|=1} q(\omega)^{-s} \, d\mu_S(\omega) \right) \right) \\
= \; & \mathrm{Res}_{z=0} \left(\mathrm{fp}_{R \to \infty} \frac{R^{-(2s+z)+d} - 1}{-(s+z)+d} \right) \left(\int_{|\omega|=1} q(\omega)^{-s} \, d\mu_S(\omega) \right) \\
= \; & \mathrm{Res}_{z=0} \frac{1}{2s+z-d} \left(\int_{|\omega|=1} q(\omega)^{-s} \, d\mu_S(\omega) \right) \\
= \; & \delta_{2s-d} \left(\int_{|\omega|=1} q(\omega)^{-s} \, d\mu_S(\omega) \right). \qquad (10.72)
\end{aligned}$$

As announced, there is therefore a pole at $z = 0$ only if $s = d/2$ in which case the residue coincides with $\int_{|\omega|=1} q(\omega)^{-s} \, d\mu_S(\omega)$. $\square$

Remark 13 *For $d = 2$ and $q(x,y) = ax^2 + bxy + cy^2$ with $4ac - b^2 > 0$, $Z_q(s)$ yields a meromorphic extension of Epstein's ζ-function $\sum_{(m,n) \in \mathbb{Z} - \{0\}} (am^2 + bmn + cn^2)^{-s}$ (see e.g.[CS]) which is known to satisfy a functional equation similar to the one satisfied by the Riemann zeta function.*
When $a = c = 1, b = 0$, $Z_q(s)$ provides a meromorphic extension of the zeta function of $\mathbb{Z}[i]$ given by (see e.g. [Ca])

$$Z_4(s) := \sum_{z \in \mathbb{Z}[i] - \{0\}} |z|^{-2s} = \sum_{(m,n) \in \mathbb{Z} - \{0\}} m^2 + n^2.$$

When $a = b = c = 1$, $Z_q(s)$ provides a meromorphic extension of the zeta function of $\mathbb{Z}[j]$ given by (see e.g. [Ca])

$$Z_3(s) := \sum_{z \in \mathbb{Z}[j] - \{0\}} |z|^{-2s} = \sum_{(m,n) \in \mathbb{Z} - \{0\}} m^2 + mn + n^2.$$

Proposition 22 *Whenever* $\mathrm{Re}(s) \leq 0$

1. *$Z_q(s) = C(\xi \mapsto q(\xi)^{-s})$,*
2. *Specifically, for any non negative integer k*

$$Z_q(-k) = 0.$$

3. *Moreover, Z_q is holomorphic at $s = -k$ for any non negative integer k and $Z'(-k) = \partial_{s|_{s=-k}} C(q^{-s})$ where the derivative at $k = 0$ stands for the derivative of the map $C(q^{-s})$ restricted to the half plane $Re(s) \leq 0$* [5] *.*

Proof:

1. When $\mathrm{Re}(s) \leq 0$, the map $\xi \mapsto q(\xi)^{-s}$ can be extended by continuity to $\xi = 0$ by[6]

$$\bar{\sigma}_{q,s}(\xi) := q(\xi)^{-s} \, \forall \xi \neq 0, \quad \bar{\sigma}_{q,s}(0) = 0.$$

[5] In contrast to the value at $s = -k$ which vanishes, one does not expect the derivative to vanish in general.

[6] Note that this extension is not smooth at 0 so that it does not define a symbol. It nevertheless has the same asymptotic behaviour as $|\xi| \to \infty$ as $\xi \mapsto \chi(\xi) \, q(\xi)^{-s}$ which is enough for our needs.

In that case, there is no need to introduce a cut-off function χ at 0 and we write:

$$Z_q(s) = \mathrm{fp}_{z=0,\mathrm{Re}(s+z)\leq 0} \sum_{\mathbb{Z}^d} q(\vec{n})^{-(s+z)} = ⨍_{\mathbb{R}^d} q(\xi)^{-s}\, d\xi + C(\xi \mapsto q(\xi)^{-s})$$

along the lines of the proof of the previous proposition. Here we take the finite part at $z = 0$ of the restriction $z \mapsto \sum_{\mathbb{Z}^d} q(\vec{n})^{-(s+z)}$ to the half plane $\mathrm{Re}(z) \leq 0$. Using polar coordinates $\xi = r\,\omega$ with $r > 0$ and ω in the unit sphere, the result then follows from the fact that the cut-off regularised integral vanishes if $\mathrm{Re}(z) \leq 0$ since we have

$$\begin{aligned}
⨍_{\mathbb{R}^d} q(\xi)^{-2s}\, d\xi &= \mathrm{fp}_{R\to\infty} \int_{|\xi|\leq R} q(\xi)^{-2s}\, d\xi \\
&= \mathrm{fp}_{R\to\infty} \int_{|\omega|=1} \int_0^R q(r\omega)^{-2s}\, r^{d-1}\, d\xi \\
&= \left(\mathrm{fp}_{R\to\infty} \int_0^R r^{-2s+d-1}\, dr\right) \left(\int_{|\omega|\leq 1} q(\omega)^{-2s}\, \mu_S(\omega)\right) \\
&= \left(\mathrm{fp}_{R\to\infty} \frac{R^{-2s+d}}{-2s+d}\, dr\right) \left(\int_{|\xi|\leq 1} q(\omega)^{-2s}\, \mu_S(\omega)\right) \quad \text{if} \quad -2s+d \neq 0 \\
&= 0 \quad \text{if} \quad -2s+d \neq 0.
\end{aligned}$$

where μ_S is the volume measure on the unit sphere induced by the canonical measure on $\mathbb{R}^d$.

2. When $s = -k$, we also have $C(\xi \mapsto q(\xi)^k) = 0$ since C vanishes on polynomials so that $Z_q(-k) = 0$.

3. By Theorem 12 there is no pole at $s = -k$ (the presence of the cut-off function χ does not affect poles) since the only pole corresponds to $s = d/2$. The map Z_q is therefore holomorphic at $s = -k$ with derivative given by the derivative of the map $C(\xi \mapsto q(\xi)^{-s})$ at $s = -k$.

PART II: Renormalisation procedures: a prologomon

We first extend the regularisation techniques described in Part I to log-polyhomogeneous symbols with occur in nested iterated integrals and nestes iterated discrete sums. We then renormalise such integrals and sums which are particular instances of more general multiple integrals and sums with linear constraints studied in the last section.

1. Renormalised evaluators
2. Integrals of log-polyhomogeneous symbols
3. A Laurent expansion for canonical integrals of holomorphic families of log-polyhomogeneous symbols
4. Renormalised nested integrals of symbols
5. Renormalised nested sums of symbols
6. Renormalised multiple discrete sums with conical constraints
7. Renormalised multiple integrals with linear constraints

11 Renormalised evaluators

11.1 Meromorphic functions in several variables

We recall some very basic definitions concerning holomorphic and meromorphic functions in several variables and refer to [GR], [Ho2] for further detials.

Definition 13 *A complex-valued function f defined on an open subset $\Omega \subset \mathbb{C}^k$ is called holomorphic in Ω if each point ω in Ω has an open neighborhood U contained in Ω such that the function f has a power series expansion:*

$$f(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha(f)(\omega)\,(z-\omega)^\alpha, \quad a_\alpha(f)(\omega) \in \mathbb{C},$$

where $\alpha = (\alpha_1, \cdot, s\alpha_k) \in \mathbb{N}_0^k$ is a multiindex, $|\alpha|$ stands for $\alpha_1 + \cdots + \alpha_k$ and $(z-\omega)^\alpha = \prod_{i=1}^k (z_i - \omega_i)$.

Osgood's lemma tells us that, in contrast to the fact that continuous functions which are differentiable at a point in each variable can well not be differentiable as functions of several variables at that point (e.g: $f(x,y) = \frac{xy}{x^2+y^2}$ continued to zero by $f(0,0) = 0$), if a complex-valued function is continuous[7] on an open set $\Omega \subset \mathbb{C}^k$ and is holomorphic in each variable separately, then it is holomorphic in Ω. This amounts to saying that when the integrand is continuous, iterated Cauchy integrals can be replaced by a single multiple Cauchy integral so that:

$$a_\alpha(f)(w) = \frac{1}{(2i\pi)^k} \int_{\bigcap_{i=1}^k |w_i - \zeta_i| = r_i} \frac{f(\zeta)\, d\zeta_1 \cdots d\zeta_k}{\prod_{i=1}^k (\zeta_i - w_i)^{\alpha_i}} = \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial z}(w) \quad \forall r_i > 0 \quad \text{sufficiently} \quad \text{small},$$

where we have set $\alpha! = \prod_{i=1}^k \alpha_i!$.
Osgood's criterion provides a bridge to extend properties of holomorphic functions in one variable to those in several variables. For example, using Osgood's criterion, holomorphic functions can be characterised via the Cauchy-Riemann equations.

Proposition 23 *A complex-valued function f defined in an open subset Ω of $\mathbb{C}^k$ which is continuously differentiable in the underlying real coordinates of $\mathbb{C}^k$, is holomorphic in Ω if and only if it satisfies the system of aprtial diferential equations:*

$$\partial_{\bar{z}_i} f(z) = 0 \quad \forall i \in \{1, \cdots, k\},$$

with $\partial_{\bar{z}_i} = \frac{\partial_{x_i} + i\partial_{y_i}}{2}$.

Using Cauchy integrals, one checks that analytic extension also holds for holomorphic functions in several variables.

Proposition 24 *If two holomorphic functions f and g in a connected open subset Ω in $\mathbb{C}^k$ coincide on an non empty open subset U of Ω, then they coincide on Ω.*

Many more properties hold for holomorphic functions in several variables, which we do not mention here since they will not be of direct use to us. Let us now turn to meromorphic functions in several variables.
For every $z \in \mathbb{C}^k$, let $\mathrm{Hom}_z(\mathbb{C}^k)$ denote set of equivalence classes of functions which are holomorphic in some neighborhood of z, under the equivalence relation $f \sim g$ if $f = g$ in some neighborhood of z. If f is holomorphic in a neighborhood of z we write $\gamma_z(f)$ for the residue class of f in $\mathrm{Hom}_z(\mathbb{C}^k)$, called the germ of f at z. We can specialise to $z = 0$ without loss of generality; elements f in $\mathrm{Hom}_0(\mathbb{C}^k)$ can be identified with the set of all power series $\sum_{|\alpha| \geq 0} a_\alpha z^\alpha$ which converge in a neighborhood of zero and the value $f(0)$ at zero corresponds to the constant term in the power series expansion. The set of units in $\mathrm{Hom}_0(\mathbb{C}^k)$, i.e. the set of invertible elements, is the set of germs of functions which do not vanish at zero. An element f in $\mathrm{Hom}_0(\mathbb{C}^k)$ is called reducible if it can be written as product $f = f_1 f_2$ with f_1, f_2 non units of $\mathrm{Hom}_0(\mathbb{C}^k)$, otherwise it is called irreducible. The following factorisation property follows from the Weierstrass preparation theorem.

[7]The assumption that the function be continuous is actually superfluous, but the proof of the statement without this assumption, which is Hartog's result, is much more difficult.

Proposition 25 $\mathrm{Hom}_0(\mathbb{C}^k)$ *is a unique factorisation domain, i.e. every non zero element in* $\mathrm{Hom}_0(\mathbb{C}^k)$ *can be written as a product of irreducible factors in one and only one way-apart from units and the order of the factors.*

For any z in a domain Ω of $\mathbb{C}^k$, $\mathrm{Hom}_z(\mathbb{C}^k)$ is a ring without divisors of zero, so that one can form the quotient field

$$\mathrm{Mer}_z(\mathbb{C}^k) := \{f/g, \quad f \in \mathrm{Hom}_z(\mathbb{C}^k), g \in \mathrm{Hom}_0(\mathbb{C}^k), \quad \gamma_0(g) \neq 0\}$$

of germs of meromorphic functions at z.
In contrast to the one variable case, it is not possible to assign values in the extended complex plane to every germ of meromorphic function in several variables in such a way that it gives rise to a continuous function with values in $\mathbb{C} \cup \infty$.
Let us make this statement more precise. We call two non zero elements in $\mathrm{Hom}_z(\mathbb{C}^k)$ relatively prime if their factorisation into irreducible factors do not present common factors apart from units. The following statement (Theorem 6.2.3 in [Ho2]) shows that it is not natural to assign any value, whether finite or infinite to the quotient f/g of two relatively prime functions f and g in $\mathrm{Hom}_0(\mathbb{C}^k)$ which vanish at zero.

Proposition 26 *Let f and g be holomorphic in a neighborhood of zero with $\gamma_0(f)$ and $\gamma_0(g)$ relatively prime. If $g(0) = g(0) = 0$, for every complex number a one can find z in any neighborhood of zero such that $g(z) \neq 0$ and $f(z)/g(z) = a$.*

Example 15 *Take $k = 2$ and $f(z_1, z_2) = z_1$, $g(z_1, z_2) = z_1 + z_2$. Then along any straight line $z_1 = \lambda z_2$ we have $f(z_1, \lambda z_1)/g(z_1, \lambda z_1) = \frac{z_1}{(1+\lambda)z_1} = \frac{1}{1+\lambda}$ for $z_1 \neq 0$ so that for every complex number $a \neq 0$, there exists $z = (z_1, \frac{1-a}{a} z_1)$ such that $f(z)/g(z) = a$ for $z \neq 0$.*

Conseauently, there is a priori no Laurent expansion representation for meromorphic functions in several variables.

11.2 Meromorphic functions in several variables with linear poles

We therefore build sets of meromorphic functions with linear pole structure.
Let us first observe that the map defined on the (Grothendieck closure of the) k-th tensor product $\mathcal{T}^k(\mathrm{Hol}_0(\mathbb{C})) := \hat{\otimes}^k \mathrm{Hol}_0(\mathbb{C})$[8] over the germ of holomorphic functions at zero by

$$\begin{array}{rcl} \mathcal{T}^k(\mathrm{Hol}_0(\mathbb{C})) & \to & \mathrm{Hol}_0(\mathbb{C}^k) \\ f_1 \otimes \cdots \otimes f_k & \mapsto & \left((z_1, \cdots, z_k) \mapsto \prod_{i=1}^k f_i(z_i) \right) \end{array}$$

is onto so that we can identify $\mathcal{T}^k(\mathrm{Hol}_0(\mathbb{C})) = \mathrm{Hol}_0(\mathbb{C}^k)$. Let us set $\mathrm{Hol}_0(\mathbb{C}^\infty) := \bigoplus_{k=0}^\infty \mathcal{T}^k(\mathrm{Hol}_0(\mathbb{C}))$.
Similarly, the (Grothendieck closure of the) k-th tensor product over the germ of meromorphic functions in a neighborhood of zero:

$$\mathcal{T}^k(\mathrm{Mer}_0(\mathbb{C})) := \hat{\otimes}^k \mathrm{Mer}_0(\mathbb{C})$$

can be viewed as a subset of $\mathrm{Mer}_0(\mathbb{C}^k)$. We therefore equip the corresponding tensor algebra

$$\mathcal{T}(\mathrm{Mer}_0(\mathbb{C})) = \oplus_{k=0}^\infty \mathcal{T}^k(\mathrm{Mer}_0(\mathbb{C}))$$

over $\mathrm{Mer}_0(\mathbb{C})$ with the product of meromorphic functions:

$$(f_1 \otimes \cdots \otimes f_k) \bigotimes (f_{k+1} \otimes \cdots \otimes f_{k+l}) = f_1 \otimes \cdots \otimes f_k \otimes f_{k+1} \otimes \cdots \otimes f_{k+l}.$$

For any positive integer j, a similar construction can be carried out to build $\mathcal{T}^k(\mathrm{Mer}_0^j(\mathbb{C})) := \hat{\otimes}^k \mathrm{Mer}_0^j(\mathbb{C})$.

A first step is to extend regularised evaluators to tensor algebras of meromorphic functions in one

[8] The symbol $\hat{\otimes}$ stands for the Grothendieck closure.

variable.
Clearly, a linear form $\lambda : \mathrm{Mer}_0^j(C) \to \mathbb{C}$ uniquely extends to a character

$$\tilde{\lambda}(f_1 \otimes \cdots \otimes f_k) := \prod_{i=1}^{k} \lambda(f_i) \tag{11.73}$$

on $\mathcal{T}\left(\mathrm{Mer}_0^j(\mathbb{C})\right)$.
A similar statement holds when dropping the superscript j altogether, allowing for meromorphic functons with any order poles at zero.

We now go beyond the tensor algebra and consider the following linear extension of $\mathcal{T}(\mathrm{Mer}_0^j(\mathbb{C}))$ which corresponds to germs at zero of meromorphic maps in severable variables with linear poles. Let for $j \in \mathbb{N}$, $\mathcal{L}\mathrm{Mer}_0^j(\mathbb{C}^\infty) := \oplus_{k=1}^\infty \mathcal{L}\mathrm{Mer}_0^j(\mathbb{C}^k)$ where

$$\mathcal{L}\mathrm{Mer}_0^j(\mathbb{C}^k) := \{\prod_{i=1}^{I} f_i \circ L_i, \quad f_i \in \mathrm{Mer}_0^j(\mathbb{C}), \quad L_i \in \left(\mathbb{C}^k\right)^*\}$$

or equivalently,

$$\mathcal{L}\mathrm{Mer}_0^j(\mathbb{C}^k) := \left\{(z_1, \cdots, z_k) \mapsto \frac{h(z_1, \cdots, z_k)}{\prod_{L \in (\mathbb{C}^k)^*} \left(L(z_1, \cdots, z_k)\right)^{m_L}}, \quad h \in \mathrm{Hol}_0\left(\mathbb{C}^k\right), \quad m_L \in \mathbb{N} \cap [0, j]\right\}. \tag{11.74}$$

Setting $I = k$ and $L_i(z_1, \cdots, z_k) = z_i$ yields a canonical injection

$$\begin{array}{rcl} i : \mathcal{T}^k(\mathrm{Mer}_0^j(\mathbb{C})) & \to & \mathcal{L}\mathrm{Mer}_0^j(\mathbb{C}^k) \\ f_1 \otimes \cdots \otimes f_k & \mapsto & \left((z_1, \cdots, z_k) \mapsto \prod_{i=1}^{k} f_i \circ L_i(z_1, \cdots, z_k)\right), \end{array}$$

and the tensor product on $\mathcal{T}(\mathrm{Mer}_0^j(\mathbb{C}))$ extends to $\mathcal{L}\mathrm{Mer}_0^j(\mathbb{C}^\infty)$, by

$$\begin{aligned} &\left((z_1, \cdots, z_k) \mapsto \prod_{i=1}^{I} f_i \circ L_i(z_1, \cdots, z_k)\right) \circledast \left((z_{k+1}, \cdots, z_{k+l}) \mapsto \prod_{j=1}^{J} f_{I+j} \circ L_{i+j}(z_{k+1}, \cdots, z_{k+l})\right) \\ = \quad &\left((z_1, \cdots, z_k, \cdots, z_{k+l}) \mapsto \prod_{i=1}^{I} f_i \circ L_i(z_1, \cdots, z_k) \prod_{j=1}^{J} f_{I+j} \circ L_{i+j}(z_{i+1}, \cdots, z_{k+l})\right) \end{aligned} \tag{11.75}$$

which makes it a graded algebra.
Specializing to linear forms $\mathcal{L}_k := \{L \in \left(\mathbb{C}^k\right)^*, \exists J \subset \{1, \cdots, k\}, \quad L(z_1, \cdots, z_k) = \sum_{j \in J} z_j\}$, gives rise to a subalgebra $\mathcal{LM}_0^j(\mathbb{C}^\infty) := \oplus_{k=1}^\infty \mathcal{LM}_0^j(\mathbb{C}^k) \subset \mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty)$ defined by:

$$\mathcal{LM}_0^j(\mathbb{C}^k) := \left\{(z_1, \cdots, z_k) \mapsto \frac{h(z_1, \cdots, z_k)}{\prod_{L \in \mathcal{L}_k} \left(L(z_1, \cdots, z_k)\right)^{m_L}}, \quad h \in \mathrm{Hol}_0\left(\mathbb{C}^k\right), \quad m_L \in \mathbb{N} \cap [0, j]\right\}. \tag{11.76}$$

We shall also consider the set

$$\mathcal{LM}_0(\mathbb{C}^k) := \bigcup_{j=1}^{\infty} \mathcal{LM}_0^j(\mathbb{C}^k). \tag{11.77}$$

For future use, we consider the diagonal map $\delta : \mathbb{C} \mapsto \mathcal{T}(\mathbb{C})$ defined by

$$\delta_k : \mathbb{C} \quad \to \quad \mathbb{C}^{\otimes k} \tag{11.78}$$

$$z \quad \mapsto \quad z \cdot 1^{\otimes k} \tag{11.79}$$

and the induced map $\delta^* : \mathcal{LM}_0^j(\mathbb{C}^k) \to \mathrm{Mer}_0^j(\mathbb{C})$

$$\begin{array}{rcl} \delta_k^\star : \mathcal{LM}_0^j(\mathbb{C}^k) & \to & \mathrm{Mer}_0^j(\mathbb{C}) \\ f & \mapsto & f \circ \delta_k. \end{array}$$

11.3 Renormalised evaluators at zero

Following [Sp] we introduce renormalised evaluators at zero.

Definition 14 *A renormalised evaluator* Λ *on a graded subalgebra* $\mathcal{B} = \oplus_{k=0}^{\infty}\mathcal{B}_k$ *of* $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty) = \oplus_{k=0}^{\infty}\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k)$ *equipped with the product* $\circledast$ *introduced in (11.75), is a character on* $\mathcal{B}$ *which is compatible with the filtration induced by the grading and extends the ordinary evaluation at zero on holomorphic maps. Equivalently,*

1. *Compatibility with the filtration: Let* $\mathcal{B}^K := \oplus_{k=0}^{K}\mathcal{B}_k$ *and* $\Lambda_K := \Lambda_{|_{\mathcal{B}^K}}$*. Then* $(\Lambda_{K+1})_{|_{\mathcal{B}^K}} = \Lambda_K$.

2. *It coincides with the evaluation map at zero on holomorphic maps:*

$$\Lambda_{|_{\mathcal{T}(\mathrm{Hol}_0(\mathbb{C}))}} = \mathrm{ev}_0.$$

3. *It fulfills a multiplicativity property:*

$$\Lambda(f \circledast g) = \Lambda(f)\,\Lambda(g) \quad \forall f, g \in \mathcal{B}.$$

We call the evaluator symmetric *if moreover for any* f *in* $\mathcal{B}_k$ *and* τ *in* Σ_k*, we have*

$$\Lambda(f_\tau) = \Lambda(f) \quad \forall \tau \in \Sigma_k,$$

where we have set $f_\tau(z_1, \cdots, z_k) := f(z_{\tau(1)}, \cdots, z_{\tau(k)})$.

Example 16 *Any regularised evaluator at zero* λ *on* $\mathrm{Mer}_0(\mathbb{C})$ *uniquely extends to a renormalised evaluator* $\tilde{\lambda}$ *on the tensor algebra* $(\mathcal{T}(\mathrm{Mer}_0(\mathbb{C})), \otimes)$ *defined by (11.73).*

Example 17 *Let* $\mathcal{B}$ *be a subalgebra of* $\mathcal{L}\mathcal{M}_0(\mathbb{C}^\infty)$ *equipped*[9] *with a coproduct which makes it a graded connected Hopf algebra. Then the map* $\delta^* : \mathcal{B} \to \mathrm{Mer}_0(\mathbb{C})$ *is a morphism of algebras*

$$\delta^*(f \circledast g) = \delta^*(f)\,\delta^*(g)$$

to which one can implement Birkhoff factorization as in the Connes and Kreimer setup ([CK], [Ma]):

$$\delta^* = \left(\delta^*_+\right)^{*-1} * \delta^*_-$$

using the convolution $*$ *associated with the product and coproduct on* $\mathcal{B}$ *combined with a minimal substraction scheme. The map* $\delta^*_+(0) : \mathcal{B} \to \mathbb{C}$ *then yields a renormalised evaluator on* $\mathcal{B}$.

Example 18 *Given a function* f *in* $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k)$*, for any* $i \in \{1, \cdots, k\}$ *and any fixed complex numbers* $z_1, \cdots, z_{i_1}, z_{i+1}, \cdots, z_k$ *outside a finite number of hyperplanes, the map* $z_i \mapsto f(z_1, \cdots, z_k)$ *lies in* $\mathrm{Mer}_0(\mathbb{C})$ *so that we can apply to it a regularised evaluator* λ*. Let* λ_{z_i} *stand for the evaluator* λ *implemented in the sole variable* z_i*, the others being kept fixed.*
Any regularised evaluator λ *on* $\mathrm{Mer}_0(\mathbb{C})$ *extends to renormalised evaluators* Λ *and* Λ' *on* $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty)$ *defined on* $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k)$ *by*

$$\Lambda := \lambda_{z_1} \circ \cdots \circ \lambda_{z_k}, \quad \Lambda' := \lambda_{z_k} \circ \cdots \circ \lambda_{z_1}$$

and to a symmetrised evaluator defined on $\mathcal{L}\mathrm{Mer}_0(\mathbb{C}^k)$ *by*

$$\Lambda^{\mathrm{sym}} := \frac{1}{k!} \sum_{\tau \in \Sigma_k} \lambda_{z_{\tau(1)}} \circ \cdots \circ \lambda_{z_{\tau(k)}}.$$

Their restrictions to $\mathcal{T}(\mathrm{Mer}_0(\mathbb{C}))$ *all coincide with* $\tilde{\lambda}$.
Let us for example check the multiplicativity property for Λ*; a similar proof holds for* Λ' *and* Λ^{sym}*. Given* $f \in \mathcal{B}_k$*,* $g \in \mathcal{B}_l$

$$\begin{aligned}
\Lambda(f \otimes g) &= \lambda_{z_1}\left(\cdots\left(\lambda_{z_{k+l}}\, f(z_1, \cdots, z_k)\, g(z_{k+1}, \cdots, z_{k+l}\right)\cdots\right) \\
&= \left(\lambda_{z_1}\left(\cdots\left(\lambda_{z_k}\, f(z_1, \cdots, z_k)\right)\cdots\right)\right)\left(\lambda^{\mathrm{reg}}_{z_{k+1}}\left(\cdots\left(\lambda_{z_{k+l}} g(z_{k+1}, \cdots, z_{k+l}\right)\cdots\right)\right) \\
&= \Lambda(f)\,\Lambda(g).
\end{aligned}$$

[9] Work is in progress to provide concrete instances of such a situation.

Example 19 *Take* $\lambda := \mathrm{ev}_0^{\mathrm{reg}}$*, and set with the above notations*

$$\mathrm{ev}_0^{\mathrm{ren}} := \Lambda; \quad \mathrm{ev}_0^{\mathrm{ren}'} := \Lambda', \quad \mathrm{ev}_0^{\mathrm{ren,sym}} := \Lambda^{\mathrm{sym}},$$

then given a holomorphic function $h(z_1, z_2)$ *in a neighborhood of* 0 *and setting* $f(z_1, z_2) := \frac{h(z_1,z_2)}{z_1+z_2}$*, we have*

$$\mathrm{ev}_0^{\mathrm{ren}}(f) = \partial_1 h(0,0); \quad \mathrm{ev}_0^{\mathrm{ren}'}(f) = \partial_2 h(0,0); \quad \mathrm{ev}_0^{\mathrm{ren,sym}}(f) = \frac{\partial_1 h(0,0) + \partial_2 h(0,0)}{2} = \mathrm{ev}_0^{\mathrm{reg}} \circ \delta^*(f).$$

But in general,

$$\mathrm{ev}_0^{\mathrm{ren,sym}} \neq \mathrm{ev}_0^{\mathrm{reg}} \circ \delta^*.$$

For example, for $f(z_1, z_2) = f_1(z_1)\, f_2(z_2)$ *we have* $\mathrm{ev}_0^{\mathrm{ren,sym}}(f) = \mathrm{ev}_0^{\mathrm{reg}}(f_1)\, \mathrm{ev}_0^{\mathrm{reg}}(f_2)$ *whereas* $\mathrm{ev}_0^{\mathrm{reg}} \circ \delta^*(f) = \mathrm{ev}_0^{\mathrm{reg}}(f_1 f_2)$.

This example illustrates how such renormalised evaluators at zero pick up linear combinations of jets of the holomorphic function h at zero. The following proposition shows that turns out to be a general feature of these particular renormalised evaluators at zero.

Proposition 27 *Let* f *be a meromorphic function with linear poles at zero of the type*

$$f = \frac{h}{\prod_{L\in\mathcal{L}_k} L^{m_L}}$$

where $h : \mathbb{C}^k \to \mathbb{C}$ *is holomorphic at zero, and* $\mathcal{L}_k$ *a set of linear forms on* $\mathbb{C}^k$*,* $m_L \in \mathbb{N}$*.*
The renormalised evaluated value of f *at zero* $\mathrm{ev}_0^{\mathrm{ren}}(f)$ *is a polynomial expression in the jets of* h *at* $\underline{0} = (0, \cdots, 0)$*. A similar property holds for* $\mathrm{ev}_0^{\mathrm{ren}'}(f)$ *and* $\mathrm{ev}_0^{\mathrm{ren,sym}}(f)$*.*

Proof: Clearly, if the property holds for $\mathrm{ev}_0^{\mathrm{ren}}(f)$, then it holds for $\mathrm{ev}_0^{\mathrm{ren}'}(f)$ and $\mathrm{ev}_0^{\mathrm{ren,sym}}(f)$. We proceed by induction on k to show it for the renormalised evaluator $\mathrm{ev}_0^{\mathrm{ren}}$. The assertion holds for $k = 1$ since $\mathrm{ev}_{z=0}^{\mathrm{reg}} \frac{h(z)}{z^m} = \frac{h^{(m)}(0)}{m!}$.
Let us assume that the statement holds for $k-1$. Up to a multiplication of h by a scalar, we can assume that all the linear forms L_k in the denominator involving z_k are of the type $z_k + L'(z_1, \cdots, z_{k-1})$ with L' possibly zero. So without loss of generality we can write $\mathcal{L}_k = \mathcal{L}_k^0 \cup \mathcal{L}'_{k-1} \cup \mathcal{L}''_k$ as a union of $\mathcal{L}_k^0$ consisting of linear forms $L(z_1, \cdots, z_k) = z_k$, of $\mathcal{L}'_{k-1}$ consisting of linear forms in $\mathcal{L}_k$ which do not involve the variable z_k, and of $\mathcal{L}''_{k-1}$ consisting of linear forms $L(z_1, \cdots, z_{k-1})$ entering in the linear forms of $\mathcal{L}_k$ of the type $a\,(z_k + L(z_1, \cdots, z_{k-1}))$ with a non zero. Up to a modification of h by a multiplicative constant, we can therefore write f in the form:

$$f(z_1 \cdots, z_k) = \frac{h(z_1, \cdots, z_k)}{\prod_{L\in\mathcal{L}_k^0} z_k^{m_L} \prod_{L\in\mathcal{L}'_k} L(z_1, \cdots, z_{k-1})^{m_L} \prod_{L\in\mathcal{L}''_{k-1}} (z_k + L(z_1, \cdots, z_{k-1}))^{m_L}},$$

with h holomorphic at zero. Applying the regularised evaluator $\mathrm{ev}_{z_k=0}^{\mathrm{reg}}$ at zero in the variable z_k, yields:

$$\mathrm{ev}_{z_k=0}^{\mathrm{reg}} f(z_1, \cdots, z_k) = \frac{\partial_{z_k}^{\sum_{L\in\mathcal{L}_k^0} m_L} h(z_1, \cdots, z_{k-1}, 0)}{\prod_{L\in\mathcal{L}'_{k-1}} L(z_1, \cdots, z_{k-1})^{m_L} \prod_{L\in\mathcal{L}''_{k-1}} L(z_1, \cdots, z_{k-1})^{m_L}}.$$

We can apply the induction assumption to the function in $k-1$ variables $z_1, \cdots, z_{k-1}$,

$$g := \mathrm{ev}_{z_k=0}^{\mathrm{reg}} f : (z_1, \cdots, z_{k-1}) \mapsto \frac{\partial_{z_k}^{\sum_{L\in\mathcal{L}_k^0} m_L} h(z_1, \cdots, z_{k-1}, 0)}{\prod_{L\in\mathcal{L}'_{k-1}\cup\mathcal{L}''_k k-1} L(z_1, \cdots, z_{k-1})^{m_L}}.$$

By assumption $\mathrm{ev}_0^{\mathrm{ren}}(g)$ is a polynomial expression in the jets at zero of the holomorphic function $(z_1, \cdots, z_{k-1}) \mapsto \partial_{z_k}^{\sum_{L\in\mathcal{L}_{k-1}^0} m_L} h(z_1, \cdots, z_{k-1}, 0)$ so that $\mathrm{ev}_0^{\mathrm{ren}}(f) = \mathrm{ev}_0^{\mathrm{ren}}(g)$ is a polynomial expression in the jets at zero of the holomorphic function h, which proves the induction step. $\square$

To end this paragraph, we point out to a discrepancy caracteristic of renormalised evaluators, similar to the ones observed for regularised evaluators. Given a function h in $\mathrm{Hol}_0(\mathbb{C}^k)$, we have the following covariance property in the parameters:

$$\mathrm{ev}_0(h \circ A) = \mathrm{ev}_0(h) \quad \forall A \in Gl_k(\mathbb{C}).$$

This does not hold for renormalised evaluators any longer. The following example shows how a change of variable modifies the renormalised value.

Example 20 *The evaluator* $\mathrm{ev}_0^{\mathrm{ren}}$ *on* $\mathcal{B}_2$ *applied to* $f(z_1, z_2) = \frac{h(z_1,z_2)}{z_1+z_2}$ *with* $h(z_1, z_2) = z_1$ *yields:*

$$\mathrm{ev}_0^{\mathrm{ren}}(f) = \mathrm{ev}_{z_1=0}^{\mathrm{reg}} \left(\mathrm{ev}_{z_2=0}^{\mathrm{reg}} \frac{h(z_1, z_2)}{z_1 + z_2} \right) = \partial_1 h(0, 0) = 1$$

whereas after a change of variable $(u_1, u_2) := (z_1, z_1 + z_2)$ *the function* f *reads* $g(u_1, u_2) = \frac{u_1}{u_2}$ *and*

$$\mathrm{ev}_0^{\mathrm{ren}}(g) = 0 \neq \mathrm{ev}_0^{\mathrm{ren}}(f).$$

12 Integrals of log-polyhomogeneous symbols

12.1 The noncommutative residue extended to log-polyhomogeneous symbols

We briefly recall some basic notions concerning log-polyhomogeneous symbols and fix the corresponding notations. A useful reference for the log-polyhomogeneous symbol calculus is [L1].
For any complex number a and any non positive integer k, let $CS^{a,k}_{\rm c.c}(\mathbb{R}^d) \subset \mathcal{S}^{\rm Re(a)}_{\rm c.c}(\mathbb{R}^d)$ be the subset of symbols σ, called log-polyhomogeneous symbols of order a and type k with constant coefficients, such that (compare with (2.11))

$$\sigma(\xi) = \sum_{j=0}^{N-1} \sigma_{a-j}(\xi) + \sigma_{(N)}(\xi) \quad \forall \xi \in \mathbb{R}^d, \quad \text{such} \quad \text{that} \quad |\xi| \geq 1 \tag{12.80}$$

where $\sigma_{(N)} \in \mathcal{S}^{{\rm Re}(a)-N}_{\rm c.c}(\mathbb{R}^d)$ and

$$\sigma_{a-j}(\xi) = \sum_{l=0}^{k} \sigma_{a-j,l}(\xi) \quad \log^l |\xi|, \quad j \in \mathbb{Z}_+ \tag{12.81}$$

with $\sigma_{a-j,l}$ positively homogeneous of degree $a-j$ for each l.
Endowed with the product of functions, the set

$$CS^{*,*}_{\rm c.c}(\mathbb{R}^d) := \bigcup_{k=0}^{\infty} CS^{*,k}_{\rm c.c}(\mathbb{R}^d), \quad \text{where} \quad CS^{*,k}(\mathbb{R}^d) = \bigcup_{a\in\mathbb{C}} CS^{a,k}_{\rm c.c}(\mathbb{R}^d),$$

generates the algebra filtered by k of log-polyhomogeneous symbols on $\mathbb{R}^d$. In particular, $CS^{*,0}_{\rm c.c}(\mathbb{R}^d)$ coincides with the algebra $CS_{\rm c.c}(\mathbb{R}^d)$ of classical symbols on $\mathbb{R}^d$ with constant coefficients.

Definition 15 *For a non negative integer k and any non negative integer $l \leq k$, the l-th noncommutative residue of a symbol $\sigma \in CS^{a,k}_{\rm c.c}(\mathbb{R}^d)$ with asymptotic expansion given by:*

$$\sigma(\xi) = \sum_{l=0}^{k} \sum_{j=0}^{N-1} \sigma_{a-j,l}(\xi) + \sigma_{(N)}(\xi) \quad \forall |\xi| \geq 1$$

reads:

$$\text{res}_l(\sigma) = \int_{S^{d-1}} \sigma_{-d,l}(\xi)\, d\!\!{}^-_S\xi. \tag{12.82}$$

Remark 14 *On $CS^{a}_{\rm c.c}(\mathbb{R}^d)$, res_0 coincides with the ordinary residue introduced in (2.20).*

12.2 The cut-off integral extended to log-polyhomogeneous symbols

The cut-off integral extends to log-polyhomogeneous symbols on the grounds of an asymptotic expansion generalising formula (4.30) (see e.g. [L1]) proved for classical symbols.

Proposition 28 *Let σ be a symbol in $CS^{*,k}_{\rm c.c}(\mathbb{R}^d)$.*

1. *Using the notations of (12.80), the integral $\int_{|\xi|\leq R} \sigma(\xi)\, d\xi$ has an asymptotic expansion as $R \to +\infty$ of the type*

$$\int_{|\xi|\leq R} \sigma(\xi) d\xi \quad \sim_{R\to\infty} \quad -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi) d\xi + \sum_{j=0, a-j+d\neq 0}^{\infty} \sum_{l=0}^{k} P_l(\sigma_{a-j,l})(\log R)\, R^{a-j+d}$$
$$+ \quad \sum_{l=0}^{k} r_l(\sigma) \log^{l+1} R \tag{12.83}$$

where the $r_l(\sigma)$ are positive constants depending on $\sigma_{l,-d}$, $P_l(\sigma_{a-j,l})(X)$ is a polynomial of degree l with coefficients depending on $\sigma_{a-j,l}$ and where the constant term $-\!\!\!\!\!\int_{\mathbb{R}^d}\sigma$ is the cut-off integral of σ corresponding to the finite part:

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma(\xi)d\xi \;:=\; \int_{\mathbb{R}^d}\sigma_{(N)}(\xi)\,d\xi+\int_{|\xi|\le 1}\sigma(\xi)\,d\xi + \sum_{j=0,a-j+d\neq 0}^{N}\sum_{l=0}^{k}\frac{(-1)^{l+1}l!}{(a-j+d)^{l+1}}\int_{|\xi|=1}\sigma_{a-j,l}(\xi)d_S\xi, \tag{12.84}$$

which is independent of $N\ge a+d-1$.

2. *The finite part is sensitive to a rescaling in the presence of residues; for any positive real number λ*

$$\mathrm{fp}_{R\to\infty}\int_{|\xi|\le R}\sigma(\xi)d\xi = -\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma+\sum_{l=0}^{l}\frac{\log^{l+1}\lambda}{l+1}\cdot \mathrm{res}_l(\sigma).$$

Proof: We split the integral $\int_{|\xi|\le R}\sigma(\xi)đ\xi$ according to the splitting in (12.80):

$$\int_{|\xi|\le R}\sigma(\xi)đ\xi=\sum_{j=0}^{N-1}\int_{|\xi|\le R}\chi(\xi)\sigma_{a-j}(\xi)đ\xi+\int_{|\xi|\le R}\sigma_{(N)}(\xi)đ\xi.$$

Choosing $N>\mathrm{Re}(a)+d$, we have that $\sigma_{(N)}\in L^1(\mathbb{R}^d)$ and the integral $\int_{|\xi|\le R}\sigma_{(N)}(\xi)đ\xi$ converges when $R\to\infty$ to $\int_{\mathbb{R}^d}\sigma_{(N)}(\xi)đ\xi$. On the other hand, for any $j\le N-1$

$$\int_{|\xi|\le R}\chi\,\sigma_{a-j}=\int_{|\xi|\le 1}\chi\,\sigma_{a-j}+\int_{1\le|\xi|\le R}\sigma_{a-j} \tag{12.85}$$

since χ is constant equal to 1 outside the unit ball.The first integral on the l.h.s. converges and since $\sigma_{a-j}(\xi)=\sum_{l=0}^{k}\sigma_{a-j,l}(\xi)\log^l|\xi| \quad \forall|\xi|\ge 1$, the second integral reads:

$$\int_{1\le|\xi|\le R}\sigma_{a-j}(\xi)đ\xi=\sum_{l=0}^{k}\int_1^R r^{a-j+d-1}\log^l r\,dr\cdot\int_{S^{d-1}}\sigma_{a-j,l}(\omega)\,đ_S\omega.$$

Hence the following asymptotic behaviours:

$$\int_{1\le|\xi|\le R}\sigma_{a-j}\sim_{R\to\infty}\sum_{l=0}^{k}\frac{\log^{l+1}R}{l+1}\cdot\int_{S^{d-1}}\sigma_{a-j,l}(\omega)d\omega=\sum_{l=0}^{k}\frac{\log^{l+1}R}{l+1}\mathrm{res}_l(\sigma)\quad \text{if } a-j=-d$$

$$\begin{aligned}\int_{1\le|\xi|\le R}\sigma_{a-j}(\xi)\,đ\xi \quad\sim_{R\to\infty}\quad & \sum_{l=0}^{k}\left(\sum_{i=0}^{l}\frac{(-1)^{i+1}\frac{l!}{(l-i)!}\log^i R}{(a-j+d)^i}\cdot R^{a-j+n}\int_{S^{n-1}}\sigma_{a-j,l}(\omega)đ_S\omega\right.\\ & + (-1)^l l!\frac{R^{a-j+d}}{(a-j+d)^{l+1}}\cdot\int_{S^{d-1}}\sigma_{a-j,l}(\omega)\,đ_S\omega\\ & \left.+\frac{(-1)^{l+1}l!}{(a-j+d)^{l+1}}\cdot\int_{S^{d-1}}\sigma_{a-j,l}(\omega)\,đ_S\omega\right)\text{ if }\quad a-j\neq -d.\end{aligned}$$

Putting together these asymptotic expansions yields (12.83) with constant term given by (12.84)

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma(\xi)d\xi:=\int_{\mathbb{R}^d}\sigma_{(N)}(\xi)\,d\xi+\sum_{j=0}^{N-1}\int_{|\xi|\le 1}\chi\,\sigma_{a-j}(\xi)\,d\xi+\sum_{j=0,a-j+d\neq 0}^{K_N}\sum_{l=0}^{k}\frac{(-1)^{l+1}l!}{(a-j+d)^{l+1}}\int_{S^{d-1}}\sigma_{a-j,l}(\omega)\,đ_S\omega.$$

The λ-dependence of the constant term follows from

$$\begin{aligned} \log^{l+1}(\lambda R) &= \left(\log R\, 1 + \frac{\log\lambda}{\log R}\right)^{l+1} \\ &\sim_{R\to\infty} \log^{l+1} R \sum_{i=0}^{l+1} C_{l+1}^i \left(\frac{\log\lambda}{\log R}\right)^i . \end{aligned}$$

The logarithmic terms $\sum_{l=0}^k \frac{\mathrm{res}_l(\sigma)}{l+1} \log^{l+1}(\lambda\, R)$ therefore contribute to the finite part by $\sum_{l=0}^l \frac{\log^{l+1}\lambda}{l+1} \cdot \mathrm{res}_l(\sigma)$ as claimed in the proposition.
□

12.3 Examples of log-polyhomogeneous symbols

12.3.1 Regularised integrals of translated symbols

The following lemma shows that $\mathbb{R}^d$ acts via translations on the set $CS^{*,k}_{\mathrm{c.c}}(\mathbb{R}^d)$ of log-polyhomogeneous symbols of type k, for any non negative integer k, thus generalising Lemma 8 which corresponded to the case $k = 0$.

Lemma 10 *Given a symbol σ in $CS^{a,k}_{\mathrm{c.c}}(\mathbb{R}^d)$ for some complex number a and some non negative integer k, for any $\eta \in \mathbb{R}^d$, the translated symbol $t_\eta^*\sigma := \sigma(\cdot + \eta)$ lies in $CS^{a,k}_{\mathrm{c.c}}(\mathbb{R}^d)$.*

Proof: We showed in the proof of Lemma 8 that $\mathbb{R}^d$ acts by translation on symbols and classical symbols while preserving the order. Since any log-polyhomogeneous symbol σ of log-type k reads

$$\sigma(\xi) = \sum_{l=0}^k \sigma_{(l)}(\xi)\, \log^l |\xi|, \quad \forall |\xi| \geq 1,$$

with $\sigma_{(l)}$ classical symbols, it suffices to show that $\log |t_\eta^*\xi|$ lies in $CS^{*,1}_{\mathrm{c.c}}(\mathbb{R}^d)$. Rescaling ξ by $\lambda > 0$ yields for large λ:

$$\log |t_{\eta^*}(\lambda\xi)| \sim \log\lambda + \log\left|\xi + \frac{\eta}{\lambda}\right| .$$

A Taylor expansion of $\eta \mapsto \log |t_\eta \xi|$ at zero yields an asymptotic expansion of $\log\left|\xi + \frac{\eta}{\lambda}\right|$ in decreasing powers of λ from which we infer that $\log |t_{\eta^*}(\lambda\xi)|$ has asymptotic expansion involving at most one power of $\log\lambda$. Thus the map $\xi \mapsto \log |t_{\eta^*}(\lambda\xi)$ is log-polyhomogeneous of log-type 1. □

The lack of translation invariance of regularised integrals observed in (8.62), naturally gives rise to log-polyhomogeneous symbols.

Proposition 29 *Given σ in $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ and a holomorphic regularisation $\mathcal{R}$ on $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ of dimensional regularisation type (8.57), the map*

$$\eta \mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} t_\eta^*\sigma(\xi)\, đ\xi$$

defines a log-polyhomogeneous symbol of log-type 1 unless σ has vanishing residue in which case it is classical.

Proof: This follows from (8.62) by which we have:

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} t_\eta^*\sigma(\xi)\, đ\xi = -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi)\, đ\xi + \mathrm{res}\left(\sigma(\xi)\ \left(\log|\xi| - \log|t_{-\eta}^*\xi|\right)\right).$$

By the above lemma, the symbol under the residue is log-polyhomogeneous of log type 1 in η. Since the residue corresponds to the integral over a compact set (the sphere) of some homogeneous component of the symbol, this symbolic behaviour in η still holds after taking the residue and the result follows. Since the logarithmic term in $|\eta|$ arises as a factor of res(σ), the symbol in η is actually classical if σ has vanishing residue. □

12.3.2 "Radial primitives" of symbols

Setting $R = |\eta|$ in (12.83) leads to the following operator which turns a classical symbol into a log-polyhomogeneous symbol of type 1, thus justifying the need for log-polyhomogeneous symbols.

Proposition 30 *[MP1] The following operator on $C^\infty(\mathbb{R}^d)$:*

$$\tilde{P}(f)(\eta) := \int_{|\xi|\leq|\eta|} f(\xi)\, d\xi \tag{12.86}$$

maps $CS^{*,k-1}_{c;c}(\mathbb{R}^d)$ to $CS^{*,k}_{c.c}(\mathbb{R}^d)$ for any positive integer k, sending a symbol of order a to a linear combination of symbols of order $a+d$ and of zero order.

Iterating this operation, we build from classical symbols a log-polyhomogeneous symbol of type k.

Corollary 6 *[MP1] Given $\sigma_1, \cdots, \sigma_k \in CS_{c.c}(\mathbb{R}^d)$, the iterated nested integration map*

$$\eta \mapsto \tilde{P}(\sigma_1\, \tilde{P}(\sigma_2 \cdots \sigma_{k-1} \tilde{P}(\sigma_k) \cdots))(\eta) = \int_{|\xi_k|\leq\cdots\leq|\xi_1|\leq|\eta|} \sigma_1(\xi_1)\cdots\sigma_k(\xi_k)\, d\xi_1 \cdots d\xi_k \tag{12.87}$$

defines a symbol in $CS^{*,k}_{c.c}(\mathbb{R}^d)$ as a linear combination of symbols of order $a_{j_1} + \cdots + a_{j_i} + i\,d, i = 1, \cdots, k$ and of zero order.

Proof: This follows from the previous proposition by induction on k. □

12.3.3 Derivatives of holomorphic families

Log-polyhomogeneous symbols also arise from differentiating holomorphic families of classical symbols.

Proposition 31 *[PS] If $\sigma(z)(\xi) \in CS^{\alpha(z),j}_{c.c}(\mathbb{R}^d)$ is a holomorphic family of log-classical symbols, then so is each derivative*

$$\sigma^{(k)}(z)(\xi) := \partial_z^k\left(\sigma(z)(\xi)\right) \ \in CS^{\alpha(z),j+k}_{c.c}(\mathbb{R}^d). \tag{12.88}$$

Precisely, $\sigma^{(k)}(z)(\xi)$ has asymptotic expansion

$$\sigma^{(k)}(z)(\xi) \sim \sum_{i\geq 0} \sigma^{(k)}(z)_{\alpha(z)-i}(\xi) \tag{12.89}$$

where as elements of $\bigcup_{l=0}^{j+k} CS^{\alpha(z)-i,\,l}_{c.c}(\mathbb{R}^d)$

$$\sigma^{(k)}(z)_{\alpha(z)-i}(\xi) = \partial_z^k\left(\sigma(z)_{\alpha(z)-i}(\xi)\right). \tag{12.90}$$

That is,

$$\left(\partial_z^k \sigma(z)\right)_{\alpha(z)-i}(\xi) = \partial_z^k\left(\sigma(z)_{\alpha(z)-i}(\xi)\right). \tag{12.91}$$

Proof: We need to show that

$$\partial_z^k\left(\sigma(z)(\xi)\right) \sim \sum_{i\geq 0} \partial_z^k\left(\sigma(z)_{\alpha(z)-i}(\xi)\right) \tag{12.92}$$

where the summands are log-polyhomogeneous of the asserted order. First, the estimate

$$\partial_z^k\left(\sigma(z)(\xi)\right) - \sum_{i=0}^{N-1} \partial_z^k\left(\sigma(z)_{\alpha(z)-i}(\xi)\right) \in \mathrm{S}^{\mathrm{Re}(\alpha(z))-N+\epsilon}_{c.c}(\mathbb{R}^d)$$

any $\epsilon > 0$, needed for (12.92) to hold follows from differentiating the remainder symbol in the asymptotic expansion of $\sigma(z)$

$$\sigma_{(N)}(z) := \sigma(z) - \sum_{i=0}^{N-1}\sum_{l=0}^{k} \sigma(z)_{\alpha(z)-i,l}, \quad i \in \mathbb{N}_0$$

which, together with its derivatives in z, lies in $\mathrm{S}_{\mathrm{c.c}}^{\mathrm{Re}(\alpha(z))-N+\epsilon}(\mathbb{R}^d)$ as a result of the locally uniform estimates in z for the remainder term of holomorphic families of symbols.
It remains to examine the form of the summands in $\sum_{i=0}^{N-1} \partial_z^k \left(\sigma(z)_{\alpha(z)-i}(\xi)\right)$. Taking differences of remainders $\sigma_{(N)}(z)(\xi)$ implies that each term $\sigma(z)_{\alpha(z)-i}(x,\xi)$ is holomorphic. In order to compute $\partial_z \left(\sigma(z)_{\alpha(z)-i}(\xi)\right)$ one must compute the derivative of each of its homogeneous components; for $|\xi| \geq 1$ and any $l \in \{0, \cdots, j\}$

$$\begin{aligned}\partial_z \left(\sigma_{\alpha(z)-i,\,l}(z)(\xi)\right) &= \partial_z \left(|\xi|^{\alpha(z)-i}\ \sigma_{\alpha(z)-i,\,l}(z)(\frac{\xi}{|\xi|})\right)\\ &= \left(\alpha'(z)\,|\xi|^{\alpha(z)-i}\ \sigma_{\alpha(z)-j,\,l}(z)(x,\frac{\xi}{|\xi|})\right)\log|\xi|\\ &\quad +\ |\xi|^{\alpha(z)-i}\ \partial_z\left(\sigma_{\alpha(z)-i,\,l}(z)(\frac{\xi}{|\xi|})\right).\end{aligned}$$

Since $\sigma_{\alpha(z)-i,\,l}(z)\left(\frac{\xi}{|\xi|}\right)$ is a symbol of constant order zero, so is its z-derivative. Hence

$$\partial_z \left(\sigma(z)_{\alpha(z)-i,\,l}(\xi)\right) = \alpha'(z)\,\sigma(z)_{\alpha(z)-i,\,l}(\xi)\ \log[\xi]\ +\ p_{\alpha(z)-i,\,l}(z)(\xi)$$

where $\sigma_{\alpha(z)-i,\,l}(z)$, $p_{\alpha(z)-i,\,l}(z) \in CS_{\mathrm{c.c}}^{\alpha(z)-i}(\mathbb{R}^d)$ are homogeneous in ξ of order $\alpha(z)-i$. Hence,

$$\partial_z \left(\sigma(z)_{\alpha(z)-i}(\xi)\right) = \alpha'(z)\,\sigma(z)_{\alpha(z)-i}(\xi)\ \log[\xi]\ +\ p_{\alpha(z)-i,}(z)(\xi) \quad \forall|\xi| \geq 1, \tag{12.93}$$

where we have set $p_{\alpha(z)-i}(z) := \sum_{l=0}^{j}\ p_{\alpha(z)-i,\,l}(z)(\xi)\ \log^l|\xi|$. Thus, the derivative $\partial_z\left(\sigma(z)_{\alpha(z)-i}\right)$ lies in $CS_{\mathrm{c.c}}^{\alpha(z)-j,\,j+1}(\mathbb{R}^d)$. Iterating (12.93), $\partial_z^k\left(\sigma_{\alpha(z)-i}(z)(\xi)\right)$ is thus seen to be a polynomial in $\log[\xi]$ of the form

$$(\alpha'(z))^k\,\sigma_{\alpha(z)-i,\,k}(z)(\xi)\ \log^{k+j}[\xi]\ +\ \ldots\ +\ \ |\xi|^{\alpha(z)-j}\ \partial_z^k(\sigma_{\alpha(z)-i,\,l}(z)(\frac{\xi}{|\xi|}))\ \log^0[\xi]$$

with each coefficient homogeneous of order $\alpha(z)-i$. This completes the proof. □

Thus, taking derivatives adds more logarithmic terms to each term $\sigma(z)_{\alpha(z)-j}(\xi)$, increasing the log-degree, but the order is unchanged. Specifically, $\sigma^{(k)}(z)_{\alpha(z)-j}$ takes the form

$$\sigma^{(k)}(z)_{\alpha(z)-j}(\xi) = \sum_{l=0}^{m+k} \sigma^{(k)}(z)_{\alpha(z)-j,\,l}(\xi)\log^l[\xi], \tag{12.94}$$

where the terms $\sigma^{(k)}(z)_{\alpha(z)-j,\,l}(\xi)$ are positively homogeneous in ξ of degree $\alpha(z)-j$ for $|\xi| \geq 1$ and can be computed explicitly from the lower order derivatives of $\sigma(z)_{\alpha(z)-j,\,m}(\xi)$. The following more precise inductive formulae will be needed in what follows.

Lemma 11 *Let $\sigma(z)(\xi) \in CS_{\mathrm{c.c}}(\mathbb{R}^d)$ be a holomorphic family of classical symbols. Then for $|\xi| \geq 1$*

$$\begin{aligned}\sigma^{(k+1)}_{\alpha(z)-j,\,k+1}(z)(\xi) &= \alpha'(z)\,\sigma^{(k)}_{\alpha(z)-j,\,k}(z)(x,\xi),\\ \sigma^{(k+1)}_{\alpha(z)-j,\,l}(z)(\xi) &= \alpha'(z)\,\sigma^{(k)}_{\alpha(z)-j,\,l-1}(z)(\xi)\\ &\quad + |\xi|^{\alpha(z)-j}\,\partial_z(\sigma^{(k)}_{\alpha(z)-j,\,l}(z)(\xi/|\xi|)), \qquad 1 \leq l \leq k,\\ \sigma^{(k+1)}_{\alpha(z)-j,\,0}(z)(\xi) &= |\xi|^{\alpha(z)-j}\,\partial_z(\sigma^{(k)}_{\alpha(z)-j,0}(z)(\,\xi/|\xi|)).\end{aligned}$$

Pproof: From the above

$$\sigma^{(k)}(z)_{\alpha(z)-j}(\xi) = \partial_z^k(\sigma(z)_{\alpha(z)-j}(\xi)) = \sum_{l=0}^{k} \sigma^{(k)}(z)_{\alpha(z)-j,\,l}(\xi)\log^l[\xi],$$

so that

$$\sigma^{(k+1)}(z)_{\alpha(z)-j}(\xi) = \sum_{l=0}^{k} \partial_z \left(\sigma^{(k)}(z)_{\alpha(z)-j,\, l}(\xi) \right) \log^l [\xi]. \tag{12.95}$$

Hence for $|\xi| \geq 1$

$$\sum_{l=0}^{k+1} \sigma^{(k+1)}(z)_{\alpha(z)-j,\, l}(\xi) \log^l |\xi| =$$

$$\sum_{r=0}^{k} \alpha'(z)\, \sigma^{(k)}(z)_{\alpha(z)-j,\, r}(\xi)\, \log^{r+1} |\xi| \;+\; |\xi|^{\alpha(z)-j}\, \partial_z \left(\sigma^{(k)}(z)_{\alpha(z)-j,\, r}(\frac{\xi}{|\xi|}) \right) \log^r |\xi|$$

where for the right-side we apply (12.88) to each of coefficient on the right-side of (12.95.) Equating coefficients completes the proof. □

12.4 Derivatives of holomorphic families of log-polyhomogeneous symbols

We derive some useful formulae for derivatives of holomorphic families of log-polyhomogeneous symbols. Let us first recall a technical result (see Lemma 1.24 in [PS]) which we then want to extend to log-polyhomogeneous symbols.

Lemma 12 *For $j \neq j_0$ one has teh following identityb of meromorphic functions*

$$\partial_z^k \left(\frac{-1}{\alpha(z)+d-j} \int_{S^{d-1}} \sigma(z)_{\alpha(z)-j}(\xi)\, đ_S\xi \right)$$

$$= \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z)-j+d)^{l+1}} \int_{S^{d-1}} \left(\partial_z^k \sigma(z)\right)_{\alpha(z)-j,\, l} (\xi)\, đ_S\xi. \tag{12.96}$$

The equality holds trivially for $k = 0$. For clarity we check the case $k = 1$ before proceeding to the general inductive step. For $k = 1$ the left-side of (12.96) is equal to

$$\frac{\alpha'(z)}{(\alpha(z)-j+d)^2} \int_{S^{d-1}} \sigma(z)_{\alpha(z)-j}(x,\xi)\, đ_S\xi$$

$$-\; \frac{1}{\alpha(z)-j+d} \int_{S^{d-1}} \partial_z \left(\sigma(z)_{\alpha(z)-j}\right)(\xi)\, đ_S\xi. \tag{12.97}$$

From (12.93), for $|\xi| \geq 1$

$$\left(\partial_z \sigma(z)\right)_{\alpha(z)-j}(\xi) = \alpha'(z)\, \sigma(z)_{\alpha(z)-j}(\xi)\, \log|\xi| \;+\; p_{\alpha(z)-j}(z)(\xi)$$

and hence $\left(\partial_z \sigma(z)\right)_{\alpha(z)-j,\, 1}(\xi) = \alpha'(z)\, \sigma(z)_{\alpha(z)-j}(x,\xi)$ for $|\xi| \geq 1$. The expression in (12.97) is therefore equal to

$$\frac{1}{(\alpha(z)-j+d)^2} \int_{S^{d-1}} \left(\partial_z \sigma(z)\right)_{\alpha(z)-j,\, 1}(\xi)\, đ_S\xi$$

$$-\; \frac{1}{\alpha(z)-j+d} \int_{S^{d-1}} \partial_z \left(\sigma(z)_{\alpha(z)-j}\right)(\xi)\, đ_S\xi$$

which is the right-side of (12.96) for $k = 1$.

Assume now that (12.96) holds for some arbitrary fixed $k \geq 0$. Then the left-side of (12.96) for $k+1$ is equal to

$$\partial_z \left(\partial_z^k \left(\frac{-1}{\alpha(z)+d-j} \int_{S^{d-1}} \sigma(z)_{\alpha(z)-j}(x,\xi)\, đ_S\xi \right) \right)$$

$$
\begin{aligned}
&= \partial_z \left(\sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z)-j+d)^{l+1}} \int_{S^{d-1}} \left(\partial_z^k \sigma(z)\right)_{\alpha(z)-j,\, l}(\xi)\, đ_S\xi \right) \\
&= \sum_{l=0}^{k} \frac{(-1)^{l}\,(l+1)!\ \alpha'(z)}{(\alpha(z)-j+d)^{l+2}} \int_{S^{d-1}} \sigma^{(k)}(z)_{\alpha(z)-j,\, l}(\xi)\, đ_S\xi \\
&\qquad + \sum_{l=0}^{k} \frac{(-1)^{l+1}\, l!}{(\alpha(z)-j+d)^{l+1}} \int_{S^{d-1}} \partial_z \left(\sigma^{(k)}(z)_{\alpha(z)-j,\, l}(\xi) \right) đ_S\xi,
\end{aligned} \tag{12.98}
$$

where for the second equality we use the property that in the notation of (12.81)

$$
\left(\partial_z^k \sigma(z)\right)_{\alpha(z)-j}(\xi) = \sum_{r=0}^{k} \sigma^{(k)}(z)_{\alpha(z)-j,\, r}(\xi) \log^r[\xi].
$$

In that notation the right-side of (12.96) for k replaced by $k+1$ reads

$$
\sum_{l=0}^{k+1} \frac{(-1)^{l+1} l!}{(\alpha(z)-j+d)^{l+1}} \int_{S^{d-1}} \sigma^{(k+1)}(z)_{\alpha(z)-j,\, l}(\xi)\, đ_S\xi, \tag{12.99}
$$

while on the sphere S^{d-1} where $|\xi| = 1$ the identities of Lemma 11 become

$$
\begin{aligned}
\sigma^{(k+1)}_{\alpha(z)-j,\, k+1}(z)(x,\xi) &= \alpha'(z)\, \sigma^{(k)}_{\alpha(z)-j,\, k}(z)(x,\xi), \\
\sigma^{(k+1)}_{\alpha(z)-j,\, l}(z)(x,\xi) &= \alpha'(z)\, \sigma^{(k)}_{\alpha(z)-j,\, l-1}(z)(x,\xi) + \partial_z(\sigma^{(k)}_{\alpha(z)-j,\, l}(z)(x,\xi)), \quad 1 \leq l \leq k, \\
\sigma^{(k+1)}_{\alpha(z)-j,\, 0}(z)(x,\xi) &= \partial_z(\sigma^{(k)}_{\alpha(z)-j,0}(z)(x,\, \xi)).
\end{aligned}
$$

Substitution of these identities in (12.99) immediately shows (12.99) to be equal to (12.98). This completes the proof of Lemma 12. □

Identity (12.96) in particular implies that for the log-polyhomogeneous symbol $\tau_k(z) := \partial_z^k \sigma(z) \in CS_{\rm c.c}^{\alpha(z),k}(\mathbb{R}^d)$ we have:

$$
\begin{aligned}
&\partial_z^j \left(\sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z)-i+d)^{l+1}} \int_{S^{d-1}} \left(\tau_k(z)\right)_{\alpha(z)-i,\, l}(\xi)\, đ_S\xi \right) \\
&= \partial_z^j \left(\sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z)-i+d)^{l+1}} \int_{S^{d-1}} \left(\partial_z^k \sigma(z)\right)_{\alpha(z)-i,\, l}(\xi)\, đ_S\xi \right) \\
&= \partial_z^{k+j} \left(\frac{-1}{\alpha(z)+d-i} \int_{S^{d-1}} \sigma(z)_{\alpha(z)-i}(\xi)\, đ_S\xi \right) \\
&= \sum_{l=0}^{k+j} \frac{(-1)^{l+1} l!}{(\alpha(z)-i+d)^{l+1}} \int_{S^{d-1}} \left(\partial_z^{k+j} \sigma(z)\right)_{\alpha(z)-i,\, l}(\xi)\, đ_S\xi \\
&= \sum_{l=0}^{k+j} \frac{(-1)^{l+1} l!}{(\alpha(z)-i+d)^{l+1}} \int_{S^{d-1}} \left(\partial_z^{j} \tau_k(z)\right)_{\alpha(z)-i,\, l}(\xi)\, đ_S\xi.
\end{aligned}
$$

The following proposition (which is unpublished joint work with Simon Scott) shows that this property holds for any log-polyhomogeneous symbol $\tau(z) \in CS_{\rm c.c}^{\alpha(z),k}(\mathbb{R}^d)$.

Proposition 32 *Let $\sigma(z) \sim \sum_{i=0}^{\infty} \sigma_{\alpha(z)-i} \in CS_{\rm c.c}^{\alpha(z),k}(\mathbb{R}^d)$ be a holomorphic family of log-polyhomogeneous symbols. For any non negative integer j,*

$$
\begin{aligned}
&\partial_z^j \left(\sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z)-i+d)^{l+1}} \int_{S^{d-1}} \sigma(z)_{\alpha(z)-i,\, l}(\xi)\, đ_S\xi \right) \\
&= \sum_{l=0}^{k+j} \frac{(-1)^{l+1} l!}{(\alpha(z)-i+d)^{l+1}} \int_{S^{d-1}} \left(\partial_z^{j} \sigma(z)\right)_{\alpha(z)-i,\, l}(\xi)\, đ_S\xi.
\end{aligned} \tag{12.100}
$$

Proof: To simplify the presentation, we only prove the case $j = 1$; as in the classical case, the case $j > 1$ can be proved by induction along the same line of reasoning.
On the one hand we have

$$\begin{aligned}
&\partial_z \left(\sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z) - i + d)^{l+1}} \int_{S^{d-1}} \sigma(z)_{\alpha(z)-i,\,l}(\xi)\, đ_S\xi \right) \\
=& \sum_{l=0}^{k} \frac{(-1)^{l+2} (l+1)!\, \alpha'(z)}{(\alpha(z) - i + d)^{l+2}} \int_{S^{d-1}} \sigma(z)_{\alpha(z)-i,\,l}(\xi)\, đ_S\xi \\
+& \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z) - i + d)^{l+1}} \int_{S^{d-1}} \partial_z \left(\sigma(z)_{\alpha(z)-i,\,l}(\xi)\right)\, đ_S\xi. \qquad (12.101)
\end{aligned}$$

On the other hand Lemma 11 (see Lemma 1.16 in [PS]) generalises to log-polyhomogeneous symbols. Indeed, for any $1 \leq l \leq k$ and $|\xi| = 1$ we have

$$d_z(\sigma_{\alpha(z)-i,\,l}(z)(\xi)) = \sigma'_{\alpha(z)-i,\,l}(z)(\xi) - \alpha'(z)\, \sigma(z)_{\alpha(z)-i,\,l-1}(\xi),$$

$$d_z(\sigma_{\alpha(z)-i,\,0}(z)(\xi)) = \sigma'_{\alpha(z)-i,\,0}(z)(\xi),$$

and

$$\sigma'_{\alpha(z)-i,\,k+1}(z)(\xi) = \alpha'\, \sigma_{\alpha(z)-i,\,k}(z)(\xi),$$

as a result of the relation:

$$\begin{aligned}
\partial_z \left(\sigma(z)_{\alpha(z)-i}(\xi)\right) &= \sum_{l=0}^{k} \partial_z \left[\sigma(z)_{\alpha(z)-i,l}(\xi)\, \log^l |\xi|\right] \\
&= \sum_{l=0}^{k} \partial_z \left[|\xi|^{\alpha(z)-i} \sigma(z)_{\alpha(z)-i,l}(\xi\, |\xi|^{-1})\, \log^l |\xi|\right] \\
&= \alpha'(z) \sum_{l=0}^{k} \sigma(z)_{\alpha(z)-i,l}(\xi)\, \log^{l+1} |\xi| \\
&+ \sum_{l=0}^{k} |\xi|^{\alpha(z)-i} \partial_z \left[\sigma(z)_{\alpha(z)-i,l}(\xi\, |\xi|^{-1})\right]\, \log^l |\xi|.
\end{aligned}$$

Hence, the last expression in equation (12.101) reads

$$\begin{aligned}
& \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z) - i + d)^{l+1}} \int_{S^{d-1}} \partial_z \left(\sigma(z)_{\alpha(z)-i,\,l}(\xi)\right)\, đ_S\xi \\
=& -\frac{1}{\alpha(z) - i + d} \int_{S^{d-1}} \sigma'_{\alpha(z)-i,\,0}(z)(\xi)\, đ_S\xi \\
+& \sum_{l=1}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z) - i + d)^{l+1}} \int_{S^{d-1}} \sigma'_{\alpha(z)-i,\,l}(z)(\xi)\, đ_S\xi \\
-& \alpha'(z) \sum_{l=1}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z) - i + d)^{l+1}} \int_{S^{d-1}} \sigma_{\alpha(z)-i,\,l-1}(z)(\xi)\, đ_S\xi
\end{aligned}$$

so that

$$
\begin{aligned}
\sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z)-i+d)^{l+1}} \int_{S^{d-1}} \partial_z \left(\sigma(z)_{\alpha(z)-i,\,l}(\xi)\right) \, đ_S\xi \;&=\; \sum_{l=0}^{k+1} \frac{(-1)^{l+1} l!}{(\alpha(z)-i+d)^{l+1}} \int_{S^{d-1}} \sigma'_{\alpha(z)-i,\,l}(z)(\xi)\, đ_S\xi \\
&+\; \frac{(-1)^{k+1}(k+1)!}{(\alpha(z)-i+d)^{k+2}} \int_{S^{d-1}} \sigma'_{\alpha(z)-i,\,k+1}(z)(\xi)\, đ_S\xi \\
&-\; \alpha'(z) \sum_{l=0}^{k-1} \frac{(-1)^{l+2}(l+1)!}{(\alpha(z)-i+d)^{l+2}} \int_{S^{d-1}} \sigma_{\alpha(z)-i,\,l}(z)(\xi)\, đ_S\xi \\
&=\; \sum_{l=0}^{k+1} \frac{(-1)^{l+1} l!}{(\alpha(z)-i+d)^{l+1}} \int_{S^{d-1}} \sigma'_{\alpha(z)-i,\,l}(z)(\xi)\, đ_S\xi \\
&-\; \alpha'(z) \sum_{l=0}^{k} \frac{(-1)^{l+1}(l+2)!}{(\alpha(z)-i+d)^{l+2}} \int_{S^{d-1}} \sigma_{\alpha(z)-i,\,l}(z)(\xi)\, đ_S\xi.
\end{aligned}
$$

Adding this to the first term $\sum_{l=0}^{k} \frac{(-1)^{l+2}(l+1)!\,\alpha'(z)}{(\alpha(z)-i+d)^{l+2}} \int_{S^{d-1}} \frac{\sigma(z)_{\alpha(z)-i,\,l}(\xi)}{\alpha(z)-i+d}\, đ_S\xi$ in (12.101) yields

$$
\begin{aligned}
&\partial_z \left(\sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z)-i+d)^{l+1}} \int_{S^{d-1}} \sigma(z)_{\alpha(z)-i,\,l}(\xi)\, đ_S\xi \right) \\
=\;& \sum_{l=0}^{k+1} \frac{(-1)^{l+1} l!}{(\alpha(z)-i+d)^{l+1}} \int_{S^{d-1}} \sigma'_{\alpha(z)-i,\,l}(z)(\xi)\, đ_S\xi,
\end{aligned}
$$

which gives (12.100) when $j = 1$. □

13 A Laurent expansion for canonical integrals of holomorphic families of log-polyhomogeneous symbols

We generalise to log-polyhomogeneous symbols, the Laurent expansion previously derived for canonical integrals of holomorphic families of classical symbols. To simplify the presentation, we only consider the case of affine holomorphic order. This is based on unpublished joint work with Simon Scott.

13.1 A Laurent expansion

Theorem 13 *Let k be a non negative integer and let $z \mapsto \sigma(z) \in CS_{\rm c.c}^{\alpha(z),k}(\mathbb{R}^d)$ be a holomorphic family of symbols parametrised by a domain $\Omega \subset \mathbb{C}$ with affine order function $z \mapsto \alpha(z)$. The map $z \mapsto \int\!\!\!\!\!\!-_{\mathbb{R}^d} \sigma(z)(\xi)\, đ\xi$ is meromorphic with poles of order $\leq k+1$ in a discrete set of points $\alpha^{-1}\left([-d, +\infty[\cap \mathbb{Z}\right)$. There is a Laurent expansion in a neighborhood of any $z_0 \in \Omega$*

$$\int\!\!\!\!\!\!-_{\mathbb{R}^d} \sigma(z)(\xi) đ\xi = \sum_{j=1}^{k+1} \frac{r_j(\sigma)(z_0)}{(z-z_0)^j} + \sum_{j=0}^{K} \frac{s_j(\sigma)(z_0)}{j!} (z-z_0)^j + o\left((z-z_0)^K\right). \tag{13.102}$$

For $1 \leq j \leq m+1$, $r_j(\sigma)(z_0)$ is locally determined and given by

$$r_j(\sigma)(z_0) := \sum_{l=j-1}^{m} \frac{(-1)^{l+1}}{(\alpha'(z_0))^{l+1}} \frac{l!}{(l+1-j)!} \operatorname{res}\left(\left(\sigma_{(l)}\right)^{(l+1-j)}(z_0)\right). \tag{13.103}$$

On the other hand, the finite part $s_0(\sigma)(z_0) = \mathrm{ev}_{z_0}^{\rm reg}\left(\int\!\!\!\!\!\!-_{\mathbb{R}^d} \sigma(z)(\xi) đ\xi\right)$ consists of a globally determinant part $\int\!\!\!\!\!\!-_{\mathbb{R}^d} \sigma(z_0)(\xi)\, đ\xi$ as well as a local term, and given by

$$\mathrm{ev}_{z_0}^{\rm reg}\left(\int\!\!\!\!\!\!-_{\mathbb{R}^d} \sigma(z)(\xi) đ\xi\right) = \int\!\!\!\!\!\!-_{\mathbb{R}^d} \sigma(z_0)(\xi)\, đ\xi + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{(\alpha'(z_0))^{l+1}} \frac{1}{l+1} \operatorname{res}\left(\left(\sigma_{(l)}\right)^{(l+1)}(z_0)\right). \tag{13.104}$$

For $1 \leq j \leq K$, $s_j(\sigma)(z_0)$ which also involves a global and a local term reads:

$$s_j(\sigma)(z_0) = \int\!\!\!\!\!\!-_{\mathbb{R}^d} \sigma^{(j)}(z_0)(\xi)\, đ\xi + \sum_{l=0}^{k} \frac{(-1)^{l+1} l!\, j!}{(\alpha'(z_0))^{l+1} (j+l+1)!} \operatorname{res}\left(\left(\sigma_{(l)}\right)^{(j+l+1)}(z_0)\right) \tag{13.105}$$

where we have set for any non negative integer l:

$$\sigma_{(l)}(\xi) \sim \sum_{j=0}^{\infty} \sigma_{a-j,l}(\xi)\, \chi(\xi)$$

for any smooth cut-off function χ which vanishes ina neighborhood of zero and is one outside the unit ball, where as before,

$$\sigma_{\alpha(z_0)-i,l}^{(r)}(z_0) := \left(\partial_z^r(\sigma_{\alpha(z)-i,l}(z))\right)|_{z=z_0}. \tag{13.106}$$

Remark 15

- *When $k=0$, then $\sigma_{(l)} = \sigma_{(0)} = \sigma$ so that we recover formulae (7.54) derived in [PS] in the case of classical symbols.*
- *In general, $\left(\sigma_{(l)}\right)^{(r)}(z_0) \neq \left(\sigma^{(r)}\right)_{(l)}(z_0)$ as can easily be seen from differentiating $\sigma(z) = \sigma \cdot |\xi|^{-z}$ at $z=0$; $\left(\sigma_{(1)}\right)'(0) = 0$ but $(\sigma')_{(1)} = -\sigma$.*

Proof: Since the orders $\alpha(z)$ define a holomorphic map, for any $z_0 \in \Omega$ such that $\alpha(z_0) \notin \mathbb{Z}$ and $\alpha'(z_0) \neq 0$, there is a ball $B(z_0, r) \subset \Omega \subset \mathbb{C}$ centered at z_0 with radius $r > 0$ such that $z \in (B(z_0, r) - \{z_0\}) \Rightarrow \alpha(z) \notin \mathbb{Z}$. In particular, for all $z \in B(z_0, r) - \{z_0\}$, the symbol $\sigma(z)$ lies in $CS_{\rm c.c}^{\alpha(z),k}(\mathbb{R}^d)$ and $d + \alpha(z) - j \neq 0 \quad \forall j \in \mathbb{N} \cup \{0\}$. The finite part integral $-\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(x,\xi) d\xi$ yields a meromorphic function with a discrete set of poles in $\alpha^{-1}\left([-d, +\infty[\cap \mathbb{Z}\right)$:

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)\, đ\xi \;=\; \int_{\mathbb{R}^d} \sigma_{(N)}(z)(\xi)\, đ\xi + \sum_{j=0}^{N} \int_{|\xi| \leq 1} \chi(\xi) \sigma_{\alpha(z)-j}(z)\, đ\xi \\ +\; \sum_{j=0}^{N} \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z) - j + d)^{l+1}} \int_{S^{d-1}} \sigma_{\alpha(z)-j,l}(z)(\xi)\, đ_S\xi. \tag{13.107}$$

Let $j_0 \in \mathbb{N} \cup \{0\}$ be the integer such that $\alpha(z_0) + d - j_0 = 0$. Choosing N large enough so that $\sigma_{(N)}(z)(\xi)$ is integrable in ξ, we have

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi)\; đ\xi \tag{13.108}$$

$$= \sum_{j=0}^{N} \int_{|\xi| \leq 1} \chi(\xi)\, \sigma_{\alpha(z)-j}(z)(\xi)\, đ\xi + \int_{\mathbb{R}^d} \sigma_{(N)}(z)(\xi)\, đ\xi \\ + \sum_{j=0}^{N} \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z) - j + d)^{l+1}} \int_{S^{d-1}} \sigma_{\alpha(z)-j,l}(z)(\xi)\, d_S\xi \\ = \sum_{j=0}^{N} \int_{|\xi| \leq 1} \chi(\xi)\, \sigma_{\alpha(z)-j}(z)(\xi)\, d\xi + \int_{\mathbb{R}^d} \sigma_{(N)}(z)(\xi)\, đ\xi \\ + \sum_{i \neq j_0}^{N} \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z) - i + d)^{l+1}} \int_{S^{d-1}} \sigma_{\alpha(z)-i,l}(z)(\xi)\, đ_S\xi \\ + \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z) - \alpha(z_0))^{l+1}} \int_{S^{d-1}} \sigma_{\alpha(z)-j_0,l}(z)(\xi)\, đ_S\xi. \tag{13.109}$$

Since $\sigma(z)$ is a holomorphic family of polyhomogeneous symbols, there is a power series expansion

$$\sigma_{\alpha(z)-j,l}(z)(\xi) = \sum_{r=0}^{\infty} \sigma^{(r)}_{\alpha(z_0)-j,l}(z_0)(\xi)\, \frac{(z - z_0)^r}{r!}. \tag{13.110}$$

The first line of (13.109) therefore gives rise to a Taylor expansion:

$$
\begin{aligned}
& \sum_{j=0}^{N} \int_{|\xi|\leq 1} \chi(\xi)\sigma_{\alpha(z)-j}(z)(\xi)\, d\xi + \int_{\mathbb{R}^d} \sigma_{(N)}(z)(\xi)\, đ\xi \\
= & \sum_{r=0}^{R} \left(\sum_{j=0}^{N} \int_{B_x^*(0,1)} \chi(\xi)\, \sigma^{(r)}_{\alpha(z_0)-j}(z)(\xi) \right) \frac{(z-z_0)^r}{r!}\, đ\xi \\
+ & \sum_{r=0}^{R} \left(\int_{\mathbb{R}^d} \sigma^{(r)}_{(N)}(z_0)(\xi)\, đ\xi \right) \frac{(z-z_0)^r}{r!} \\
= & \sum_{j=0}^{N} \int_{B_x^*(0,1)} \chi(\xi)\sigma_{\alpha(z_0)-j}(z)(\xi) + \int_{\mathbb{R}^d} \sigma_{(N)}(z_0)(\xi)\, đ\xi \\
+ & \sum_{r=1}^{R} \left(\sum_{j=0}^{N} \int_{|\xi|\leq 1} \chi(\xi)\sigma^{(r)}_{\alpha(z_0)-j}(z)(\xi) \right) \frac{(z-z_0)^r}{r!}\, đ\xi \\
+ & \sum_{r=1}^{R} \left(\int_{\mathbb{R}^d} \sigma^{(r)}_{(N)}(z_0)(\xi)\, đ\xi \right) \frac{(z-z_0)^r}{r!} \\
+ & \; o\left((z-z_0)^R\right).
\end{aligned} \tag{13.111}
$$

On the other hand, the third line in (13.109) yields a meromorphic function with poles at points z_0 such that $\alpha(z_0) \in \mathbb{Z}$. Since the order function $z \mapsto \alpha(z)$ is affine, we have

$$
\begin{aligned}
& \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z)-\alpha(z_0))^{l+1}} \int_{S^{d-1}} \sigma_{\alpha(z)-j_0,l}(z)(\xi)\, đ_S\xi \\
= & \sum_{l=0}^{k}\sum_{r=0}^{R} \left[\frac{(-1)^{l+1} l!}{(\alpha'(z_0))^{l+1}} \int_{S^{d-1}} \sigma^{(r)}_{-d,l}(z_0)(\xi)\, d_S\xi \right] \frac{(z-z_0)^{r-l-1}}{r!} \\
+ & \sum_{l=0}^{k}\sum_{r=0}^{R} \left[\frac{(-1)^{l+1} l!}{(\alpha'(z_0))^{l+1}} \int_{S^{d-1}} \sigma^{(r+l+1)}_{-d,l}(z_0)(\xi)\, d_S\xi \right] \frac{(z-z_0)^{r}}{(r+l+1)!} \\
+ & \; o\left((z-z_0)^R\right).
\end{aligned} \tag{13.112}
$$

When $j \neq j_0$, the expression $\alpha(z) - j + d$ does not vanish so that the second line of equation (13.109) is also holomorphic as a function of z. It follows from Lemma **??** that

$$
\begin{aligned}
& \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z)-j+d)^{l+1}} \int_{S^{d-1}} \sigma_{\alpha(z)-j,l}(z)(\xi)\, đ_S\xi \\
= & \sum_{l=0}^{k}\sum_{r=0}^{R} \partial_z^r \left(\frac{(-1)^{l+1} l!}{(\alpha(z)-j+d)^{l+1}} \int_{S^{d-1}} \sigma_{\alpha(z)-j,l}(z)(\xi) đ_S\xi \right) \frac{(z-z_0)^r}{r!} + o\left((z-z_0)^R\right) \\
= & \sum_{r=0}^{R} \left[\sum_{l=0}^{k+r} \frac{(-1)^{l+1} l!}{(\alpha(z)-j+d)^{l+1}} \int_{S^{d-1}} đ_S\xi\, \sigma^{(r)}_{\alpha(z)-j,l}(z)(\xi) \right] \frac{(z-z_0)^r}{r!} + o\left((z-z_0)^R\right).
\end{aligned} \tag{13.113}
$$

Inserting (13.111), (13.112), (13.113) back into (13.109) and applying equation (13.107) to $\sigma^{(r)} \in CS_{\rm c.c}^{\alpha(z),k+r}(\mathbb{R}^d)$, we find:

$$\begin{aligned}
&-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi) đ\xi \\
= \;& \sum_{l=0}^{k}\sum_{t=0}^{T}\left[\frac{(-1)^{l+1} l!}{(\alpha'(z_0))^{l+1}} \int_{S^{d-1}} \sigma_{-d,l}^{(t)}(z_0)(\xi)\, đ_S\xi\right] \frac{(z-z_0)^{t-l-1}}{t!}\, đ_S\xi \\
+ \;& -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z_0)(\xi)\, đ\xi \\
+ \;& \sum_{l=0}^{k} \frac{(-1)^{l+1}}{(l+1)\,(\alpha'(z_0))^{l+1}} \int_{S^{d-1}} \sigma_{-d,l}^{(l+1)}(z_0)(\xi)\, đ_S\xi \\
+ \;& \sum_{r=1}^{R} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma^{(r)}(z_0)\, \frac{(z-z_0)^r}{r!} \\
+ \;& \sum_{r=1}^{R}\left[\sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha'(z_0))^{l+1}} \frac{\int_{S^{d-1}} \sigma_{-d,l}^{(r+l+1)}(z_0)(\xi)\, đ_S\xi}{(r+l+1)!}\right](z-z_0)^r \\
+ \;& o\left((z-z_0)^R\right].
\end{aligned}$$

Relabelling the terms, this gives the expansions stated in the theorem at the level of local symbols. □

The highest complex residue in the Laurent expansion relates to the highest noncommutative residue [L1] of the symbol $\sigma(z_0)$. The following corollary provides a slight extension already proved in [**?**] of the result derived in [L1] where it was asumed that $\alpha'(0) = 1$.

Corollary 7 *[L1] Let k be a non negative integer. For any holomorphic family $z \mapsto \sigma(z) \in CS_{\rm c.c}^{*,k}(\mathbb{R}^d)$ of symbols parametrised by a domain $\Omega \subset \mathbb{C}$ with order function $z \mapsto \alpha(z)$ as in Theorem 13, then the map $z \mapsto -\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi) d\xi$ is meromorphic with poles of order at most $k+1$ at any point $z_0 \in \Omega$. The pole of order $k+1$ of $-\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi) d\xi$ at a point z_0 reads:*

$$\begin{aligned}
\mathrm{Res}_{z_0}^{k+1} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi)\, đ\xi &= \frac{(-1)^{k+1} k!}{(\alpha'(z_0))^{k+1}} \int_{S^{d-1}} \sigma_{-d,k}(z_0)(\xi)\, đ\xi \\
&= \frac{(-1)^{k+1} k!}{(\alpha'(z_0))^{k+1}} \mathrm{res}_k\left(\sigma(z_0)\right). \qquad (13.114)
\end{aligned}$$

13.2 Regularised integrals of log-polyhomogeneous symbols

Given a holomorphic regularisation $\mathcal{R}$, Theorem 13 provides a meromorphic extension $-\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma)(z)(\xi)\, đ\xi$ of the holomorphic function $\int_{\mathbb{R}^d} \mathcal{R}(\sigma)(z)(\xi)\, đ\xi$ on the half plane $\mathrm{Re}(\alpha(z)) < -d$ with Laurent epxansion whose coefficients are noncommutative residues of the jets at zero $\mathcal{R}^{(m)}(0)(\sigma_{(l)})$ (see (13.106)) of the classical components $\sigma_{(l)}$ of the symbol.

Proposition 33 *Let $\mathcal{R} : \sigma \mapsto \sigma(z)$ be a holomorphic regularisation which sends a symbol σ in $CS_{\rm c.c}^{*,k}(\mathbb{R}^d)$ to a holomorphic family $\sigma(z)$ with affine order $\alpha(z) = -qz + \alpha(0)$ for some $q > 0$. We have the following Laurent expansion:*

$$\begin{aligned}
-\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma)(z)(\xi) đ\xi &= \sum_{l=0}^{k}\sum_{t=0}^{T} \frac{l!}{q^{l+1}} \mathrm{res}\left(\mathcal{R}^{(t)}(0)(\sigma_{(l)})\right) \frac{z^{t-l-1}}{t!} \\
&+ -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi)\, đ\xi + \sum_{l=0}^{k} \frac{1}{(l+1)\, q^{l+1}} \mathrm{res}\left(\mathcal{R}^{(l+1)}(0)(\sigma_{(l)})\right) \\
&+ \sum_{r=1}^{R} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}^{(r)}(0)(\sigma)\, \frac{z^r}{r!} + \sum_{r=1}^{R}\left[\sum_{l=0}^{k} \frac{l!}{q^{l+1}} \frac{\mathrm{res}\left(\mathcal{R}^{(r+l+1)}(0)(\sigma_{(l)})\right)}{(r+l+1)!}\right] z^r + o(z^R).
\end{aligned}$$

In contrast to the other coefficients of the Laurent expansion which depend on the holomorphic regularisation $\mathcal{R}$, the highest order residue given by (13.114) is independent of the choice of $\mathcal{R}$.

On teh grounds of this proposition, we set the following defintion.

Definition 16 *Given a holomorphic regularisation $\mathcal{R} : \sigma \mapsto \sigma(z)$ which sends a symbol σ to a holomorphic family $\sigma(z)$ with affine order $\alpha(z)$, the constant term in the Laurent expansion given by*

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi)\, đ\xi := \mathrm{ev}_0^{\mathrm{reg}} \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma)(z)(\xi)\, đ\xi \right)$$

is called the $\mathcal{R}$-regularised integral of σ.
If $\mathcal{R} : \sigma \mapsto \sigma(z)$ is a holomorphic regularisation of type (8.57):

$$\mathcal{R}(\sigma)(z)(\xi) = H(z)\sigma(\xi)\, |\xi|^{-z} \quad \forall |\xi| \geq 1, \quad H(0) = 1,$$

we set

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathrm{dim.reg,H}} \sigma(\xi)\, đ\xi = -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi)\, đ\xi,$$

called a dimensionally regularised type integral of σ. If $H(z) := \frac{\frac{2\pi^{\frac{d-z}{2}}}{\Gamma\left(\frac{d-z}{2}\right)}}{\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}}$, this corresponds to a dimensionally regularised integral which we denote by $-\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathrm{dim.reg}} \sigma(\xi)\, đ\xi$.
If moreover $H \equiv 1$, we set

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathrm{Riesz}} \sigma(\xi)\, đ\xi = -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R}} \sigma(\xi)\, đ\xi,$$

called the Riesz regularised integral of σ.

Specialising to dimensional type regularisation, leads to the following.

Proposition 34 *Let $\mathcal{R} : \sigma \mapsto \sigma(z)$ be a holomorphic regularisation of type (8.57):*

$$\mathcal{R}(\sigma)(z)(\xi) = H(z)\sigma(\xi)\, |\xi|^{-z} \quad \forall |\xi| \geq 1, \quad H(0) = 1,$$

and σ a symbol in $CS^{*,k}_{\mathrm{c.c}}(\mathbb{R}^d)$. There is a Laurent expansion in a neighborhood of 0 with coefficients given in terms of the jets of H at zero:

$$\begin{aligned}
-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi) đ\xi \quad &= \quad \sum_{j=1}^{k+1} \sum_{l=j-1}^{k} \frac{l!}{(l+1-j)!}\, H^{(l+1-j)}(0) \frac{\mathrm{res}(\sigma_{(l)})}{z^j} \\
&+ \quad -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathrm{dim.reg,H}} \sigma(\xi)\, đ\xi \\
&+ \quad \sum_{j=0}^{K} \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma^{(j)}(0)(\xi)\, đ_S\xi + \sum_{l=0}^{k} \frac{l!\, j!}{(j+l+1)!}\, H^{(j+l+1)}(0)\, \frac{\mathrm{res}\left(\sigma_{(l)}\right)}{j!}\, z^j \right) \\
&+ \quad o\left(z^K\right), \qquad (13.115)
\end{aligned}$$

with

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathrm{dim.reg,H}} \sigma(\xi) đ\xi = -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(\xi)\, đ\xi + \sum_{l=0}^{k} \frac{1}{l+1}\, H^{(l+1)}(0)\, \mathrm{res}\left(\sigma_{(l)}\right).$$

Moreover, we have

$$\mathrm{Res}_0^j \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi) đ\xi \right) = \sum_{l=j-1}^{k} \frac{l!}{(l+1-j)!}\, H^{(l+1-j)}(0)\, \mathrm{res}(\sigma_{(l)}),$$

which for $j = k+1$ reads:

$$\mathrm{Res}_0^{k+1} \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(z)(\xi) đ\xi \right) = k!\, \mathrm{res}_k(\sigma),$$

independently of the jets of H at zero.

Setting $H \equiv 1$ in the above proposition shows that Riesz regularised integrals coincide with cut-off regularised integrals of log-polyhomogeneous symsbols, thus generalising a property already observed for classical symbols.

Corollary 8 *Riesz regularised integrals coincide with cut-off regularised integrals.*

14 Renormalised nested integrals of symbols

This section based on joint work with Dominique Manchon [MP1], is dedicated to the renormalisation of nested integrals of symbols which obey shuffle relations.

14.1 Rota-Baxter relations and shuffle product

Recall from Paragraph 1.1 that a Rota-Baxter operator on an algebra $\mathcal{A}$ over a field k is a linear operator $P : \mathcal{A} \to \mathcal{A}$ such that the relation:

$$P(a)P(b) = P\big(P(a)b + bP(a)\big) + \lambda P(a)P(b) \quad \forall \sigma, \tau \in CS^{*,*}_{\rm c.c}(\mathbb{R}^d). \tag{14.116}$$

holds for any a, b in $\mathcal{A}$. Here λ is a scalar in the field k called the *weight*[10].
The operator $\widetilde{P}$ introduced in (12.86) satisfies the weight zero Rota-Baxter relation which corresponds to an integration by parts in disguise.

Proposition 35 *The map $\sigma \mapsto \tilde{P}(\sigma)$ defined by(12.86) obeys the following Rota-Baxter relation [EGK]:*

$$\tilde{P}(\sigma)\,\tilde{P}(\tau) = \tilde{P}(\sigma\,\tilde{P}(\tau)) + \tilde{P}(\tau\,\tilde{P}(\sigma)). \tag{14.117}$$

Proof: The Rota-Baxter relation follows from:

$$\begin{aligned}
\tilde{P}(\sigma)(\eta)\,\tilde{P}(\tau)(\eta) &= \int_{|\xi|\le|\eta|} \sigma(\xi)\,d\xi \int_{|\xi|\le|\eta|} \tau(\xi)\,d\xi \\
&= \int_{|\xi|\le|\eta|} \sigma(\xi)\,d\xi \int_{|\tilde{\xi}|\le|\xi|} \tau(\tilde{\xi})\,d\tilde{\xi} + \int_{|\xi|\le|\eta|} \tau(\xi)\,d\xi \int_{|\tilde{\xi}|\le|\xi|} \sigma(\tilde{\xi})\,d\tilde{\xi} \\
&= \tilde{P}(\sigma\,\tilde{P}(\tau))(\eta) + \tilde{P}(\tau\,\tilde{P}(\sigma))(\eta).
\end{aligned}$$

□

Let us now recall the definition of a shuffle Hopf algebra following the presentation in [MP2].
Let V be a linear space and $\mathcal{T}(V) = \bigoplus_{k\ge 0} V^{\otimes k}$ be the associated tensor algebra. The shuffle product ш is defined by:

$$(v_1 \otimes \cdots \otimes v_k) \text{ ш } (v_{k+1} \otimes \cdots \otimes v_{k+l}) := \sum_{\tau \in \Sigma_{k;l}} v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(k+l)}$$

where τ runs over the set $\Sigma_{k;l}$ of (k,l)-shuffles, i.e.

$$\Sigma_{k,l} = \{\tau \in \Sigma_{k+l}, \quad \text{s.t.} \quad \tau(1) < \cdots < \tau(k) \quad \text{and} \quad \tau(k+1) < \cdots < \tau(k+l).\}$$

For $k = 2$ this reads:

$$v_1 \text{ ш }_2 = v_1 \otimes v_2 + v_2 \otimes v_1.$$

The shuffle product and the deconcatenation coproduct:

$$\Delta\,(v_1 \otimes \cdots \otimes v_k) := \sum_{j=0}^{k} (v_1 \otimes \cdots \otimes v_j) \bigotimes (v_{j+1} \otimes \cdots \otimes v_k)$$

endow $\mathcal{T}(V)$ with a structure of connected graded commutative Hopf algebra $(\mathcal{T}(V), \text{ш}, \Delta)$ [H1].
In terms of the shuffle product, relation (14.117) reads:

$$\left(\int_{0\le|\xi_1|\le|\eta|} \sigma(\xi_1)\,d\xi_1\right)\left(\int_{0\le|\xi_2|\le|\eta|} \tau(\xi_2)\,d\xi_2\right) = \int_{0\le|\xi_1|\le|\xi_2|\le|\eta|} (\sigma \text{ ш } \tau)(\xi_1,\xi_2)\,d\xi_1\,d\xi_2. \tag{14.118}$$

The aim of this section is to extract finite parts of the expressions on either side as $|\eta| \to \infty$ while preserving the identity, i.e. while preserving the shuffle relations. Naively extracting a finite part as $|\eta| \to \infty$ for each of the integrals involved in these expressions does not do the job since the finite part of a product does not generally coincide with the product of the finite parts. A more subtle renormalisation procedure is needed.

[10] Some authors use the opposite sign convention for the weight.

14.2 Nested integrals of non integer order symbols

We introduce the following nested integrals.

Definition 17 *Given $R>0$, for symbols $\sigma_i \in CS_{\rm c.c}(\mathbb{R}^d)$, $i\in\{1,\cdots,k\}$ we set*

$$\begin{aligned}&\int_{B(0,R)}^{\rm nested} \sigma_1\otimes\cdots\otimes\sigma_k\\ :=\ &\int_{0\leq|\xi_k|\leq|\xi_{k-1}|\leq\cdots\leq|\xi_1|\leq R} \sigma_1(\xi_1)\,\sigma_2(\xi_2)\cdots\sigma_k(\xi_k)\,d\xi_1\cdots d\xi_k\\ =\ &\int_{r\leq r_k\leq r_{k-1}\leq\cdots\leq r_1\leq R} f_1(r_1)\cdots f_k(r_k)\,dr_1\cdots dr_k\end{aligned}$$

where $f_i(r):=r^{d-1}\int_{|\xi|=1}\sigma_i(r\xi)\,d_S\xi$ and $B(0,R):=\{\xi\in\mathbb{R}^d, |\xi|\leq R\}$.

These nested integrals correspond to ordinary nested integrals $\int_{r\leq r_k\leq\cdots\leq r_1\leq R}\omega_1\wedge\cdots\wedge\omega_k$, with $\omega_i(t)=f_i(t)dt$. As such they enjoy the usual properties of one-dimensional nested integrals (see e.g [Ch], or Appendix XIX.11 in [Ka2]). In particular, they obey shuffle relations. For symbols $\sigma_i\in CS_{\rm c.c}(\mathbb{R}^d)$, i varying from 1 to k, we have:

$$⨍_{B(0,R)}^{\rm nested}(\sigma_1\otimes\cdots\otimes\sigma_k)\,ш\,(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l})=\left(⨍_{B(0,R)}^{\rm nested}\sigma_1\otimes\cdots\otimes\sigma_k\right)\left(⨍_{B(0,R)}^{\rm nested}\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l}\right). \tag{14.119}$$

In view of Corollary 6, we control the asymptotic behaviour of the map $R\longrightarrow ⨍_{B(0,R)}^{\rm nested}\sigma$ as $R\to\infty$. Picking the constant term in the expansion leads to the following definition.

Definition 18 *Given $\sigma_i\in CS_{\rm c.c}(\mathbb{R}^d)$, with $i\in\{1,\cdots,k\}$, we set*

$$\begin{aligned}⨍_{\mathbb{R}^d}^{\rm nested}\sigma_1\otimes\cdots\otimes\sigma_k\ &:=\ {\rm fp}_{R\to\infty}⨍_{B(0,R)}^{\rm nested}\sigma_1\otimes\cdots\otimes\sigma_k\\ &=\ ⨍_{\mathbb{R}^d}\sigma_1(\xi)\tilde P\left(\sigma_2\,\tilde P(\cdots\tilde P(\sigma_{k-1}\tilde P(\sigma_k))\cdots)\right)(\xi)\,đ\xi\\ &=\ ⨍_{\mathbb{R}^d}d\xi_1\int_{|\xi_2|\leq|\xi_1|}d\xi_2\cdots\int_{|\xi_k|\leq|\xi_{k-1}|}d\xi_k\ \sigma_1(\xi_1)\cdots\sigma_k(\xi_k).\end{aligned}$$

Proposition 36 *Given symbols $\sigma_i\in CS_{\rm c.c}(\mathbb{R}^d)$ with $i\in\{1,\cdots,k\}$ whose orders a_i satisfy*

$$a_{\gamma(1)}+\cdots+a_{\gamma(i)}+i\,d\notin\mathbb{N}_0,\quad\forall\gamma\in\Sigma_k,\quad\forall i\in\{1,\cdots,k\},$$

(this holds in particular if none of the partial sums of the orders is a multiple of d) then

$$\prod_{i=1}^k⨍_{\mathbb{R}^d}\sigma_i(\xi)\,d\xi=\sum_{\gamma\in\Sigma_k}⨍_{\mathbb{R}^d}^{\rm nested}\sigma_{\gamma(1)}\otimes\cdots\otimes\sigma_{\gamma(k)}. \tag{14.120}$$

Similarly, for symbols $\sigma_i\in CS_{\rm c.c}(\mathbb{R}^d)$ with order a_i, the index i varying from 1 to $k+l$, provided

$$a_{\gamma(1)}+\cdots+a_{\gamma(i)}+i\,d\notin\mathbb{N}_0,\quad\forall\gamma\in\Sigma_{k+l},\quad\forall i\in\{1,\cdots,k+l\},$$

(in particular if none of the partial sums of the orders a_i's is a multiple of d) we have:

$$\begin{aligned}&⨍_{\mathbb{R}^d}^{\rm nested}(\sigma_1\otimes\cdots\otimes\sigma_k)\,ш\,(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l})\\ =\ &\left(⨍_{\mathbb{R}^d}^{\rm nested}\sigma_1\otimes\cdots\otimes\sigma_k\right)\left(⨍_{\mathbb{R}^d}^{\rm nested}\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l}\right).\end{aligned} \tag{14.121}$$

Proof: The equality

$$\prod_{i=1}^k \int_{B(0,R)} \sigma_i(\xi)\, d\xi = \sum_{\gamma\in\Sigma_k} \int_{B(0,R)} \sigma_{\gamma(1)} \otimes \cdots \otimes \sigma_{\gamma(k)}$$

follows from writing the product space $\prod_{i=1}^k B(0,R)$ as a union of sets $\Delta_\gamma := \{\xi \in \mathbb{R}^d, |\xi_{\gamma(k)}| \leq |\xi_{\gamma(k-1)}| \leq \cdots \leq |\xi_{\gamma(1)}| \leq R\}$.
Both sides have asymptotic expansions of type (12.83) as $R\to\infty$ involving products of powers $R^{a_i-j_i+d}$ and logarithmic powers $\log^l R$ where a_i is the order of σ_i and j_i are non negative integers. If $a_{\gamma(1)}+\cdots+a_{\gamma(i)}+id\notin \mathbb{N}_0$ for any $i\in\{1,\cdots,k\}$ and any $\gamma\in\Sigma_k$, no extra finite contribution other than the product of the constant terms, can arise from products of such powers. In this case, taking finite parts on either side yields:

$$\begin{aligned}
\prod_{i=1}^k -\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma_i &= \prod_{i=1}^k \mathrm{fp}_{R\to\infty} \int_{|\xi|\leq R} \sigma_i(\xi)\, d\xi \\
&= \mathrm{fp}_{R\to\infty} \prod_{i=1}^k \int_{|\xi|\leq R} \sigma_i(\xi)\, d\xi \\
&= \mathrm{fp}_{R\to\infty} \sum_{\gamma\in\Sigma_k} \int_{B(0,R)}^{\text{nested}} \sigma_{\gamma(1)} \otimes\cdots\otimes \sigma_{\gamma(k)} \\
&= \sum_{\gamma\in\Sigma_k} -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\text{nested}} \sigma_{\gamma 1)} \otimes\cdots\otimes \sigma_{\gamma(k)}
\end{aligned}$$

which leads to (14.120). One derives (14.121) from (14.119) along the same line of proof. □

14.3 Nested integrals of holomorphic families

A holomorphic regularisation $\mathcal{R}$ on $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ induces one on the tensor algebra $\mathcal{T}\left(CS_{\mathrm{c.c}}(\mathbb{R}^d)\right)$:

$$\widetilde{\mathcal{R}}(\sigma_1\otimes\cdots\otimes\sigma_k)(z_1,\cdots,z_k) := \mathcal{R}(\sigma_1)(z_1)\otimes\cdots\otimes\mathcal{R}(\sigma_k)(z_k) \tag{14.122}$$

which is compatibile relation with the shuffle product

$$\widetilde{\mathcal{R}}\left((\sigma_1\otimes\cdots\otimes\sigma_k) ш (\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l})\right) = \widetilde{\mathcal{R}}\left(\sigma_1\otimes\cdots\otimes\sigma_k\right) ш \widetilde{\mathcal{R}}\left(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l}\right) \tag{14.123}$$

for any $\sigma_i\in CS_{\mathrm{c.c}}(\mathbb{R}^d)$, $i\in\{1,\cdots,k+l\}$.

Remark 16 *Note that in the case $k=2$ compatibility with the shuffle product means*

$$\widetilde{\mathcal{R}}(\sigma_1 ш \sigma_2)(z_1,z_2) = \mathcal{R}(\sigma_1) ш \mathcal{R}(\sigma_2)(z_1,z_2)$$

in spite of the fact that:

$$\begin{aligned}
&\widetilde{\mathcal{R}}(\sigma_1 ш \sigma_2)(z_1,z_2) = \mathcal{R}(\sigma_1)(z_1)\otimes\mathcal{R}(\sigma_2)(z_2) + \mathcal{R}(\sigma_2)(z_1)\otimes\mathcal{R}(\sigma_1)(z_2) \\
\neq\; &\mathcal{R}(\sigma_1)(z_1) ш \mathcal{R}(\sigma_2)(z_2) = \mathcal{R}(\sigma_1)(z_1)\otimes\mathcal{R}(\sigma_2)(z_2) + \mathcal{R}(\sigma_2)(z_2)\otimes\mathcal{R}(\sigma_1)(z_1).
\end{aligned}$$

The holomorphic regularisation $\mathcal{R}$ induces a one parameter holomorphic regularisation:

$$\left(\delta^*\circ\widetilde{\mathcal{R}}\right)(\sigma_1\otimes\cdots\otimes\sigma_k)(z) := \mathcal{R}(\sigma_1)(z)\otimes\cdots\otimes\mathcal{R}(\sigma_k)(z), \tag{14.124}$$

which is also compatible with the shuffle product.

$$\begin{aligned}
&\left(\delta^*\circ\widetilde{\mathcal{R}}\right)\left((\sigma_1\otimes\cdots\otimes\sigma_k) ш (\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l})\right) \\
=\; &\left(\delta^*\circ\widetilde{\mathcal{R}}\right)(\sigma_1\otimes\cdots\otimes\sigma_k) ш \left(\delta^*\circ\widetilde{\mathcal{R}}\right)(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l}), \qquad (14.125)
\end{aligned}$$

Here, δ is the diagonal map introduced in (11.78).
The following theorem describes the pole structure of nested integrals.

Theorem 14 *Let $\mathcal{R} : \sigma \mapsto \sigma(z)$ be a holomorphic regularisation procedure on $CS_{\rm c.c}(\mathbb{R}^d)$ such that $\sigma(z)$ has affine order $\alpha(z) = -q\,z + \alpha(0)$ with $q > 0$.*
For any $\sigma_i \in CS_{\rm c.c}(\mathbb{R}^d), i = 1, \cdots, k$, with orders $a_i, i = 1, \cdots, k$,

1. *the map*

$$(z_1, \cdots, z_k) \mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \widetilde{\mathcal{R}}\left(\sigma_1 \otimes \cdots \otimes \sigma_k\right)(z_1, \cdots, z_k)$$

is a meromorphic function in several variables with poles on a countable set of hyperplanes

$$z_{j_1} + \cdots + z_{j_i} = \frac{a_{j_1} + \cdots + a_{j_i} + id - n}{q}, \quad n \in \mathbb{N}_0 \tag{14.126}$$

amongst which those passing through zero:

$$z_{\tau(1)} + \cdots + z_{\tau(i)} = 0, \quad i \in \{1, \cdots, k\}, \tau \in \Sigma_k.$$

2. *The following identities of meromorphic functions hold:*

$$\prod_{i=1}^{k} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma_i)(z_i)(\xi)\, d\xi = \sum_{\tau \in \Sigma_k} -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \mathcal{R}\left(\sigma_{\tau(1)}\right)(z_{\tau(1)}) \otimes \cdots \otimes \mathcal{R}\left(\sigma_{\tau(k)}\right)(z_{\tau(k)}) \tag{14.127}$$

and for symbols $\sigma_{i+k} \in CS_{\rm c.c}(\mathbb{R}^d)$ of order a_{i+k} with i varying from 1 to l,

$$\begin{aligned} & -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \left(\mathcal{R}(\sigma_1)(z_1) \otimes \cdots \otimes \mathcal{R}(\sigma_k)(z_k)\right) ш \left(\mathcal{R}(\sigma_{k+1})(z_{k+1}) \otimes \cdots \otimes \mathcal{R}(\sigma_{k+l})(z_{k+l})\right) \\ = & \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \mathcal{R}(\sigma_1)(z_1) \otimes \cdots \otimes \mathcal{R}(\sigma_k)(z_k)\right) \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \mathcal{R}(\sigma_{k+1})(z_{k+1}) \otimes \cdots \otimes \mathcal{R}(\sigma_{k+l})(z_{k+l})\right). \end{aligned} \tag{14.128}$$

3. *Hence the following identities of meromorphic functions*

$$\left[\prod_{i=1}^{k} -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathcal{R}(\sigma_i)(z_i)(\xi)\, d\xi\right]_{\rm sym} = \left[\sum_{\tau \in \Sigma_k} -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \widetilde{\mathcal{R}}\left(\sigma_{\tau(1)} \otimes \cdots \otimes \sigma_{\tau(k)}\right)(z_1, \cdots, z_k)\right]_{\rm sym} \tag{14.129}$$

and

$$\begin{aligned} & \left[-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \left(\mathcal{R}(\sigma_1)(z_1) \otimes \cdots \otimes \mathcal{R}(\sigma_k)(z_k)\right) ш \left(\mathcal{R}(\sigma_{k+1})(z_{k+1}) \otimes \cdots \otimes \mathcal{R}(\sigma_{k+l})(z_{k+l})\right)\right]_{\rm sym} \\ = & \left[\left(-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \widetilde{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k)\right) \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \widetilde{\mathcal{R}}(\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})(z_{k+1}, \cdots, z_{k+l})\right)\right]_{\rm sym} \end{aligned} \tag{14.130}$$

where as before "sym" stands for symmetrisation in the complex variables.

4. *Provided*

$$a_{\tau(1)} + \cdots + a_{\tau(i)} + id \notin \mathbb{N}_0, \quad \forall \tau \in \Sigma_k, \quad \forall i \in \{1, \cdots, k\}$$

(which holds in particular if the partial sums of the orders are not multiples of d), then (14.127) holds as an identity of holomorphic functions in a neighborhood of zero, which when evaluated at zero, gives back (14.120).
Provided

$$a_{\gamma(1)} + \cdots + a_{\gamma(i)} + id \notin \mathbb{N}_0, \quad \forall \gamma \in \Sigma_{k+l}, \quad \forall i \in \{1, \cdots, k+l\}$$

(which holds in particular if the partial sums of the orders are not multiples of d), then (14.128) holds as an identity of holomorphic functions in a neighborhood of zero, which when evaluated at zero, gives back (14.121).

Proof: To simplify notations we set $\sigma_i(z) = \mathcal{R}(\sigma_i)(z)$.

1. The symbol $\sigma_1 \tilde{P}(\sigma_2 \tilde{P}(\cdots \tilde{P}(\sigma_k)\cdots))$ is a linear combination of symbols of order $a_{j_1}+\cdots+a_{j_i}+(i-1)\,d$, with $i\in\{1,\cdots,k\}$. This can be shown by induction on k writing $\sigma_1\tilde{P}(\sigma_2\tilde{P}(\cdots\tilde{P}(\sigma_{k+1})\cdots)) = \sigma_1\tilde{P}(\sigma)$ and applying Lemma 30 to $\sigma := \sigma_2\tilde{P}(\sigma_3\tilde{P}(\cdots\tilde{P}(\sigma_{k+1})\cdots))$ which by the induction assumption, lies in $CS^{*,k-2}_{\rm c.c}(\mathbb{R}^d)$ as a linear combination of log-polyhomogeneous symbols of order $a_{j_1}+\cdots+a_{j_i}+(i-1)\,d$ with $i\in\{1,\cdots,k-1\}$ and log-type $k-2$.
   In the same manner, the symbol $\sigma_1(z_1)\tilde{P}(\sigma_2(z_2)\tilde{P}(\cdots\tilde{P}(\sigma_k(z_k))\cdots))$ can be viewed as a linear combination of holomorphic log-polyhomogeneous symbols of log type $k-1$ and order $\alpha_{j_1}(z_{j_1})+\cdots+\alpha_{j_i}(z_{j_i})+(i-1)\,d$ where $\alpha_i(z_i)$ is the order of $\sigma_i(z_i)$. Since by assumption,
   $$\alpha_{j_1}(z_{j_1})+\cdots+\alpha_{j_i}(z_{j_i})+(i-1)\,d=\alpha_{j_1}(0)+\cdots+\alpha_{j_i}(0)+(i-1)\,d-q\,(z_{j_1}+\cdots+z_{j_i}),$$
   we can apply Theorem 13 with complex parameter $z=z_{j_1}+\cdots+z_{j_i}$ to each of the holomorphic families of symbols with order $\alpha_{j_1}(z_{j_1})+\cdots+\alpha_{j_i}(z_{j_i})+(i-1)\,d$ arising in this linear combination. This shows that their cut-off sums are meromorphic with poles of order k on a countable set of hyperplanes $\alpha_{j_1}(z_{j_1})+\cdots+\alpha_{j_i}(z_{j_i})+(i-1)\,d\in[-d,+\infty[\cap\mathbb{Z}$, i.e
   $$z_{j_1}+\cdots+z_{j_i}=\frac{a_{j_1}+\cdots+a_{j_i}+id-n}{q},\quad n\in\mathbb{N}_0.$$
   Hyperplanes of poles through zero are therefore of the type $z_{\tau(1)}+\cdots+z_{\tau(i)}=0,\quad i\in\{1,\cdots,k\},\tau\in\Sigma_k$, as anounced.
2. We now know that the expressions on either side of identity (14.127)
   $$\prod_{i=1}^k -\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma_i(z_i)(\xi)\,d\xi=\sum_{\tau\in\Sigma_k} -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested}\sigma_{\tau(1)}(z_{\tau(1)})\otimes\cdots\otimes\sigma_{\tau(k)}(z_{\tau(k)})$$
   are meromorphic functions. By Proposition 36, this identity holds outside the discrete set of points $\alpha_{\tau(1)}(z_{\tau(1)})+\cdots+\alpha_{\tau(i)}(z_{\tau(i)})+id\in\mathbb{N}_0$ and hence outside the hyperplanes of poles. Thus, the identity holds as an equality of meromorphic functions.
3. By the second item of the theorem we have:
   $$\prod_{i=1}^k -\!\!\!\!\!\!\int_{\mathbb{R}^d}\sigma_i(z_i)(\xi)\,d\xi=\sum_{\tau\in\Sigma_k} -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested}\widetilde{\mathcal{R}}(\sigma_{\tau(1)}\otimes\cdots\otimes\sigma_{\tau(k)})(z_{\tau(1)},\cdots,z_{\tau(k)}).$$
   The third item in the theorem then follows from
   $$\left(\widetilde{\mathcal{R}}(\sigma_{\tau(1)}\otimes\cdots\otimes\sigma_{\tau(k)})(z_{\tau(1)},\cdots,z_{\tau(k)})\right)_{sym}=\left(\mathcal{R}(\sigma_{\tau(1)}\otimes\cdots\otimes\sigma_{\tau(k)})(z_1,\cdots,z_k)\right)_{\rm sym}\quad\forall\tau\in\Sigma_k.$$
4. If
   $$a_{\tau(1)}+\cdots+a_{\tau(i)}+id\notin\mathbb{N}_0,\quad\tau\in\Sigma_k, i\in\{1,\cdots,k\}$$
   the functions on either side of (14.127) are holomorphic at zero. Evaluating them at zero yields back (14.120).
   A similar proves shows (14.128) and the related statement at the end of the theorem.

□

14.4 Nested integrals renormalised via evaluators

Given a holomorphic regularisation $\mathcal{R}$ which takes a symbol σ to a symbol $\mathcal{R}(\sigma)(z)$ with holomorphic order $\alpha(z)=\alpha(0)-qz$ for some positive real number q, we infer from Theorem 14 that the map

$$\sigma_1\otimes\cdots\otimes\sigma_k\mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested}\widetilde{\mathcal{R}}(\sigma_1\otimes\cdots\otimes\sigma_k)$$

defined on the tensor algebra of classical symbols, takes its values in the algebra $\mathcal{LM}_0(\mathbb{C}^\infty)$ (introduced in (11.77)) of meromorphic functions with linear poles at zero given by

$$z_{\tau(1)}+\cdots+z_{\tau(i)}=0\quad\forall\tau\in\Sigma_k,\quad\forall i\in\{1,\cdots,k\}.$$

We set the following definitions.

Definition 19 *Let $\mathcal{R}:\sigma\mapsto\sigma(z)$ be a holomorphic regularisation procedure on $CS_{\rm c.c}(\mathbb{R}^d)$ such that $\sigma(z)$ has affine order $\alpha(z)=-q\,z+\alpha(0)$ with $q>0$.*
Given a renormalised evaluator Λ at zero, we set for any $\sigma\in CS^{*,*}_{\rm c.c}(\mathbb{R}^d)$

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R},\Lambda}\sigma(\xi)\,d\xi:=\Lambda\left(z\mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}\mathcal{R}(\sigma)(z)(\xi)\,d\xi\right)$$

and for any $\sigma_i\in CS_{\rm c.c}(\mathbb{R}^d)$, $\quad i\in\{1,\cdots,k\}$

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\text{nested},\mathcal{R},\Lambda}\sigma_1\otimes\cdots\otimes\sigma_k:=\Lambda\left((z_1,\cdots,z_k)\mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\text{nested}}\widetilde{\mathcal{R}}(\sigma_1\otimes\cdots\otimes\sigma_k)(z_1,\cdots,z_k)\right).$$

Proposition 37 *Let $\mathcal{R}:\sigma\mapsto\sigma(z)$ be a holomorphic regularisation procedure on $CS_{\rm c.c}(\mathbb{R}^d)$ such that $\sigma(z)$ has affine order $\alpha(z)=-q\,z+\alpha(0)$ with $q>0$.*
Given a symmetric *renormalised evaluator Λ at zero, the following identity holds:*

$$\prod_{i=1}^k -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R},\Lambda}\sigma_i(\xi)\,d\xi=\sum_{\tau\in\Sigma_k}-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\text{nested},\mathcal{R},\Lambda}\sigma_{\tau(1)}\otimes\cdots\otimes\sigma_{\tau(k)}. \tag{14.131}$$

Provided

$$a_{\tau(1)}+\cdots+a_{\tau(i)}+id\notin\mathbb{N}_0,\quad\forall\tau\in\Sigma_k,\quad\forall i\in\{1,\cdots,k\}$$

(in particular if the partial sums of the orders are not multiples of d) then this boilds down to (14.120).

Proof: Equation (14.131) follows from implementing the evaluator Λ on either side of (14.127). Indeed, compatibility of any evaluator with the product $\circledast$ (defined in (11.75)) ensures that

$$\begin{aligned}\Lambda\left((z_1,\cdots,z_k)\mapsto\prod_{i=1}^k -\!\!\!\!\!\!\int_{\mathbb{R}^d}\mathcal{R}(\sigma_i)(z_i)(\xi)\,d\xi\right)&=\prod_{i=1}^k\Lambda\left(z_i\mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}\mathcal{R}(\sigma_i)(z_i)(\xi)\,d\xi\right)\\&=\prod_{i=1}^k -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R},\Lambda}\sigma_i(\xi)\,d\xi,\end{aligned}$$

which yields the left hand side of (14.131).
Using the linearity of Λ on the r.hs. of (14.127) combined with the symmetry of Λ which implies that for any $\tau\in\Sigma_k$, for any $\sigma_1,\cdots,\sigma_k\in CS_{\rm c.c}(\mathbb{R}^d)$

$$\begin{aligned}&\Lambda\left((z_1,\cdots,z_k)\mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\text{nested}}\widetilde{\mathcal{R}}(\sigma_{\tau(1)}\otimes\cdots\otimes\sigma_{\tau(k)})(z_1,\cdots,z_k)\right)\\=\;&\Lambda\left((z_1,\cdots,z_k)\mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\text{nested}}\widetilde{\mathcal{R}}(\sigma_{\tau(1)}\otimes\cdots\otimes\sigma_{\tau(k)})(z_{\tau(1)},\cdots,z_{\tau(k)})\right),\end{aligned}$$

then yields the right hand side of (14.131). $\square$

This gives rise to a character on the tensor algebra $\mathcal{T}\left(CS_{\rm c.c}(\mathbb{R}^d)\right)$. We first need a technical result.

Lemma 13 *Let $\mathcal{R}:\sigma\mapsto\sigma(z)$ be a holomorphic regularisation procedure on $CS_{\rm c.c}(\mathbb{R}^d)$ such that $\sigma(z)$ has affine order $\alpha(z)=-q\,z+\alpha(0)$ with $q>0$.*
The map

$$\begin{aligned}\Phi^{\mathcal{R}}:\mathcal{T}\left(CS_{\rm c.c}(\mathbb{R}^d)\right)&\to\mathcal{LM}_0(\mathbb{C}^\infty)\\ \sigma_1\otimes\cdots\otimes\sigma_k&\mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\text{nested}}\widetilde{\mathcal{R}}(\sigma_1\otimes\cdots\otimes\sigma_k)\end{aligned}\tag{14.132}$$

satisfies the following identity of meromorphic functions. For any symbols $\sigma_1, \cdots, \sigma_{k+l}$ *in* $CS_{\rm c.c.}(\mathbb{R}^d)$

$$\begin{aligned} & \left[\Phi^{\mathcal{R}}\left((\sigma_1 \otimes \cdots \otimes \sigma_k) \amalg (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})\right)\right]_{\rm sym} \\ = & \left[\Phi^{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k) \circledast \Phi^{\mathcal{R}}(\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})\right]_{\rm sym} \end{aligned} \tag{14.133}$$

where $\circledast$ *is as in (11.75) and the subscript* sym *stands for the symmetrised expression in the complex parameters* z_i*'s.*

Proof: By (14.130) we have

$$\begin{aligned} & \left[-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \left(\widetilde{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k)\right) \amalg \left(\widetilde{\mathcal{R}}(\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})(z_{k+1}, \cdots, z_{k+l})\right)\right]_{\rm sym} \\ = & \left[\left(-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \widetilde{\mathcal{R}}(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k)\right)\left(-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \widetilde{\mathcal{R}}(\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})(z_{k+1}, \cdots, z_{k+l})\right)\right]_{\rm sym}, \end{aligned}$$

from which we infer (14.133). □

Proposition 38 *Let* $\mathcal{R} : \sigma \mapsto \sigma(z)$ *be a holomorphic regularisation procedure on* $CS^{*,*}(U)$ *such that* $\sigma(z)$ *has affine order* $\alpha(z) = -q\, z + \alpha(0)$ *with* $q > 0$*.*
Given any symmetrised *renormalised evaluator* Λ *at zero, the map:*

$$\begin{aligned} \phi^{\mathcal{R},\Lambda} : \left(\mathcal{T}\left(CS_{\rm c.c}(\mathbb{R}^d)\right), \amalg\right) & \to \mathbb{C} \\ \sigma_1 \otimes \cdots \otimes \sigma_k & \mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{{\rm nested}\mathcal{R},\Lambda} \sigma_1 \otimes \cdots \otimes \sigma_k \end{aligned} \tag{14.134}$$

defines a character i.e. for any symbols $\sigma_1, \cdots, \sigma_{k+l}$ *in* $CS_{\rm c.c}(\mathbb{R}^d)$

$$-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R},\Lambda} (\sigma_1 \otimes \cdots \otimes \sigma_k) \amalg (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}) = \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R},\Lambda} \sigma_1 \otimes \cdots \otimes \sigma_k\right) \cdot \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\mathcal{R},\Lambda} \sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}\right). \tag{14.135}$$

Proof: This follows from applying the evaluator Λ on either side of (14.133) using the fact that

$$\Lambda(f) = \Lambda(f_{\rm sym}) \quad \forall f \in \mathcal{LM}_0(\mathbb{C}^\infty)$$

since Λ is symmetric. □

14.5 Nested integrals renormalised via Birkhoff factorisation

An alternative method to renormalise is to consider the map $\delta^* \circ \Phi^{\mathcal{R}}$. The following statement is a straightforward corollary of Lemma 14.

Proposition 39 *Let* $\mathcal{R} : \sigma \mapsto \sigma(z)$ *be a holomorphic regularisation procedure on* $CS_{\rm c.c}(\mathbb{R}^d)$ *such that* $\sigma(z)$ *has affine order* $\alpha(z) = -q\, z + \alpha(0)$ *with* $q > 0$*.*
The map

$$\begin{aligned} \delta^* \circ \Phi^{\mathcal{R}} : \left(\mathcal{T}\left(CS_{\rm c.c}(\mathbb{R}^d)\right), \amalg\right) & \to ({\rm Mer}_0(\mathbb{C}^\infty), \cdot) \\ \sigma_1 \otimes \cdots \otimes \sigma_k & \mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}^{\rm nested} \left(\delta^* \circ \widetilde{\mathcal{R}}\right)(\sigma_1 \otimes \cdots \otimes \sigma_k)(z) \end{aligned} \tag{14.136}$$

is an algebra morphism, i.e. for any symbols $\sigma_1, \cdots, \sigma_{k+l}$ *in* $CS_{\rm c.c.}(\mathbb{R}^d)$

$$\begin{aligned} & \left(\delta^* \circ \Phi^{\mathcal{R}}\right)\left((\sigma_1 \otimes \cdots \otimes \sigma_k) \amalg (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})\right) \\ = & \left(\delta^* \circ \Phi^{\mathcal{R}}\right)(\sigma_1 \otimes \cdots \otimes \sigma_k) \cdot \left(\delta^* \circ \Phi^{\mathcal{R}}\right)(\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}), \end{aligned} \tag{14.137}$$

where $\cdot$ *stands for the ordinary product of functions.*

We know that the tensor algebra $\left(\mathcal{T}(CS_{\rm c.c}(\mathbb{R}^d)), ш\right)$ equipped with the deconcatenation coproduct:

$$\Delta\left(\sigma_1\otimes\cdots\otimes\sigma_k\right) := \sum_{j=0}^{k}\left(\sigma_1\otimes\cdots\otimes\sigma_j\right)\bigotimes\left(\sigma_{j+1}\otimes\cdots\otimes\sigma_k\right)$$

inherits a structure of connected graded commutative Hopf algebra [H1]. Using the convolution product $*$ associated with the product ш and coproduct Δ on $\mathcal{T}(CS_{\rm c.c}(\mathbb{R}^d))$, we can implement a Birkhoff factorization to the map $\left(\delta^*\circ\Phi^{\mathcal{R}}\right)$ as in the Connes and Kreimer setup ([CK], [Ma])

$$\left(\delta^*\circ\Phi^{\mathcal{R}}\right) = \left(\delta^*\circ\Phi^{\mathcal{R}}\right)_+ * \left(\delta^*\circ\Phi^{\mathcal{R}}\right)_-$$

associated with the minimal substraction scheme to build a character

$$\left(\delta^*\circ\Phi^{\mathcal{R}}\right)_+(0) : \left(\mathcal{T}\left(CS_{\rm c.c}(\mathbb{R}^d)\right), ш\right)\to\mathbb{C}.$$

Proposition 40 *[MP2] Let $\mathcal{R}$ be a holomorphic regularisation which sends a symbol σ to a symbol $\sigma(z)$ with order $\alpha(z)=\alpha(0)-q\,z$ for some positive real number q. The map*

$$\begin{array}{rcl}\phi^{\mathcal{R},\rm Birk} : \left(\mathcal{T}\left(CS_{\rm c.c}(\mathbb{R}^d)\right), ш\right) & \to & \mathbb{C}\\ \sigma_1\otimes\cdots\otimes\sigma_k & \mapsto & \left(\delta^*\circ\Phi^{\mathcal{R}}\right)_+(0)(\sigma_1\otimes\cdots\otimes\sigma_k)\end{array}$$

defines a character. In other words, $\phi^{\mathcal{R},\rm Birk}$ satisfies shuffle relations:

$$\begin{array}{rl} & \phi^{\mathcal{R},\rm Birk}\left((\sigma_1\otimes\cdots\otimes\sigma_k)\,ш\,(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l})\right)\\ = & \phi^{\mathcal{R},\rm Birk}\left(\sigma_1\otimes\cdots\otimes\sigma_k\right)\phi^{\mathcal{R},\rm Birk}\left(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l}\right)\end{array}$$

This yields an alternative set of renormalised nested integrals of symbols

$$⨍_{\mathbb{R}^d}^{\rm nested,\mathcal{R},Birk}\sigma_1\otimes\cdots\otimes\sigma_k := \phi^{\mathcal{R},\rm Birk}(\sigma_1\otimes\cdots\otimes\sigma_k)$$

which obey stuffle relations:

$$⨍_{\mathbb{R}^d}^{\mathcal{R},\rm Birk}(\sigma_1\otimes\cdots\otimes\sigma_k)\,ш\,(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l}) = \left(⨍_{\mathbb{R}^d}^{\mathcal{R},\rm Birk}\sigma_1\otimes\cdots\otimes\sigma_k\right)\cdot\left(⨍_{\mathbb{R}^d}^{\mathcal{R},\rm Birk}\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l}\right). \tag{14.138}$$

15 Renormalised nested sums of symbols

This section which is based on joint work with Dominique Manchon [MP2], closely follows the pattern of the previous section. Here, we renormalise nested discrete sums instead of the nested integrals which were the object of study of the previous section.
We restrict to dimension $d = 1$. For any complex number a and any non negative integer k the notation $\mathcal{P}^{a,k}$ stands for *positively supported* log-polyhomogeneous symbols of log-type, i.e. symbols in $CS_{c.c}^{a,k}(\mathbb{R})$ with support in $]0, +\infty[$. We keep *mutatis mutandis* the notations of subsection 1.1; in particular $\mathcal{P}^{*,0}$ is a subalgebra of the filtered algebra $\mathcal{P}^{*,*}$.

15.1 A Rota-Baxter operator

The operator P defined on sequences $\sigma : \mathbb{N} \to \mathbb{C}$ by:

$$P(f)(n) = \sum_{k=0}^{n} f(k) \tag{15.139}$$

satisfies the Rota-Baxter relation with weight 1. Similarly, the operator $Q = P - I$ which acts on sequences $f : \mathbb{N} \to \mathbb{C}$ by:

$$Q(f)(n) = \sum_{k=0}^{n-1} f(k) \tag{15.140}$$

satisfies the Rota-Baxter relation with weight -1.
The Rota-Baxter operators P defined in (15.139) and $\widetilde{P}$ defined in (12.86) relate by means of the Euler-MacLaurin formula (6.40) which compares discrete sums with integrals and provides an interpolation of $P(\sigma)$ for some symbol σ by a symbol $\widetilde{P}(\sigma)$.

Proposition 41 *[MP2] For any $\sigma \in \mathcal{P}^{a,k}$, the discrete sum $P(\sigma)$ can be interpolated by a symbol $\overline{P}(\sigma)$ in $\mathcal{P}^{a+1,k+1} + \mathcal{P}^{0,k+1}$ (i.e. $\overline{P}(\sigma)(n) = P(\sigma)(n) = \sum_{k=0}^{n} \sigma(k) \quad \forall n \in \mathbb{N}$) such that*

$$\overline{P}(\sigma) - \widetilde{P}(\sigma) \in \mathcal{P}^{a,k}.$$

The operator $\overline{Q} := \overline{P} - I : \mathcal{P}^{a,k} \to \mathcal{P}^{a+1,k+1} + \mathcal{P}^{0,k+1}$ interpolates Q.

Proof: By the Euler-MacLaurin formula (6.40) we have for a positive integer n

$$\begin{aligned} P(\sigma)(n) &= \frac{\sigma(n)+\sigma(0)}{2} + \int_1^n \sigma(x)\,dx + \sum_{k=2}^{K} (-1)^k \frac{B_k}{k!} \left(\sigma^{(k-1)}(n) - \sigma^{(k-1)}(0)\right) \\ &+ \frac{(-1)^{K-1}}{K!} \int_0^n \overline{B_K}(x)\,\sigma^{(K)}(x)\,dx. \end{aligned} \tag{15.141}$$

For any positive η, the expression

$$\begin{aligned} \overline{P}(\sigma)(\eta) &= \frac{\sigma(\eta)+\sigma(0)}{2} + \int_1^\eta \sigma(x)\,dx + \sum_{k=2}^{K} (-1)^k \frac{B_k}{k!} \left(\sigma^{(k-1)}(\eta) - \sigma^{(k-1)}(0)\right) \\ &+ \frac{(-1)^{K-1}}{K!} \int_0^\eta \overline{B_K}(x)\,\sigma^{(K)}(x)\,dx \end{aligned} \tag{15.142}$$

defines a symbol which interpolates $P(\sigma)$. More precisely, the sum $\frac{1}{2}\sigma(\eta) + \sum_{j=2}^{2K} \frac{B_j}{j!}\sigma^{(j-1)}(\eta)$ lies in $\mathcal{P}^{\alpha,k}$, whereas the integral $\tilde{P}(\sigma)$ lies in $\mathcal{P}^{\alpha+1,k+1} + \mathcal{P}^{0,k+1}$. The result then follows from splitting the integral remainder term into $\int_0^{+\infty}(...) - \int_\eta^{+\infty}(...)$: the first term in the sum is a constant for large enough K, and the second term is a symbol (with respect to the variable η) with order $a - (2K+1)$ whose real part is arbitrarily small as K grows, which lies in $\mathcal{P}^{a,k}$. $\square$

15.2 Stuffle relations

We recall the definition of a stuffle Hopf algebra, following the presentation in [MP2].

Definition 20 *Let $k, l, r \in \mathbb{N}$ with $k+l-r>0$. A (k,l)-*quasi-shuffle *of type r is a surjective map π from $\{1,\ldots,k+l\}$ onto $\{1,\ldots,k+l-r\}$ such that $\pi(1)<\cdots<\pi(k)$ and $\pi(k+1)<\cdots<\pi(k+l)$. Let us denote by* $\mathrm{mixsh}(k,l;r)$ *the set of (k,l)-quasi-shuffles of type r. The elements of* $\mathrm{mixsh}(k,l;0)$ *are the ordinary (k,l)-shuffles. Quasi-shuffles are also called* mixable shuffles *or* stuffles. *We denote by* $\mathrm{mixsh}(k,l)$ *the set of (k,l)-quasi-shuffles (of any type).*

Let $\mathcal{A}$ be a commutative (non necessarily unital) algebra equipped with a product $\bullet$ and let $\star_\bullet$ be the product on $\mathcal{T}(\mathcal{A})$ defined by:

$$(v_1\otimes\cdots\otimes v_k)\star_\bullet(v_{k+1}\otimes\cdots\otimes v_{k+l})=\sum_{\pi\in\mathrm{mixsh}(k,l)} w_1^\pi\otimes\cdots\otimes w_{k+l-r}^\pi,$$

with :

$$w_j^\pi=\prod_{i\in\{1,\ldots,k+l\},\pi(i)=j} v_i,$$

where the product above given by the product $\bullet$ of $\mathcal{A}$, contains only one or two terms.
For $k=2$ this reads:

$$v_1\star_\bullet v_2=v_1 ш\, v_2+v_1\bullet v_2.$$

Notation: In the following we set $\star_+$ when the product $\bullet$ is the ordinary product i.e. $v_1\bullet v_2=v_1v_2$, and $\star_-$ when it is the opposite of the product i.e. $v_1\bullet v_2=-v_1v_2$.

Theorem 15 (M. Hoffman, [H] Theorems 3.1 and 3.3)

- $\big(\mathcal{T}(\mathcal{A}),\star_\bullet,\Delta\big)$ *is a commutative connected filtered Hopf algebra.*
- *There is an isomorphism of Hopf algebras :*

$$\exp:\big(T(\mathcal{A}), ш\,,\Delta\big)\xrightarrow{\sim}\big(T(\mathcal{A}),\star_\bullet,\Delta\big).$$

In [H2] M. Hoffman gives a detailed proof in a slightly more restricted context, which can be easily adapted in full generality (see also [EG]). Hoffman's isomorphism is built explicitly as follows: let $\mathcal{P}(n)$ be the set of compositions of the integer n, i.e. the set of sequences $I=(i_1,\ldots,i_k)$ of positive integers such that $i_1+\cdots+i_k=n$. For any $u=v_1\otimes\cdots\otimes v_n\in T(\mathcal{A})$ and any composition $I=(i_1,\ldots,i_k)$ of n we set:

$$I[u]:=(v_1\bullet\cdots\bullet v_{i_1})\otimes(v_{i_1+1}\bullet\cdots\bullet v_{i_1+i_2})\otimes\cdots\otimes(v_{i_1+\cdots+i_{k-1}+1}\bullet\cdots\bullet v_n).$$

We then further define:

$$\exp u=\sum_{I=(i_1,\cdots,i_k)\in\mathcal{P}(n)}\frac{1}{i_1!\cdots i_k!}I[u].$$

Moreover ([H2], Lemma 2.4), the inverse log of exp is given by :

$$\log u=\sum_{I=(i_1,\cdots,i_k)\in\mathcal{P}(n)}\frac{(-1)^{n-k}}{i_1\cdots i_k}I[u].$$

For example for $v_1,v_2,v_3\in\mathcal{A}$ we have :

$$\begin{aligned}
\exp v_1 &= v_1 \quad , \quad \log v_1=v_1,\\
\exp(v_1\otimes v_2) &= v_1\otimes v_2+\frac12 v_1\bullet v_2 \quad , \quad \log(v_1\otimes v_2)=v_1\otimes v_2-\frac12 v_1\bullet v_2,\\
\exp(v_1\otimes v_2\otimes v_3) &= v_1\otimes v_2\otimes v_3 + \frac12(v_1\bullet v_2\otimes v_3+v_1\otimes v_2\bullet v_3)+\frac16 v_1\bullet v_2\bullet v_3,\\
\log(v_1\otimes v_2\otimes v_3) &= v_1\otimes v_2\otimes v_3 - \frac12(v_1\bullet v_2\otimes v_3+v_1\otimes v_2\bullet v_3)+\frac13 v_1\bullet v_2\bullet v_3.
\end{aligned}$$

15.3 Nested sums of non integer order symbols

By Proposition 41, given a symbol σ in $\mathcal{P}^{a,k}$, the interpolating symbol $\overline{P}(\sigma)$ lies in $\mathcal{P}^{a+1,k+1}+\mathcal{P}^{0,k+1}$. It follows that the discrete sum $P(\sigma)(N)=\widetilde{P}(\sigma)(N)$ has an asymptotic behaviour for large N given by finite linear combinations of expressions of the type (12.80) with k replaced by $k+1$ and a by $a+1$ or 0.
Picking the finite part, for any $\sigma\in\mathcal{P}^{*,*}$ we define the following cut-off sum:

$$-\!\!\!\!\!\!\sum_0^\infty \sigma := \mathrm{fp}_{N\to\infty} P(\sigma)(N) = \mathrm{fp}_{N\to\infty}\sum_{k=0}^N \sigma(k), \tag{15.143}$$

which extends the ordinary discrete sum $\sum_0^\infty$ defined on L^1-symbols. If σ has non integer order, we have $-\!\!\!\!\!\sum_0^\infty \sigma = \mathrm{fp}_{N\to\infty}\sum_{k=0}^{N+K}\sigma(k)$ for any integer K, so that $-\!\!\!\!\!\sum_0^\infty \sigma = \mathrm{fp}_{N\to\infty} Q(\sigma)(N)$.
With the help of the interpolation map described in Proposition 41, we can assign to a tensor product $\sigma:=\sigma_1\otimes\cdots\otimes\sigma_k$ of (positively supported) classical symbols, two log-polyhomogeneous symbols defined inductively in the degree k of the tensor product, which interpolate the nested iterated sum

$$\sum_{0\le n_k\le n_{k-1}\le\cdots\le n_2\le n_1} \sigma_1(n_1)\cdots\sigma_k(n_k) = \sigma_1\, P\Big(\cdots\sigma_{k-2}\, P\big(\sigma_{k-1}\, P(\sigma_k)\big)...\Big),$$

$$\sum_{0\le n_k< n_{k-1}<\cdots< n_2< n_1} \sigma_1(n_1)\cdots\sigma_k(n_k) = \sigma_1\, Q\Big(\cdots\sigma_{k-2}\, Q\big(\sigma_{k-1}\, P(\sigma_k)\big)...\Big).$$

Theorem 16 *[MP2] Given $\sigma_i\in\mathcal{P}^{a_i,0}$, $a_i\in\mathbb{C}$, $i=1,\dots k$, setting $\sigma:=\sigma_1\otimes\cdots\otimes\sigma_k$, the functions $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ defined by:*

$$\widetilde{\sigma} := \sigma_1\,\overline{P}\Big(\cdots\sigma_{k-2}\,\overline{P}\big(\sigma_{k-1}\,\overline{P}(\sigma_k)\big)...\Big);\quad \widetilde{\sigma}' := \sigma_1\,\overline{Q}\Big(\cdots\sigma_{k-2}\,\overline{Q}\big(\sigma_{k-1}\,\overline{Q}(\sigma_k)\big)...\Big) \tag{15.144}$$

which interpolate nested sums in the following way:

$$\begin{aligned}\widetilde{\sigma}(n_1) &= \sum_{0\le n_k\le n_{k-1}\le\cdots\le n_2\le n_1}\sigma_1(n_1)\cdots\sigma_k(n_k)\quad \forall n_1\in\mathbb{N},\\ \widetilde{\sigma}'(n_1) &= \sum_{0\le n_k< n_{k-1}<\cdots< n_2< n_1}\sigma_1(n_1)\cdots\sigma_k(n_k)\quad \forall n_1\in\mathbb{N},\end{aligned}$$

both lie in $\mathcal{P}^{*,k-1}$ as linear combinations of (positively supported) symbols in $\mathcal{P}^{a_1+\cdots+a_j+j-1,j-1}$, $j\in\{1,\dots,k\}$.

On the grounds of this result, we define the cut-off nested discrete sum of a tensor product of (positively supported) classical symbols.

Definition 21 *For $\sigma_1,\dots,\sigma_k\in\mathcal{P}^{*,0}$ and $\sigma:=\sigma_1\otimes\cdots\otimes\sigma_k$ we call*

$$-\!\!\!\!\!\!\sum_{\le}^{\text{nested}}\sigma := -\!\!\!\!\!\!\sum_{n\in\mathbb{N}}\widetilde{\sigma}(n) = \mathrm{fp}_{N\to\infty}\sum_{\le}^{\text{nested},N}\sigma_1\otimes\cdots\otimes\sigma_k,$$

the cut-off nested sum of $\sigma_1\otimes\cdots\otimes\sigma_k$, where, for any positive integer N we have set

$$\sum_{\le}^{\text{nested},N}\sigma_1\otimes\cdots\otimes\sigma_k := \sum_{0<n_k\le\cdots\le n_1\le N}\sigma_1(n_1)\cdots\sigma_k(n_k) = P\left(\widetilde{\sigma}\right)(N),$$

with the notations of (15.144).
The strict inequality version is defined by:

$$-\!\!\!\!\!\!\sum_{<}^{\text{nested}}\sigma := -\!\!\!\!\!\!\sum_{n\in\mathbb{N}}\widetilde{\sigma}'(n) = \mathrm{fp}_{N\to\infty}-\!\!\!\!\!\!\sum_{<}^{\text{nested},N}\sigma_1\otimes\cdots\otimes\sigma_k,$$

where, for any positive integer N we further set:

$$\sum_{<}^{\text{nested},N} \sigma_1 \otimes \cdots \otimes \sigma_k := \sum_{0<n_k<\cdots<n_1<N} \sigma_1(n_1)\cdots\sigma_k(n_k) = Q\left(\widetilde{\sigma}'\right)(N)$$

with the notations of (15.144).

Proposition 42 *Given symbols $\sigma_i \in CS_{\text{c.c}}(\mathbb{R}^d)$ with order a_i, the index i varying from 1 to $k+l$ and provided* [11]

$$a_{\gamma(1)} + \cdots + a_{\gamma(i)} + i \notin \mathbb{N}_0, \quad \forall \gamma \in \Sigma_{k+l}, \quad \forall i \in \{1, \cdots, k+l\},$$

we have:

$$\begin{aligned} & \operatorname{-\!\!\!\!\sum}_{\leq}^{\text{nested}} (\sigma_1 \otimes \cdots \otimes \sigma_k) \star_- (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}) \\ = & \left(\operatorname{-\!\!\!\!\sum}_{\leq}^{\text{nested}} \sigma_1 \otimes \cdots \otimes \sigma_k \right) \left(\operatorname{-\!\!\!\!\sum}_{\leq}^{\text{nested}} \sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l} \right). \end{aligned} \tag{15.145}$$

Similarly,

$$\begin{aligned} & \operatorname{-\!\!\!\!\sum}_{<}^{\text{nested}} (\sigma_1 \otimes \cdots \otimes \sigma_k) \star_+ (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}) \\ = & \left(\operatorname{-\!\!\!\!\sum}_{<}^{\text{nested}} \sigma_1 \otimes \cdots \otimes \sigma_k \right) \left(\operatorname{-\!\!\!\!\sum}_{<}^{\text{nested}} \sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l} \right). \end{aligned} \tag{15.146}$$

Proof:

1. We first observe that stuffle relations hold for finite nested sums. Let us prove this statement for the weak inequality case. We want to show that

$$\begin{aligned} & \sum_{\leq}^{\text{nested},N} (\sigma_1 \otimes \cdots \otimes \sigma_k) \star_- (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}) \\ = & \left(\sum_{\leq}^{\text{nested},N} \sigma_1 \otimes \cdots \otimes \sigma_k \right) \left(\sum_{\leq}^{\text{nested},N} \sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l} \right), \end{aligned} \tag{15.147}$$

   To do so, we partition the domain:

$$P_{k,l} := \{n_1 > \cdots > n_k > 0\} \times \{n_{k+1} > \cdots > n_{k+l} > 0\} \subset (\mathbb{N} - \{0\})^{k+l}$$

   into:

$$P_{k,l} = \coprod_{\pi \in \text{mixsh}(k,l)} P_\pi,$$

   where the domain P_π is defined by:

$$P_\pi = \{(n_1, \ldots, n_{k+l}) \,/\, n_{\pi_m} > n_{\pi_p} \text{ if } m > p \text{ and } \pi_m \neq \pi_p, \text{ and } n_m = n_p \text{ if } \pi_m = \pi_p\}.$$

   As we must replace strict inequalities by large ones, let us consider the "closures"

$$\overline{P_\pi} := \{(n_1, \ldots, n_{k+l}) \,/\, n_{\pi_m} \geq n_{\pi_p} \text{ if } m \geq p \text{ and } n_m = n_p \text{ if } \pi_m = \pi_p\}.$$

[11] This holds in particular when all the partial sums of the orders a_i's are not integer valued or $\text{Re}(a_i) < -1$ for any $i \in \{1, \cdots, k\}$.

which then overlap. By the inclusion-exclusion principle we have:

$$\overline{P_{k,l}} = \coprod_{0\le r\le \min(k,l)} (-1)^r \coprod_{\pi\in \mathrm{mixsh}(k,l;r)} \overline{P_\pi}, \tag{15.148}$$

where we have set:

$$\overline{P_{k,l}} := \{n_1 \geq \cdots \geq n_k > 0\} \times \{n_{k+1} \geq \cdots \geq n_{k+l} > 0\} \subset (\mathbb{N} - \{0\})^{k+l}$$

Each term in equation (15.148) must be added if r is even, and removed if r is odd. Considering the summation of $\sigma_1\otimes\cdots\otimes\sigma_{k+l}$ over each $\overline{P_\pi}$, this decomposition immediately yields the equality:

$$\left(\sum_{0\le n_k\le\cdots\le n_1\le N} \sigma_1(n_1)\cdots\sigma_i(n_k)\right)\left(\sum_{0\le n_{k+l}\le\cdots\le n_{k+1}\le N} \sigma_{k+1}(n_{k+1})\cdots\sigma_{k+l}(n_{k+l})\right)$$
$$= \sum_{\le}^{N,\text{nested}} \sum_{\pi\in\mathrm{mixsh}(k,l)} f^\pi, \tag{15.149}$$

where $\sigma^\pi = \sigma_1^\pi \otimes \cdots \otimes \sigma_{k+l-r}^\pi$ is the tensor product defined by:

$$\sigma_j^\pi = \bullet_{i\in\{1,\ldots,k+l\},\, \pi(i)=j}\sigma_i.$$

The stuffle relations (15.147) are then a re-writing of equality (15.149) using the commutative algebra $(V, \bullet)$. Taking the limit as $N\to\infty$ provides the second statement of the theorem. The proof is similar for the strict inequality case, using the domains P_π rather than the "closures" $\overline{P_\pi}$. As there are no overlaps the signs disappear in the formula (15.148).

2. Stuffle relations for the cut-off nested sums are obtained by taking the finite part in (15.147) as $N\to\infty$. Since

$$\sum_{\le}^{\text{nested},N} \sigma_1\otimes\cdots\otimes\sigma_k = P(\tilde\sigma)(N)$$

is interpolated by a linear comination of symbols in $\mathcal{P}^{a_1+\cdots+a_i+i-1,i-1}$, $i\in\{1,\ldots,k\}$ and $\sum_{\le}^{\text{nested},N}\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l}$ is interpolated in a similar manner by a linear combination of symbols in $\mathcal{P}^{a_{k+1}+\cdots+a_{k+j}+j-1,j-1}$, $j\in\{1,\ldots,l\}$, the asymptotics of the r.h.s of (15.147) as $N\to\infty$ involve powers $N^{a_1+\cdots+a_i+a_{k+1}+\cdots+a_{k+j}+i+j-m}$ with $m\in\mathbb{N}_0$. Coefficients of such powers of N in the expansion can only contribute to the finite part when $a_1+\cdots+a_i+a_{k+1}+\cdots+a_{k+j}+i+j\in\mathbb{N}_0$. In all other cases we have

$$\mathrm{fp}_{N\to\infty}\left(\left(\sum_{\le}^{\text{nested},N}\sigma_1\otimes\cdots\otimes\sigma_k\right)\left(\sum_{\le}^{\text{nested},N}\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l}\right)\right)$$
$$= \mathrm{fp}_{N\to\infty}\left(\sum_{\le}^{\text{nested},N}\sigma_1\otimes\cdots\otimes\sigma_k\right)\mathrm{fp}_{N\to\infty}\left(\sum_{\le}^{\text{nested},N}\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l}\right).$$

The stuffle relations (15.145) then follow from (15.147) by taking the cut-off limit as $N\to\infty$ on either side.
A similar reasoning yields (15.146).

□

15.4 Nested sums of holomorphic symbols

The results derived in Theorem 10 extend to log-polyhomogeneous symbols. We state this generalisaiton in the one dimensional context needed here, but it also holds in higher dimensions. We provide a proof which although similar in the spirit of the one of Theorem 10 since it uses the Euler-Maclaurin formula, is simpler because we are in dimension 1.

Proposition 43 *Given a holomorphic regularisation $\mathcal{R} : \sigma \longmapsto \sigma(z)$ on $\mathcal{P}^{*,k}$, for any $\sigma \in \mathcal{P}^{*,k}$, the map*

$$z \mapsto -\!\!\!\!\!\!\int_{\mathbb{R}} \sigma(z) - \sum \sigma(z)$$

is holomorphic for any $\sigma \in \mathcal{P}^{*,k}$.
Consequently, the map $z \mapsto \sum \sigma(z)$ is meromorphic with the same poles (of order $\leq k+1$) as the map $z \mapsto -\!\!\!\!\!\int_{\mathbb{R}} \sigma(z)$. These poles lie in the discrete set $\alpha^{-1}(\{-1, 0, 1, 2, \cdots\})$ whenever $\sigma(z)$ is a holomorphic family of order $\alpha(z)$.

Proof: Let $\sigma \mapsto \sigma(z)$ be a holomorphic perturbation in $\mathcal{P}^{*,k}$. By Proposition 41, the difference

$$-\!\!\!\!\!\!\int_{\mathbb{R}} \sigma(z) - \sum \sigma(z) = \mathrm{fp}_{N\to\infty} \left(\tilde{P}(\sigma(z))(N) - \overline{P}(\sigma(z)) \right)(N)$$

is a holomorphic expression since $\widetilde{P}(\sigma(z)) - \overline{P}(\sigma(z))$ is a holomorphic symbol. On the other hand, the map $z \mapsto -\!\!\!\!\!\int \sigma(z) = \mathrm{fp}_{N\to\infty} \tilde{P}(\sigma(z))(N)$ is meromorphic with poles of order $\leq k+1$ in the discrete set $\alpha^{-1}(\{-1, 0, 1, 2, \cdots\})$ where $\alpha(z)$ stands for the order of $\sigma(z)$. Thus, the same property holds for $z \mapsto \sum \sigma(z)$. $\square$

Let $\mathcal{R}$ be a holomorphic regularisation on $CS_{\mathrm{c.c}}(\mathbb{R}^d)$, and the associated holomorphic regularisation $\delta^* \circ \tilde{\mathcal{R}}$ (see (14.124)) on the tensor algebra $\mathcal{T}(CS_{\mathrm{c.c}}(\mathbb{R}^d))$ which we saw was compatible with the shuffle product. Let us twist $\delta^* \circ \tilde{\mathcal{R}}$ by Hoffman's isomorphism to build another holomorphic regularisation [MP2] [12]

$$\left(\delta^* \circ \tilde{\mathcal{R}}\right)^\star := \exp \circ \left(\delta^\star \circ \tilde{\mathcal{R}}\right) \circ \log,$$

on the tensor algebra $\mathcal{T}(CS_{\mathrm{c.c}}(\mathbb{R}^d))$, which is compatible with the stuffle product:

$$\left(\delta^* \circ \tilde{\mathcal{R}}\right)^\star (\sigma \star_\bullet \tau) = \left(\delta^* \circ \tilde{\mathcal{R}}\right)^\star (\sigma) \star_\bullet \left(\delta^* \circ \tilde{\mathcal{R}}\right)^\star (\tau) \quad \forall \sigma, \tau \in \mathcal{T}(\mathcal{A}), \tag{15.150}$$

where $\bullet$ stands here for the ordinary product $\cdot$ or the opposite of the ordinary product.
The following induced regularisation

$$\widetilde{\mathcal{R}}^*(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k) = \exp \circ \widetilde{\mathcal{R}} \circ \log(\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k) \tag{15.151}$$

is therefore compatible with stuffle relations after symmetrization[13] in the complex variables z_i :

$$\left(\widetilde{\mathcal{R}}^\star(\sigma \star_\bullet \tau)\right)_{\mathrm{sym}} = \left(\widetilde{\mathcal{R}}^\star(\sigma) \star_\bullet \widetilde{\mathcal{R}}^\star(\tau)\right)_{\mathrm{sym}} \quad \forall \sigma, \tau \in \mathcal{T}(\mathcal{A}), \tag{15.152}$$

where the subscript sym stands for symmetrisation over all the complex variables $z_1, \cdots, z_{k+l}$ if σ is a tensor of degree k and τ a tensor of degree l.

Remark 17 *Note that e.g. when $k = 2$*

$$\begin{aligned}
\tilde{\mathcal{R}}^\star(\sigma_1 \star_\bullet \sigma_2)(z_1, z_2) &= \tilde{\mathcal{R}}(\sigma_1 \amalg \sigma_2)(z_1, z_2) - \frac{1}{2}\mathcal{R}(\sigma_1 \bullet \sigma_2)(z_1) + \frac{1}{2}\mathcal{R}(\sigma_1)(z_1) \bullet \mathcal{R}(\sigma_1)(z_1) \\
&= \mathcal{R}(\sigma_1)(z_1) \otimes \mathcal{R}(\sigma_2)(z_2) + \mathcal{R}(\sigma_2)(z_1) \otimes \mathcal{R}(\sigma_1)(z_2) \\
&- \frac{1}{2}\mathcal{R}(\sigma_1 \bullet \sigma_2)(z_1) + \frac{1}{2}\mathcal{R}(\sigma_1)(z_1) \bullet \mathcal{R}(\sigma_1)(z_1) \\
&= \left(\tilde{\mathcal{R}}(\sigma_1) \star_\bullet \mathcal{R}^\star(\sigma_2)\right)(z_1, z_2)
\end{aligned}$$

whereas

$$\tilde{\mathcal{R}}^\star(\sigma_1 \star_\bullet \sigma_2)(z_1, z_2) \neq \mathcal{R}(\sigma_1)(z_1) \star_\bullet \mathcal{R}(\sigma_2)(z_2)$$

and

$$\left(\tilde{\mathcal{R}}^\star(\sigma_1 \star_\bullet \sigma_2)(z_1, z_2)\right)_{\mathrm{sym}} = \left(\mathcal{R}(\sigma_1)(z_1) \star_\bullet \mathcal{R}(\sigma_2)(z_2)\right)_{\mathrm{sym}}.$$

[12] Our notations slightly differ from those of [MP2] where $\mathcal{R}^*$ stands for $(\delta^* \circ \mathcal{R})^*$.
[13] This later compels us to choosing a symmetrised evaluator at zero.

Setting $z_1 = \cdots = z_{k+l} = z$ in (15.152) yields back (15.150) so that (15.152) can be seen as a polarisation of (15.150).

Theorem 17 *Let $\mathcal{R} : \sigma \mapsto \sigma(z)$ be a holomorphic regularisation procedure on $CS_{\rm c.c}(\mathbb{R}^d)$ such that $\sigma(z)$ has affine order $\alpha(z) = -q\, z + \alpha(0)$ with $q > 0$.*
For any $\sigma_i \in CS_{\rm c.c}(\mathbb{R}^d), i = 1, \cdots, k$, with orders $a_i, i = 1, \cdots, k$,

1. *the maps*
$$(z_1, \cdots, z_k) \mapsto \sum_{\leq}^{\rm nested} \widetilde{\mathcal{R}}^* (\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k)$$
and
$$(z_1, \cdots, z_k) \mapsto \sum_{<}^{\rm nested} \widetilde{\mathcal{R}}^* (\sigma_1 \otimes \cdots \otimes \sigma_k)(z_1, \cdots, z_k)$$
are meromorphic functions in several variables with poles on a countable set of hyperplanes
$$z_{j_1} + \cdots + z_{j_i} = \frac{a_{j_1} + \cdots + a_{j_i} + i - n}{q}, \quad n \in \mathbb{N}_0 \tag{15.153}$$
of order i, amongst which the ones passing through zero:
$$z_{\tau(1)} + \cdots + z_{\tau(i)} = 0, \quad i \in \{1, \cdots, k\}, \tau \in \Sigma_k.$$

2. *The following identities of meromorphic functions hold for symbols $\sigma_{i+k} \in CS_{\rm c.c}(\mathbb{R}^d)$ of order a_{i+k} with i varying from 1 to l,*
$$\begin{aligned}&\sum_{\leq}^{\rm nested} (\sigma_1(z_1) \otimes \cdots \otimes \sigma_k(z_k)) \star_- (\sigma_{k+1}(z_{k+1}) \otimes \cdots \otimes \sigma_{k+l}(z_{k+l})) \\ = &\left(\sum_{\leq}^{\rm nested} \sigma_1(z_1) \otimes \cdots \otimes \sigma_k(z_k)\right) \left(\sum_{\leq}^{\rm nested} \sigma_{k+1}(z_{k+1}) \otimes \cdots \otimes \sigma_{k+l}(z_{k+l})\right),\end{aligned} \tag{15.154}$$
and similarly with strict inequalities
$$\begin{aligned}&\sum_{<}^{\rm nested} (\sigma_1(z_1) \otimes \cdots \otimes \sigma_k(z_k)) \star_- (\sigma_{k+1}(z_{k+1}) \otimes \cdots \otimes \sigma_{k+l}(z_{k+l})) \\ = &\left(\sum_{<}^{\rm nested} \sigma_1(z_1) \otimes \cdots \otimes \sigma_k(z_k)\right) \left(\sum_{\leq}^{\rm nested} \sigma_{k+1}(z_{k+1}) \otimes \cdots \otimes \sigma_{k+l}(z_{k+l})\right),\end{aligned} \tag{15.155}$$
where as before "sym" stands for symmetrisation in the complex variables.

3. *Provided*
$$a_{\gamma(1)} + \cdots + a_{\gamma(i)} + i \notin \mathbb{Z}_+, \quad \forall \gamma \in \Sigma_{k+l}, \quad \forall i \in \{1, \cdots, k+l\}$$
(in particular the partial sums of the orders are non integer) then (15.154) and (15.155) hold as identities of holomorphic functions in a neighborhood of zero, which when evaluated at zero, give back (15.145) and (15.146).

Proof: The proof goes as in the continuous summation case (see Theorem 14).

1. The nested cut-off sum
$$\sum_{\leq}^{\rm nested} \sigma_1(z_1) \otimes \cdots \otimes \sigma_k(z_k) = \sum \widetilde{\sigma}(z_1, \cdots, z_k),$$

where we have set

$$\widetilde{\sigma}(z_1,\cdots,z_k) := \sigma_1(z_1)\,\overline{P}\Big(\cdots\sigma_{k-2}(z_{k-2})\,\overline{P}\big(\sigma_{k-1}(z_{k-1})\,\overline{P}(\sigma_k(z_k))\big)...\Big),$$

is a linear combination of ordinary cut-off regularised sums of symbols in $\mathcal{P}^{\alpha_1(z_1)+\cdots+\alpha_i(z_i)+i-1,i-1}$ where $\alpha_i(z_i)$ is the order of $\sigma_i(z_i)$. Applying Proposition 43 yields the anounced pole structure.

2. Applying (15.145) to the symbols $\sigma_i(z_i) := \mathcal{R}(\sigma_i)(z_i)$ we have:

$$\begin{aligned}&\sum_{\leq}^{\text{nested}} (\sigma_1(z_1)\otimes\cdots\otimes\sigma_k(z_k)) \star_- (\sigma_{k+1}(z_{k+1})\otimes\cdots\otimes\sigma_{k+l}(z_{k+l}))\\ =\;& \left(\sum_{\leq}^{\text{nested}} \sigma_1(z_1)\otimes\cdots\otimes\sigma_k(z_k)\right)\left(\sum_{\leq}^{\text{nested}} \sigma_{k+1}(z_{k+1})\otimes\cdots\otimes\sigma_{k+l}(z_{k+l})\right),\end{aligned}$$

whenever $\alpha_{\gamma(1)}(z_{\gamma(1)}+\cdots+\alpha_{\gamma(i)}(z_{\gamma(i)}+i\notin\mathbb{N}_0 \quad \forall\gamma\in\Sigma_{k+l}, \quad \forall i\in\{1,\cdots,k+l\}$. By the first part of the proof, both sides of the equality are meromorphic maps with poles satisfying the above requirement so that this equality holds as an identity of meromorphic maps, which proves (15.154). The other identity (15.155) is proved similarly.

□

15.5 Nested sums of symbols renormalised via evaluators

Given a holomorphic regularisation $\mathcal{R}$ which takes a symbol σ to a symbol $\mathcal{R}(\sigma)(z)$ with holomorphic order $\alpha(z)=\alpha(0)-qz$ for some positive real number q, we infer from Theorem 17 that the maps

$$(z_1,\cdots,z_k)\mapsto\sigma_1\otimes\cdots\otimes\sigma_k\mapsto\sum_{\leq}^{\text{nested}}\widetilde{\mathcal{R}}^*(\sigma_1\otimes\cdots\otimes\sigma_k)(z_1,\cdots,z_k)$$

and

$$(z_1,\cdots,z_k)\mapsto\sigma_1\otimes\cdots\otimes\sigma_k\mapsto\sum_{<}^{\text{nested}}\widetilde{\mathcal{R}}^*(\sigma_1\otimes\cdots\otimes\sigma_k)(z_1,\cdots,z_k)$$

defined on the tensor algebra of positively supported classical symbols on $\mathbb{R}$, takes its values in the algebra $\mathcal{LM}_0(\mathbb{C}^\infty)$ (introduced in (11.77)) of meromorphic functions with linear poles at zero given by

$$z_{\tau(1)}+\cdots+z_{\tau(i)}=0 \quad \forall\tau\in\Sigma_k, \quad \forall i\in\{1,\cdots,k\}.$$

We set the following definition which extends, in the one dimensional case [14], the regularised discrete sums defined for classical symbols in (9.64).

Definition 22 *Let $\mathcal{R}:\sigma\mapsto\sigma(z)$ be a holomorphic regularisation procedure on $\mathcal{P}^{*,*}$ such that $\sigma(z)$ has affine order $\alpha(z)=-q\,z+\alpha(0)$ with $q>0$.*
Given a renormalised evaluator Λ at zero, we set for any $\sigma\in\mathcal{P}^{*,*}$

$$\sum^{\mathcal{R},\Lambda}\sigma := \Lambda\left(z\mapsto\sum_{\mathbb{R}^d}\mathcal{R}(\sigma)(z)\right)$$

and for any $\sigma_i\in\mathcal{P}^{*,0}, \quad i\in\{1,\cdots,k\}$

$$-\sum_{\leq}^{\text{nested},\mathcal{R},\Lambda}\sigma_1\otimes\cdots\otimes\sigma_k := \Lambda\left((z_1,\cdots,z_k)\mapsto\sum_{\leq}^{\text{nested}}\widetilde{\mathcal{R}}(\sigma_1\otimes\cdots\otimes\sigma_k)(z_1,\cdots,z_k)\right),$$

and similarly in the strict inequality case:

$$-\sum_{<}^{\text{nested},\mathcal{R},\Lambda}\sigma_1\otimes\cdots\otimes\sigma_k := \Lambda\left((z_1,\cdots,z_k)\mapsto\sum_{<}^{\text{nested}}\widetilde{\mathcal{R}}(\sigma_1\otimes\cdots\otimes\sigma_k)(z_1,\cdots,z_k)\right).$$

[14] This could be extended to any dimension.

As in the case of continuous sums, we build a character on the tensor algebra $\mathcal{T}\left(\mathcal{P}^{*,0}\right)$. We first need a technical result.

Lemma 14 *Let $\mathcal{R}:\sigma\mapsto\sigma(z)$ be a holomorphic regularisation procedure on $\mathcal{P}^{*,0}$ such that $\sigma(z)$ has affine order $\alpha(z)=-q\,z+\alpha(0)$ with $q>0$.*
The map

$$\begin{aligned}\Psi^{\mathcal{R}}:\mathcal{T}\left(\mathcal{P}^{*,0}\right) &\rightarrow \mathcal{LM}_0(\mathbb{C}^\infty)\\ \sigma_1\otimes\cdots\otimes\sigma_k &\mapsto \sum_{\leq}^{\text{nested}}\widetilde{\mathcal{R}}^*(\sigma_1\otimes\cdots\otimes\sigma_k)\end{aligned}\tag{15.156}$$

satisfies the following identity of meromorphic functions. For any symbols $\sigma_1,\cdots,\sigma_{k+l}$ in $CS_{\text{c.c.}}(\mathbb{R}^d)$

$$\begin{aligned}&\left[\Psi^{\mathcal{R}}\left((\sigma_1\otimes\cdots\otimes\sigma_k)\star_-(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l})\right)\right]_{\text{sym}}\\ =\;&\left[\Psi^{\mathcal{R}}(\sigma_1\otimes\cdots\otimes\sigma_k)\bullet\Psi^{\mathcal{R}}(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l})\right]_{\text{sym}},\end{aligned}\tag{15.157}$$

*where the symbol $\bullet$ was defined in (**??**) and where the subscript* sym *stands for the symmetrised expression in the complex parameters z_i's.*
Similarly, the map

$$\begin{aligned}\Psi'^{\mathcal{R}}:\mathcal{T}\left(\mathcal{P}^{*,0}\right) &\rightarrow \mathcal{LM}_0(\mathbb{C}^\infty)\\ \sigma_1\otimes\cdots\otimes\sigma_k &\mapsto \sum_{<}^{\text{nested}}\widetilde{\mathcal{R}}^*(\sigma_1\otimes\cdots\otimes\sigma_k)(\end{aligned}\tag{15.158}$$

satisfies the following identity of meromorphic functions. For any symbols $\sigma_1,\cdots,\sigma_{k+l}$ in $CS_{\text{c.c.}}(\mathbb{R}^d)$

$$\begin{aligned}&\left[\Psi'^{\mathcal{R}}\left((\sigma_1\otimes\cdots\otimes\sigma_k)\star(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l})\right)\right]_{\text{sym}}\\ =\;&\left[\Psi'^{\mathcal{R}}(\sigma_1\otimes\cdots\otimes\sigma_k)\circledast\Psi'^{\mathcal{R}}(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l})\right]_{\text{sym}}.\end{aligned}\tag{15.159}$$

Proof: By (15.154) we have

$$\begin{aligned}&\left[\sum_{<}^{\text{nested}}\left(\widetilde{\mathcal{R}}^*(\sigma_1\otimes\cdots\otimes\sigma_k)(z_1,\cdots,z_k)\right)\star_-\left(\widetilde{\mathcal{R}}^*(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l})(z_{k+1},\cdots,z_{k+l})\right)\right]_{\text{sym}}\\ =\;&\left[\left(\sum_{<}^{\text{nested}}\widetilde{\mathcal{R}}^*(\sigma_1\otimes\cdots\otimes\sigma_k)(z_1,\cdots,z_k)\right)\left(\sum_{<}^{\text{nested}}\widetilde{\mathcal{R}}^*(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l})(z_{k+1},\cdots,z_{k+l})\right)\right]_{\text{sym}}.\end{aligned}$$

from which we infer (15.157). A similar proof yields (15.159). □
The following statement yields renormalised nested sums which satisfy stuffle relations.

Proposition 44 *Let $\mathcal{R}:\sigma\mapsto\sigma(z)$ be a holomorphic regularisation procedure on $\mathcal{P}^{*,0}$ such that $\sigma(z)$ has affine order $\alpha(z)=-q\,z+\alpha(0)$ with $q>0$.*
Given any symmetrised *renormalised evaluator Λ at zero, the map:*

$$\begin{aligned}\psi^{\mathcal{R},\Lambda}:\left(\mathcal{T}\left(\mathcal{P}^{*,0}\right),\star_-\right) &\rightarrow \mathbb{C}\\ \sigma_1\otimes\cdots\otimes\sigma_k &\mapsto -\!\!\!\!\!\!\sum_{\leq}^{\text{nested}\,\mathcal{R},\Lambda}\sigma_1\otimes\cdots\otimes\sigma_k\end{aligned}\tag{15.160}$$

defines a character i.e. for any symbols $\sigma_1,\cdots,\sigma_{k+l}$ in $\mathcal{P}^{*,0}$

$$\begin{aligned}&-\!\!\!\!\!\!\sum_{\leq}^{\text{nested}\,\mathcal{R},\Lambda}(\sigma_1\otimes\cdots\otimes\sigma_k)\star_-(\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l})\\ =\;&\left(-\!\!\!\!\!\!\sum_{\leq}^{\text{nested}\,\mathcal{R},\Lambda}\sigma_1\otimes\cdots\otimes\sigma_k\right)\cdot\left(-\!\!\!\!\!\!\sum_{\leq}^{\text{nested}\,\mathcal{R},\Lambda}\sigma_{k+1}\otimes\cdots\otimes\sigma_{k+l}\right),\end{aligned}$$

with $\sum_{\leq}^{\text{nested}\,\mathcal{R},\Lambda}$ replaced by $\sum_{<}^{\text{nested}\,\mathcal{R},\Lambda}$ and $\star_-$ by $\star_+$.

Proof: This follows from applying the evaluator Λ on either side of (15.157) and (15.159) using the fact that

$$\Lambda(f) = \Lambda(f_{\rm sym}) \quad \forall f \in \mathcal{LM}_0(\mathbb{C}^\infty)$$

since Λ is symmetric. $\square$

15.6 Nested sums renormalised via Birkhoff factorisation

As in the case of continuous sums, an alternative method to renormalise is to consider the map $\delta^* \circ \Psi^{\mathcal{R}}$. The following statement is a straightforward corollary of Lemma 14.

Proposition 45 *Let $\mathcal{R} : \sigma \mapsto \sigma(z)$ be a holomorphic regularisation procedure on $CS_{\rm c.c}(\mathbb{R}^d)$ such that $\sigma(z)$ has affine order $\alpha(z) = -q\,z + \alpha(0)$ with $q > 0$.*
The map

$$\begin{aligned} \delta^* \circ \Psi^{\mathcal{R}} : \left(\mathcal{T}\left(\mathcal{P}^{*,0}\right), \star_-\right) &\rightarrow (\mathrm{Mer}_0(\mathbb{C}^\infty), \cdot) \\ \sigma_1 \otimes \cdots \otimes \sigma_k &\mapsto \sum_{\leq}^{\rm nested} \left(\delta^* \circ \widetilde{\mathcal{R}}^*\right)(\sigma_1 \otimes \cdots \otimes \sigma_k)(z) \end{aligned} \tag{15.161}$$

and

$$\begin{aligned} \delta^* \circ \Psi'^{\mathcal{R}} : \left(\mathcal{T}\left(\mathcal{P}^{*,0}\right), \star_-\right) &\rightarrow (\mathrm{Mer}_0(\mathbb{C}^\infty), \cdot) \\ \sigma_1 \otimes \cdots \otimes \sigma_k &\mapsto \sum_{<}^{\rm nested} \left(\delta^* \circ \widetilde{\mathcal{R}}^*\right)(\sigma_1 \otimes \cdots \otimes \sigma_k)(z) \end{aligned} \tag{15.162}$$

are algebra morphisms, i.e. for any symbols $\sigma_1, \cdots, \sigma_{k+l}$ in $\mathcal{P}^{*,0}$ we have

$$\begin{aligned} & \left(\delta^* \circ \Psi^{\mathcal{R}}\right)\left((\sigma_1 \otimes \cdots \otimes \sigma_k) \star_- (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})\right) \\ = & \left(\delta^* \circ \Psi^{\mathcal{R}}\right)(\sigma_1 \otimes \cdots \otimes \sigma_k) \cdot \left(\delta^* \circ \Psi^{\mathcal{R}}\right)(\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}), \end{aligned} \tag{15.163}$$

and

$$\begin{aligned} & \left(\delta^* \circ \Psi'^{\mathcal{R}}\right)\left((\sigma_1 \otimes \cdots \otimes \sigma_k) \star_+ (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})\right) \\ = & \left(\delta^* \circ \Psi'^{\mathcal{R}}\right)(\sigma_1 \otimes \cdots \otimes \sigma_k) \cdot \left(\delta^* \circ \Psi'^{\mathcal{R}}\right)(\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}), \end{aligned} \tag{15.164}$$

where $\cdot$ stands for the ordinary product of functions.

We know by results of Hoffman, that the tensor algebras $\left(\mathcal{T}(\mathcal{P}^{*,0}), *_-\right)$ and $\left(\mathcal{T}(\mathcal{P}^{*,0}), *_+\right)$ equipped with the deconcatenation coproduct:

$$\Delta\left(\sigma_1 \otimes \cdots \otimes \sigma_k\right) := \sum_{j=0}^{k} (\sigma_1 \otimes \cdots \otimes \sigma_j) \bigotimes (\sigma_{j+1} \otimes \cdots \otimes \sigma_k)$$

inherit a structure of connected graded commutative Hopf algebra [H1]. Using the convolution product $*$ associated with the product and coproduct on $\left(\mathcal{T}(\mathcal{P}^{*,0}), \star_-\right)$ (resp. $\left(\mathcal{T}(\mathcal{P}^{*,0}), \star_+\right)$), we can implement a Birkhoff factorisation to the map $\left(\delta^* \circ \Psi^{\mathcal{R}}\right)$ (resp. $\left(\delta^* \circ \Psi'^{\mathcal{R}}\right)$) as in the Connes and Kreimer setup ([CK], [Ma])

$$\left(\delta^* \circ \Psi^{\mathcal{R}}\right) = \left(\delta^* \circ \Psi^{\mathcal{R}}\right)_+ * \left(\delta^* \circ \Psi^{\mathcal{R}}\right)_-$$

(resp.

$$\left(\delta^* \circ \Psi'^{\mathcal{R}}\right) = \left(\delta^* \circ \Psi'^{\mathcal{R}}\right)_+ * \left(\delta^* \circ \Psi'^{\mathcal{R}}\right)_-)$$

associated with the minimal substraction scheme to build a character

$$\left(\delta^* \circ \Psi^{\mathcal{R}}\right)_+(0) : \left(\mathcal{T}\left(\mathcal{P}^{*,0}\right), \star_-\right) \rightarrow \mathbb{C},$$

(resp.

$$\left(\delta^* \circ \Psi'^{\mathcal{R}}\right)_+(0) : \left(\mathcal{T}\left(\mathcal{P}^{*,0}\right), \star_+\right) \rightarrow \mathbb{C}).$$

Proposition 46 *[MP2] Let $\mathcal{R}$ be a holomorphic regularisation which sends a symbol σ to a symbol $\sigma(z)$ with order $\alpha(z) = \alpha(0) - q\,z$ for some positive real number q. The map*

$$\begin{array}{rcl}\psi^{\mathcal{R},\mathrm{Birk}} : \left(\mathcal{T}\left(CS_{\mathrm{c.c}}(\mathbb{R}^d)\right), \star_-\right) & \to & \mathbb{C} \\ \sigma_1 \otimes \cdots \otimes \sigma_k & \mapsto & (\delta^* \circ \Phi)_+ (0)(\sigma_1 \otimes \cdots \otimes \sigma_k)\end{array}$$

defines a character. In other words, $\psi^{\mathcal{R},\mathrm{Birk}}$ satisfies stuffle relations:

$$\begin{array}{rl} & \psi^{\mathcal{R},\mathrm{Birk}}\left((\sigma_1 \otimes \cdots \otimes \sigma_k) \star_- (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})\right) \\ = & \psi^{\mathcal{R},\mathrm{Birk}}(\sigma_1 \otimes \cdots \otimes \sigma_k)\, \psi^{\mathcal{R},\mathrm{Birk}}(\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l})\end{array}$$

and similarly for ${\psi'}^{\mathcal{R},\mathrm{Birk}}$ with $\star_-$ replaced by $\star_+$.

This yields an alternative set of renormalised nested sums of symbols

$$-\!\!\!\!\!\!\sum_{\leq}^{\mathrm{nested},\mathcal{R},\mathrm{Birk}} \sigma_1 \otimes \cdots \otimes \sigma_k := \psi^{\mathcal{R},\mathrm{Birk}}(\sigma_1 \otimes \cdots \otimes \sigma_k)$$

and

$$-\!\!\!\!\!\!\sum_{<}^{\mathrm{nested},\mathcal{R},\mathrm{Birk}} \sigma_1 \otimes \cdots \otimes \sigma_k := {\psi'}^{\mathcal{R},\mathrm{Birk}}(\sigma_1 \otimes \cdots \otimes \sigma_k)$$

which obey stuffle relations:

$$-\!\!\!\!\!\!\sum_{\leq}^{\mathcal{R},\mathrm{Birk}} (\sigma_1 \otimes \cdots \otimes \sigma_k) \star_- (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}) = \left(-\!\!\!\!\!\!\sum_{\leq}^{\mathcal{R},\mathrm{Birk}} \sigma_1 \otimes \cdots \otimes \sigma_k\right) \cdot \left(-\!\!\!\!\!\!\sum_{\leq}^{\mathcal{R},\mathrm{Birk}} \sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}\right), \tag{15.165}$$

and

$$-\!\!\!\!\!\!\sum_{<}^{\mathcal{R},\mathrm{Birk}} (\sigma_1 \otimes \cdots \otimes \sigma_k) \star_+ (\sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}) = \left(-\!\!\!\!\!\!\sum_{<}^{\mathcal{R},\mathrm{Birk}} \sigma_1 \otimes \cdots \otimes \sigma_k\right) \cdot \left(-\!\!\!\!\!\!\sum_{<}^{\mathcal{R},\mathrm{Birk}} \sigma_{k+1} \otimes \cdots \otimes \sigma_{k+l}\right), \tag{15.166}$$

15.7 An algebra of symbols

The above constructions carry out to the following subalgebra of $\mathcal{P}^{*,0}$.
Since we want to consider both zeta and Hurwitz zeta functions, let us first observe that for any non negative number v and any σ in $\mathcal{P}^{*,k}$, the map $\xi \mapsto t_v^* \sigma(\xi) := \sigma(\xi + v)$ defines a symbol in $\mathcal{P}^{*,k}$.
Let $\widetilde{\mathcal{A}}$ be the subalgebra of $\mathcal{P}^{*,0}$ generated by the continuous functions with support inside the interval $]0,1[$ and the set

$$\{f \in \mathcal{P}^{*,0},\, \exists v \in [0,+\infty[, \exists s \in \mathbb{C},\, \sigma(\xi) = (\xi+v)^{-s} \text{ when } \xi \geq 1\}.$$

Consider the ideal $\mathcal{N}$ of $\widetilde{\mathcal{A}}$ of continuous functions with support inside the interval $]0,1[$. The quotient algebra $\mathcal{A} = \widetilde{\mathcal{A}}/\mathcal{N}$ is then generated by the elements $\sigma_{s,v}$, where $\sigma_{s,v}$ is the class of any $\sigma \in \widetilde{\mathcal{A}}$ such that $\sigma(\xi) = (\xi+v)^{-s}$ for $|\xi| \geq 1$. For any $v \in \mathbb{R}_+$ the subspace $\mathcal{A}_v$ of $\mathcal{A}$ generated by $\{\sigma_{s,v},\, s \in \mathbb{C}\}$ is a subalgebra of $\mathcal{A}$. It is therefore natural to equip $\mathcal{A}$ with the following holomorphic regularization on an open neighbourhood Ω of 0 in $\mathbb{C}$:

$$\begin{array}{rcl}\mathcal{R} : \mathcal{A} & \to & \mathrm{Hol}_\Omega(\mathcal{A}) \\ \sigma & \mapsto & (z \mapsto (1-\chi)\,\sigma + \chi\, \sigma_{s+z,v})\end{array}$$

where χ is any smooth cut-off function which is identically one outside the unit ball and vanishes in a small neighborhodd of 0.
We choose the product $\bullet$ as the opposite of the ordinary product, so that we have:

$$\sigma \bullet \sigma' = -\sigma\sigma' \quad \forall(\sigma,\sigma') \in \mathcal{A}^2; \quad \text{resp.} \quad \sigma_{s;v} \bullet \sigma_{s';v} = -\sigma_{s+s';v} \quad \forall(\sigma_{s;v}, \sigma_{s';v}) \in \mathcal{A}_v^2.$$

Let $\mathcal{W}$ be the $\mathbb{C}$-vector space freely spanned by sequences $(u_1, \dots, u_k)$ of real numbers. Let us define the stuffle product on $\mathcal{W}$ by:

$$(u_1, \dots, u_k) \star (u_{k+1}, \dots, u_{k+l}) = \sum_{0 \le r \le \min(k,l)} (-1)^r \sum_{\pi \in \mathrm{mixsh}(k,l;r)} (u_1^\pi, \dots, u_{k+l-r}^\pi), \tag{15.167}$$

with:

$$u_j^\pi = \sum_{i \in \{1,\dots,k+l\}, \pi(i)=j} u_i$$

(the sum above contains only one or two terms).
The map $u \mapsto \sigma_{u;v}$ from $\mathcal{W}$ to $\mathcal{T}(\mathcal{A}_v)$:

$$\sigma_{(u_1,\dots,u_k;\, v)} := \sigma_{u_1;v} \otimes \cdots \otimes \sigma_{u_k;v}$$

induces a stuffle product on $\mathcal{T}(\mathcal{A}_v)$:

$$\sigma_{u;v} \star_- \sigma_{u';v} = \sigma_{u \star u';v}.$$

The same holds with $\star_-$ replaced by $\star_+$ provided we drop the signs $(-1)^r$ in equation (15.167) defining the stuffle product on $\mathcal{W}$.
As before, we twist the regularisation $\widetilde{\mathcal{R}}$ induced by $\mathcal{R}$ on $\mathcal{T}(\mathcal{A}_v)$ by a Hoffman isomorphism to build a twisted holomorphic regularisation $\widetilde{\mathcal{R}}^*$ in several variables which satisfies

$$\left(\widetilde{\mathcal{R}}^*(\sigma_{u;v}) \star_- \widetilde{\mathcal{R}}^*(\sigma_{u';v})\right)_{\mathrm{sym}} = \left(\widetilde{\mathcal{R}}^*(\sigma_{u\star u';v})\right)_{\mathrm{sym}},$$

and a twisted holomorphic regularisation $\delta^* \circ \widetilde{\mathcal{R}}^*$ in one variable compatible with the stuffle product:

$$\left(\delta^* \circ \widetilde{\mathcal{R}}^*(\sigma_{u;v})\right) \star_- \left(\delta^* \circ \widetilde{\mathcal{R}}^*(\sigma_{u';v})\right) = \delta^* \circ \widetilde{\mathcal{R}}^*(\sigma_{u\star u';v}),$$

and similarly with $\star_-$ replaced by $\star_+$.

15.8 Multiple zeta values renormalised via evaluators

Let Ω be an open neighbourhood of 0 in $\mathbb{C}$ and let $\mathcal{R} : \sigma \mapsto \{\sigma(z)\}_{z\in\Omega}$ be the holomorphic regularization procedure on $\widetilde{\mathcal{A}}$ previously introduced. The multiple Hurwitz zeta functions defined by:

$$\begin{aligned} \zeta(s_1, \dots, s_k;\, v_1, \dots, v_k) &:= \Psi^{\mathcal{R}}(\sigma_{s_1,v_1} \otimes \cdots \otimes \sigma_{s_k,v_k}), \\ \overline{\zeta}(s_1, \dots, s_k;\, v_1, \dots, v_k) &:= \Psi'^{\mathcal{R}}(\sigma_{s_1,v_1} \otimes \cdots \otimes \sigma_{s_k,v_k}) \end{aligned}$$

are meromorphic in all variables with poles[15] on a countable family of hyperplanes $s_1 + \cdots + s_j \in]-\infty, j] \cap \mathbb{Z}$, j varying from 1 to k. When $v_1 = \cdots = v_k = v$, we set

$$\zeta(s_1, \dots, s_k;\, v) := \zeta(s_1, \dots, s_k;\, v_1, \dots, v_k); \quad \overline{\zeta}(s_1, \dots, s_k;\, v) := \overline{\zeta}(s_1, \dots, s_k;\, v_1, \dots, v_k)$$

in which case they satisfy the following relations :

$$\left(\overline{\zeta}(u \star_- u';\, v)\right)_{\mathrm{sym}} = \left(\overline{\zeta}^{\mathcal{E}}(u;v)\, \overline{\zeta}^{\mathcal{E}}(u';v)\right)_{\mathrm{sym}} \tag{15.168}$$

with the stuffle product $\star_-$ defined by (15.167), and:

$$\left(\zeta^{\mathcal{E}}(u \star_+ u';v)\right)_{\mathrm{sym}} = \left(\zeta^{\mathcal{E}}(u;v)\, \zeta^{\mathcal{E}}(u';v)\right)_{\mathrm{sym}}, \tag{15.169}$$

with the stuffle product $\star_+$ defined by (15.167) with signs $(-1)^r$ removed.
The renormalised multiple Hurwitz zeta values derived from a *symmetrised* renormalised evaluator $\mathcal{E}$ on $\mathcal{LM}_0(\mathbb{C}^\infty)$:

$$\begin{aligned} \zeta^{\mathcal{E}}(s_1, \dots, s_k;\, v_1, \dots, v_k) &:= \Psi^{\mathcal{R},\mathcal{E}}(\sigma_{s_1,v_1} \otimes \cdots \otimes \sigma_{s_k,v_k}), \\ \overline{\zeta}^{\mathcal{E}}(s_1, \dots, s_k;\, v_1, \dots, v_k) &:= \Psi'^{\mathcal{R},\mathcal{E}}(\sigma_{s_1,v_1} \otimes \cdots \otimes \sigma_{s_k,v_k}) \end{aligned}$$

[15] When $k = 2$ and $v_1 = \cdots = v_l = v$ a more refined analysis actually shows that for some any negative real number v, poles actually only arise for $s_1 = -1$ or $s_1 + s_2 \in \{-2, -1, 0, 2, 4, 6, \cdots\}$.

denoted by $\zeta^{\mathcal{R},\mathcal{E}}(s_1,\ldots,s_k;\,v)$ and $\overline{\zeta}^{\mathcal{R},\mathcal{E}}(s_1,\ldots,s_k;\,v)$ when $v_1=\cdots=v_k=v$, satisfy stuffle relations in that case:

$$\overline{\zeta}^{\mathcal{E}}(u\star_- u';\,v)=\overline{\zeta}^{\mathcal{E}}(u;v)\,\overline{\zeta}^{\mathcal{E}}(u';v) \tag{15.170}$$

and:

$$\zeta^{\mathcal{E}}(u\star_+ u';v)=\zeta^{\mathcal{E}}(u;v)\,\zeta^{\mathcal{E}}(u';v). \tag{15.171}$$

One can show along the lines of the proof of Theorem 10 in [MP2] that enormalised multiple zeta values at non positive arguments obtained this way with v rational, are rational linear combinations of Bernoulli numbers, and hence rational numbers.

15.9 Multiple zeta values renormalised via Birkhoff factorisation

Renormalised multiple Hurwitz zeta values derived from a Birkhoff factorisation:

$$\begin{aligned}\zeta^{\mathrm{Birk}}(s_1,\ldots,s_k;\,v_1,\ldots,v_k) &:= \Psi^{\mathcal{R},\mathrm{Birk}}(\sigma_{s_1,v_1}\otimes\cdots\otimes\sigma_{s_k,v_k}),\\ \overline{\zeta}^{\mathrm{Birk}}(s_1,\ldots,s_k;\,v_1,\ldots,v_k) &:= \Psi'^{\mathcal{R},\mathrm{Birk}}(\sigma_{s_1,v_1}\otimes\cdots\otimes\sigma_{s_k,v_k})\end{aligned}$$

denoted by $\zeta^{\mathrm{Birk}}(s_1,\ldots,s_k;\,v)$ and $\overline{\zeta}^{\mathrm{Birk}}(s_1,\ldots,s_k;\,v)$ when $v_1=\cdots=v_k=v$, satisfy stuffle relations in that case:

$$\overline{\zeta}^{\mathrm{Birk}}(u\star_- u';\,v)=\overline{\zeta}^{\mathrm{Birk}}(u;v)\,\overline{\zeta}^{\mathrm{Birk}}(u';v) \tag{15.172}$$

with the stuffle product $\star_-$ defined by (15.167), and:

$$\zeta^{\mathrm{Birk}}(u\star_+ u';v)=\zeta^{\mathrm{Birk}}(u;v)\,\zeta^{\mathrm{Birk}}(u';v). \tag{15.173}$$

with the stuffle product $\star_+$ defined by (15.167) with signs $(-1)^r$ removed.
A striking holomorphy property arises at non positive integer arguments [MP2] after implementing the diagonal map δ.

Proposition 47 *At non positive integer arguments s_i, and for a rational parameter v, the maps*

$$z\mapsto\psi^{\mathcal{R}}\left(\sigma_{s_1,v}\otimes\cdots\otimes\sigma_{s_k,v}\right)(z)$$

and

$$z\mapsto\psi'^{\mathcal{R}}\left(\sigma_{s_1,v}\otimes\cdots\otimes\sigma_{s_k,v}\right)(z)$$

are holomorphic at zero.

Consequently,

$$\zeta^{\mathrm{Birk}}(s_1,\ldots,s_k;\,v)=\lim_{z\to 0}\psi^{\mathcal{R},\mathrm{Birk}}(\sigma_{s_1,v_1}\otimes\cdots\otimes\sigma_{s_k,v_k})$$

and

$$\overline{\zeta}^{\mathrm{Birk}}(s_1,\ldots,s_k;\,v)=\lim_{z\to 0}\psi'^{\mathcal{R},\mathrm{Birk}}(\sigma_{s_1,v_1}\otimes\cdots\otimes\sigma_{s_k,v_k}).$$

Explicit computations show that renormalised double zeta values at non positive integers obtained by two different methods – using the symmetrised renormalised evaluator $\mathrm{ev}_0^{\mathrm{ren,sym}}$ or a Birkhoff factorisation– coincide. However the table of values for depth 2 derived in [MP2] differs from the one derived in [GZ] using a heat-kernel type approach, with which it however matches for arguments (a_1,a_2) with a_1+a_2 odd and $a_2\neq 0$ and for diagonal arguments $(-a,-a)$.

16 Renormalised multiple sums of symbols with conical constraints

The convergent nested discrete sums of positively supported symbols we previously investigated can be interpreted as multiple sums with conical constraints:

$$\begin{aligned}\sum_{0\leq n_k\leq\cdots\leq n_1}\sigma_1(n_1)\cdots\sigma_k(n_k) &= \sum_{(n_1,\cdots,n_k)\in C\cap\mathbb{Z}^k}\sigma_1(n_1)\cdots\sigma_k(n_k)\\ &= \sum_{n_1=0}^{\infty}\cdots\sum_{n_k=0}^{\infty}(\sigma_1\otimes\cdots\otimes\sigma_k)\circ A(n_1,\cdots,n_k),\end{aligned}$$

where C is the cone $0\leq x_k\leq\cdots\leq x_1$ and A is the upper triangular $k\times k$ matrix

$$A:=\begin{pmatrix}1&1&\cdots&1\\0&1&\cdots&1\\\cdots&\cdots&\cdots&\cdots\\\cdots&\cdots&\cdots&\cdots\\0&\cdots&0&1\end{pmatrix}.$$

We investigate discrete sums of symbols associated with general convex cones in $\mathbb{R}_+^k$.

16.1 Convex cones

We consider the filtered vector space $\mathbb{R}^\infty=\cup_{k=1}^\infty\mathbb{R}^k$ with standard basis $\{e_1,\cdots,e_k,\cdots\}$ and standard orientation.

Definition 23 *A closed (resp. open) convex cone C in $\mathbb{R}^k$ is a closed (resp. open) subset of $\mathbb{R}^k$ stable under any nonnegative combination of elements of the set. The rank of a cone is the dimension of the linear subspace spanned by the cone.*
The convex cone C together with an ordered set of generators $\vec{v}:=(v_1,\cdots,v_J)$ with $v_j\in\mathbb{R}^k$ is denoted by

$$(C,\vec{v}):=\langle v_1,\cdots,v_J\rangle_+.$$

According to whether the cone is closed or open, we take non negative or positive coefficients in the linear combinations.
Closed (resp. open) convex cones $(C,\vec{v})$ are in one to one correspondence with $k\times J$ matrices

$$A=(a_{ij})_{1\leq i\leq k,1\leq j\leq J}\Longleftrightarrow\mathcal{C}_A:=\langle\sum_{i=1}^k a_{i1}e_i,\cdots,\sum_{i=1}^k a_{iJ}e_i\rangle_+.$$

A **subdivision** of a closed cone is a finite collection of cones

- which contains the faces of any cone in this collection,
- such that the intersection of two elements of the collection is a face of both elements,
- and such that the cone is the union of the elements in this collection.

A cone C is **simplicial** if it is spanned by independent vectors $\vec{v}=(v_1,\cdots,v_J)$ in which case the matrix A corresponding to $(C,\vec{v})$ lies in $GL_k(\mathbb{Q})$.

Remark 18 *Any cone admits a subdivision into simplicial cones so that in practice we often consider simplicial cones.*

A cone is **pointed** if it does not contain a straight line; any cone can be subdivided into pointed cones. A cone is **rational** if it is spanned by vectors in $\mathbb{Q}^k$, in which case the matrix associated with a given set of generators $v_j\in\mathbb{Q}^k, j=1,\cdots,J$ has rational coefficients. We simply call a pointed rational convex cone a cone. Such a cone is **smooth** if it is spanned by part of a basis in $\mathbb{Z}^\infty$.

Example 21 *We call open (resp. closed)* **Chen cone** *of dimension k the k-dimensional smooth simplicial open (resp. closed) cone $0<x_k<\cdots<x_1$ (resp. $0\leq x_k\leq\cdots\leq x_1$) associated with the upper triangular matrix (17.180).*

16.2 Multiple zeta functions associated with cones

Theorem 18 *Given a cone* $C \subset \mathbb{R}_+^k$*, the map*

$$\vec{s} := (s_1, \cdots, s_k) \mapsto \sum_{\vec{x} \in C \cap \mathbb{Z}_+^k} x_1^{-s_1} \cdots x_k^{-s_k}$$

is holomorphic on the intersection of half planes $Re(s_{\tau(1)} + \cdots + s_{\tau(i)}) > r_i$, $i = 1, \cdots, k$, $\tau \in \Sigma_k$, *where the* r_i*'s are postive integers depending on the shape of the cone.*
It extends to a meromorphic map

$$\vec{s} \mapsto \sum_{\vec{x} \in C \cap \mathbb{Z}_+^k}^{\text{mer}} x_1^{-s_1} \cdots x_k^{-s_k} := \frac{1}{\Gamma(s_1) \cdots \Gamma(s_k)} \sum_{\tau \in \Sigma_k} \frac{H_{\tau, \underline{m}}(s_1, \cdots, s_k)}{\prod_{i=1}^k \left[\left(s_{\tau(1)} + \cdots + s_{\tau(i)} - r_i\right) \cdots \left(s_{\tau(1)} + \cdots + s_{\tau(i)} - r_i + m_i\right) \right]}$$

with poles $\vec{s} := (s_1, \cdots, s_k) \in \mathbb{C}^k$ *on a countable set of affine hyperplanes* $s_{\tau(1)} + \cdots + s_{\tau(i)} - r_i \in -\mathbb{N}_0$ *with* i *varying in* $=\{1, \cdots, k\}$ *and* τ *in* Σ_k.
Here the multiindex $\vec{m} = (m_1, \cdots, m_k)$ *lies in* $\mathbb{N}_0^k$ *and* $H_{\tau, \underline{m}}, \tau \in \Sigma_I$, *is a holomorphic map on the domain* $\cap_{i=1}^k \{\mathrm{Re}(s_{(\tau(1)} + \cdots + s_{\tau(i)}) + m_i > r_i\}$.

Remark 19 *Note that a permutation* $\tau \in \Sigma_k$ *on the arguments* s_i *boils down to changing the cone* C *to* $\tau_*^{-1}\mathcal{C}_: = \{\vec{x}, \quad (x_{\tau^{-1}(1)}, \cdots, x_{\tau^{-1}(k)}) \in C\}$ *which also lies in* $\mathbb{R}_+^k$.

Proof: For $\mathrm{Re}(s_i)$ sufficiently large we write

$$\begin{aligned}
&\sum_{\vec{x} \in C \cap \mathbb{Z}_+^k} x_1^{-s_1} \cdots x_k^{-s_k} \\
= \quad & \frac{1}{\Gamma(s_1) \cdots \Gamma(s_k)} \int_0^\infty d\epsilon_1\, \epsilon_1^{s_1 - 1} \cdots \int_0^\infty d\epsilon_k\, \epsilon_k^{s_k - 1} \sum_{\vec{x} \in C \cap \mathbb{Z}_+^k} e^{-\sum_{i=1}^k \epsilon_i \cdot x_i} \\
= \quad & \frac{1}{\Gamma(s_1) \cdots \Gamma(s_k)} \int_0^\infty d\epsilon_1\, \epsilon_1^{s_1 - 1} \cdots \int_0^\infty d\epsilon_k\, \epsilon_k^{s_k - 1} \sum_{\vec{x} \in \mathcal{C} \cap \mathbb{Z}_+^{\parallel}} e^{-\langle \vec{\epsilon}, \vec{x} \rangle} \\
= \quad & \frac{1}{\Gamma(s_1) \cdots \Gamma(s_k)} \int_0^\infty d\epsilon_1 \cdots \int_0^\infty d\epsilon_k \prod_{i=1}^k \epsilon_i^{s_i - 1} \prod_{i=1}^J \langle v_j, \vec{\epsilon} \rangle^{-1} h(\vec{\epsilon})\, e^{-\sum_{i=1}^k e_i}
\end{aligned}$$

for some entire map h. Let us decompose the space $\mathbb{R}_+^k$ of parameters $(\epsilon_1, \cdots, \epsilon_k)$ in regions D_τ defined by $\epsilon_{\tau(1)} \leq \cdots \leq \epsilon_{\tau(k)}$ for permutations $\tau \in \Sigma_k$.
This splits the integral $\int_0^\infty d\epsilon_1 \cdots \int_0^\infty d\epsilon_k \prod_{i=1}^k \epsilon_i^{a_i - 1} \prod_{i=1}^J \langle v_j, \underline{\epsilon} \rangle^{-1} h(\underline{\epsilon})\, e^{-\sum_{i=1}^k \epsilon_i}$ into a sum of integrals $\int_{D_\tau} \prod_{i=1}^k \epsilon_i^{a_i - 1} \prod_{i=1}^J \langle v_j, \underline{\epsilon} \rangle^{-1} h(\underline{\epsilon})\, e^{-\sum_{i=1}^k \epsilon_i} d\epsilon_1 \cdots d\epsilon_k$.
Let us focus on the integral over the domain D given by $\epsilon_1 \leq \cdots \leq \epsilon_k$; the results can then be transposed to other domains applying a permutation $s_i \to s_{\tau(i)}$ on the s_i's as a result of the above remark. Setting $\epsilon_i = t_k \cdots t_i$ on this domain introduces new variables $\vec{t} = (t_1, \cdots, t_k)$ which vary in the domain

$$\Delta := \prod_{i=1}^{k-1} [0, 1] \times [0, \infty).$$

Since $v_j := \sum_{i=1}^k a_{ij} e_i \neq 0 \quad \forall j \in \{1, \cdots, k\}$, for any $j \in \{1, \cdots, J\}$, we can define $i_j \in \{1, \cdots, k\}$ to be the largest index i such that $a_{ij} \neq 0$. Performing the change of variable $(\epsilon_1, \cdots, \epsilon_k) \mapsto (t_1, \cdots, t_k)$ in the integral, which introduces a jacobian determinant $\prod_{i=1}^k t_i^{i-1}$, we have

$$\langle v_j, \vec{\epsilon} \rangle = \sum_{i=1}^k a_{ij}\, \epsilon_i = t_k \cdots t_{ij} \left(\sum_{i=1}^{i_j - 1} a_{ij} t_{i_j - 1} \cdots t_1 + a_{i_j j} \right).$$

We can therefore write the sum as follows:

$$\begin{aligned}
&\sum_{\vec{x}\in C\cap \mathbb{Z}_+^k} x_1^{-s_1}\cdots x_k^{-s_k}\\
=\;& \frac{1}{\Gamma(s_1)\cdots\Gamma(s_k)}\int_0^\infty d\epsilon_1\,\epsilon_1^{s_1-1}\cdots\int_0^\infty d\epsilon_k\,\epsilon_k^{s_k-1}\sum_{\vec{x}\in \mathcal{C}\cap \mathbb{Z}_+^k} e^{-\langle\vec{\epsilon},\vec{x}\rangle}\\
=\;& \frac{1}{\Gamma(s_1)\cdots\Gamma(s_k)}\int_0^\infty dt_k\int_0^1 dt_1\cdots\int_0^1 dt_{k-1}\prod_{i=1}^k t_i^{i-1}\prod_{i=1}^k (t_k\cdots t_i)^{s_i-1}\prod_{j=1}^J (t_k\cdots t_{i_j})^{-1}\,\tilde{h}(\underline{t})\\
=\;& \frac{1}{\Gamma(s_1)\cdots\Gamma(s_k)}\int_0^\infty dt_k\int_0^1 dt_1\cdots\int_0^1 dt_{k-1}\prod_{j=1}^k t_j^{s_1+\cdots+a_j-1}\prod_{j=1}^J (t_k\cdots t_{i_j})^{-1}\,\tilde{h}(\underline{t})\\
=\;& \frac{1}{\Gamma(s_1)\cdots\Gamma(s_k)}\int_\Delta \prod_{i=1}^k t_i^{s_1+\cdots+s_i-r_i-1}\,\tilde{h}(\vec{t})
\end{aligned}$$

where we have set

$$\tilde{h}(\vec{t}) := e^{-\sum_{i=1}^k t_k\cdots t_i}\,h(t_k\cdots t_1, t_k\cdots t_2,\cdots,t_k)\prod_{j=1}^J\left(\sum_{i=1}^{i_j-1} a_{ij}t_{i_j-1}\cdots t_1 + a_{i_j\,j}\right)^{-1}$$

and where the r_i's, $i=1,\cdots,k$ are positive integers depending on the shape of the matrix $A=(a_{ij})$ via the integers $i_j, j=1,\cdots,J$. Integrating by parts with respect to each $t_i, i=1,\cdots,k$ introduces factors $\frac{1}{s_1+\cdots+s_i-r_i+m_i}, j_i\in \mathbb{N}_0$ when taking primitives of $t_i^{s_1+\cdots+s_i-r_i-1}$ and differentiating $h(\vec{t})$. Note that $\tilde{h}$ is infinitely smoothing in the domain Δ and that the integral in t_k converges at infinity since the expression $\tilde{h}(\vec{t})$ involves exponentials $e^{-\sum_{i=1}^k t_k\cdots t_i}$.
Summing the various integrals over the regions D_τ, τ varying in Σ_k, which amounts to summing over D integrals with s_i replaced by $s_{\tau(i)}$, we thereby build a meromorphic extension $\sum_{\vec{x}\in C\cap\mathbb{Z}_+^k}\prod_{i=1}^k x_i^{-s_i}$ to the whole complex plane as a sum over permutations $\tau\in\Sigma_k$ of expressions:

$$\frac{1}{\Gamma(s_1)\cdots\Gamma(s_k)}\left(\frac{\int_\Delta\prod_{i=1}^k t_i^{s_{\tau(1)}+\cdots+s_{\tau(i)}-r_i+m_i}\,\widetilde{h}_\tau^{(m_1+\cdots+m_k)}(\underline{t})}{\prod_{i=1}^k\left((s_{\tau(1)}+\cdots+s_{\tau(i)}-r_i)\cdots(s_{\tau(1)}+\cdots+s_{\tau(i)}-r_i+m_i)\right)}+\text{boundary}\quad\text{terms}\right)$$

where the boundary terms on the domain Δ are produced by the iterated m_i integrations by parts in each variable t_i.
Here we have chosen the m_i's sufficiently large for the term $\int_\Delta\prod_{i=1}^k t_i^{s_{\tau(1)}+\cdots+s_{\tau(i)}-r_i+m_i}\,\widetilde{h}_\tau^{(m_1+\cdots+m_k)}(\vec{t})$ to converge. The boundary terms are of the same type, namely they are proportional to

$$\frac{\int_{\Delta'}\prod_{i=1}^k t_i^{s_{\tau(1)}+\cdots+s_{\tau(i)}-r_i+m_i'}\,\widetilde{h}_\tau^{(m_1'+\cdots+m_k')}(\vec{t})}{\prod_{i=1}^k\left((s_{\tau(1)}+\cdots+s_{\tau(i)}-r_i)\cdots(s_{\tau(1)}+\cdots+s_{\tau(i)}-r_i+m_i')\right)}$$

for some domain $\Delta'=\prod_{i=1}^{I'-1}[0,1]\times[0,\infty[$ for some $I'<I$ or $\Delta'=\prod_{i=1}^{I'-1}[0,1]$ for some $I'\leq I$ and some non negative integers $m_i'\leq m_i$ with at least one $m_{i_0}'<m_{i_0}$.

This produces a meromorphic map which, on the domain $\cap_{\tau\in\Sigma_k}\cap_{i=1}^k\{\mathrm{Re}(s_{\tau(1)}+\cdots+s_{\tau(i)})-r_i+m_i>0\}$ (here $\vec{m}=(m_1,\cdots,m_k)$ is a fixed multiindex of non inegative integers) is a sum over permutations $\tau\in\Sigma_k$ of expressions

$$\frac{1}{\Gamma(s_1)\cdots\Gamma(s_k)}\,\frac{H_{\tau,\vec{m}}(a_1,\cdots,a_k)}{\prod_{i=1}^k\left((s_{\tau(1)}+\cdots+s_{\tau(i)}-r_i)\cdots(s_{\tau(1)}+\cdots+s_{\tau(i)}-r_i+m_i)\right)}$$

where $H_{\tau,\vec{m}}$ is a holomorphic map on the domain $\cap_{i=1}^k\{\mathrm{Re}(s_{\tau(1)}+\cdots+s_{\tau(i)})+m_i>r_i\}$.
It therefore extends to a meromorphic map on the whole complex space $\mathbb{C}^k$ with simple poles on a countable set of affine hyperplanes $\{s_{\tau(1)}+\cdots+s_{\tau(i)}-r_i\in-\mathbb{N}_0\}$, (with i varying in $\{1,\cdots,k\}$ and τ in Σ_k) and where the r_i's are integers which depend on the size $k\times J$ of the matrix and on its shape but not on the actual coefficients of the matrix. □

Example 22 *For Chen cones we have $J = k$ and $v_j = e_1 + \cdots + e_j$ so that with the notations of the proof, $i_j = j$ and $s_i = i$ and the poles lie on hyperplanes $s_{\tau(1)} + \cdots + s_{\tau(i)} - i = -l, \quad l = 0, \cdots, m_i, i = 1, \cdots, k, \quad \tau \in \Sigma_k$. If $Re(s_i) > 1$ for any $i \in \{1, \cdots, k\}$ then there is no hyperplane of poles passing through* 0.
More precise results on the location of the poles [MP2] can be derived on direct inspection of these sums on Chen cones using an Euler-MacLaurin formula.

16.3 Cut-off conical sums of symbols

We now specialize to symbols with irrational order, which have the property that the sums of their orders is not an integer that plays a role in the following. From Theorem 18, we derive the following property.

Lemma 15 *Let $C \subset \mathbb{R}_+^k$ be a cone and $s_1, \cdots, s_k$ complex numbers. Whenever the partial sums $s_{j_1} + \cdots + s_{j_i}$ are non integers for any $\{j_1, \cdots, j_i\} \subset \{1, \cdots, k\}$, then the map*

$$(z_1, \cdots, z_k) \mapsto \sum_{\vec{x} \in C \cap \mathbb{Z}_+^k}^{\text{mer}} \prod_{i=1}^k x_i^{-(s_i + z_i)}$$

is holomorphic on the complex plane.
This holds in particular if the s_i's are irrational.

We can therefore set the following definition.

Definition 24 *Let $C \subset \mathbb{R}_+^k$ be a cone and $s_1, \cdots, s_k$ complex numbers with non integer partial sums. We define the cut-off multiple sum with conical constraints by:*

$$-\!\!\!\!\!\!\sum_{\vec{x} \in C \cap \mathbb{Z}_+^k} \prod_{i=1}^k x_i^{-s_i} := \left(\sum_{\vec{x} \in C \cap \mathbb{Z}_+^k}^{\text{mer}} \prod_{i=1}^k x_i^{-s_i - z_i} \right)_{z_i = 0, i = 1, \cdots, k} . \tag{16.174}$$

Remark 20 *When $k = 1$ and $C = \mathbb{R}_+$ we recover the zeta function at a non integer argument s as an ordinary limit:*

$$\zeta(s) = -\!\!\!\!\!\!\sum_{n \in \mathbb{Z}_+} n^{-s} := \lim_{z \to 0} \sum_{n \in \mathbb{Z}_+}^{\text{mer}} n^{-s-z}.$$

Remark 21 *If C is an open Chen cone and $s_{j_1} + \cdots + s_{j_i} \notin \mathbb{Z}$ or $\mathrm{Re}\,(s_{j_1} + \cdots, s_{j_i}) > i$ for any $\{j_1, \cdots, j_i\} \subset \{1, \cdots, k\}$,*

$$-\!\!\!\!\!\!\sum_{\vec{x} \in C \cap \mathbb{Z}_+^k} \prod_{i=1}^k x_i^{-s_i} = -\!\!\!\!\!\!\sum_{0 < n_k < \cdots < n_1} n_1^{-s_1} \cdots n_k^{-s_k} = \tilde{\zeta}(s_1, \cdots, s_k)$$

which corresponds to the multiple zeta values at $(s_1, \cdots, s_k)$ familiar to number theorists.

With these notations, the meromorphy result of Theorem 18 says that the map

$$\vec{s} \mapsto -\!\!\!\!\!\!\sum_{\vec{x} \in C \cap \mathbb{Z}_+^k} x_1^{-s_1} \cdots x_k^{-s_k}$$

provides a meromorphic extension to the whole complex plane of the holomorphic map $\vec{s} \mapsto \sum_{\vec{x} \in C \cap \mathbb{Z}_+^k} x_1^{-s_1} \cdots x_k^{-s_k}$ defined on an intersection of hyperplanes.
Our aim is to extend this meromorphy result to cut-off conical sums of symbols.

Lemma 16 *Let $C \subset \mathbb{R}_+^k$ be a cone and let $s_1, \cdots, s_k$ be complex numbers whose partial sums $s_{j_1} + \cdots + s_{j_i}$ are non integers for any $\{j_1, \cdots, j_i\} \subset \{1, \cdots, k\}$. Then for any $i_0 \in \{1, \cdots, k\}$*

$$\lim_{N \to \infty} -\!\!\!\!\!\!\sum_{\vec{x} \in C \cap \mathbb{Z}_+^k} x_1^{-s_1} \cdots x_{i_0}^{-(s_{i_0} + N)} \cdots x_k^{-s_k} = 0.$$

Proof: By Theorem 18 and with the notations of the theorem,

$$-\!\!\!\!\!\!\sum_{\vec{x}\in C\cap \mathbb{Z}_+^k} x_1^{-s_1}\cdots x_{i_0}^{-(s_{i_0}+N)}\cdots x_k^{-s_k} = \frac{1}{\Gamma(s_1)\cdots\Gamma(s_{i_0}+N)\cdots\Gamma(s_k)}\int_\Delta \prod_{i=1}^k t_i^{s_1+\cdots+s_i-r_i-1}\,\tilde{h}(\vec{t})$$

with

$$\tilde{h}(\vec{t}) := e^{-\sum_{i=1}^k t_k\cdots t_i}\, h(t_k\cdots t_1, t_k\cdots t_2,\cdots,t_k)\prod_{j=1}^J\left(\sum_{i=1}^{i_j-1} a_{ij}t_{i_j-1}\cdots t_1 + a_{i_j\,j}\right)^{-1}$$

independent of N. We split the product

$$\prod_{i=1}^k t_i^{s_1+\cdots+s_i-r_i-1} = \prod_{i=1}^{i_0-1} t_i^{s_1+\cdots+s_i-r_i-1}\prod_{i=i_0}^k t_i^{s_1+\cdots+s_{i_0}+N+s_{i_0+1}+\cdots+s_k-r_i-1}.$$

For large N, the integrals in the variable $t_i, i\geq i_0$ converge so that integration by parts is only needed in the remaining variables $t_i; i=1,\cdots,i_0-1$. But these give rise to Gamma functions in the denominator which do not involve N. Thus, the only N-dependent factor in the denominator is $\Gamma(s_{i_0}+N)$. As $N\to\infty$ the numerator converges since $t_i^{s_i+N}\leq t_i^{s_i}$ for $0\leq t_i\leq 1$ to a finite quantity whereas the denominator diverges to infinity so that the whole expression tends to 0. □

Using the Fréchet topology on classical symbols of constant order, we now extend this cut-off sum with conical constraints by continuity to tensor products $\sigma_1\otimes\cdots\otimes\sigma_k$ of polyhomogeneous symbols $x\mapsto\sigma(x)$ in $\mathcal{P}$, the algebra of positively supported classical symbols on $\mathbb{R}$ introduced in Section 15. Let $\sigma_1,\cdots,\sigma_k$ be symbols in $\mathcal{P}$ with orders $\alpha_1,\cdots,\alpha_k$ respectively, which we write according to (2.11)

$$\begin{aligned}\sigma_i(x_i) &= \sum_{j_i=0}^{N_i-1}\sigma_{i,\alpha_i-j_i}(x_i)\,\chi(x_i)+\sigma_i^{(N_i)}(x_i)\\ &= \sum_{j_i=0}^{N_i-1} c_{j_i}^i\,x_i^{\alpha_i-j_i}\,\chi(x_i)+\sigma_i^{(N_i)}(x_i)\end{aligned}\qquad(16.175)$$

where $N_i, i=1,\cdots,k$ are positive integers, $\sigma_{i,\alpha_i-j_i}, i=1,\cdots,k$ are homogeneous functions of degree α_i-j_i, $\sigma_i^{(N_i)}, i=1,\cdots,k$ polyhomogeneous symbols of order whose real part is no larger than α_i-N_i and where we have set $c_{j_i}^i := \sigma_{i,\alpha_i-j_i}(1), i=1,\cdots,k$. Here χ is a smooth cut-off function which vanishes in a neighborhood of 0 and is identically one outside the unit interval.
We have

$$\lim_{N\to\infty}\prod_{i=1}^I(\sigma_i-\sigma_i^{(N)})(x_i) = \otimes_{i=1}^I\sigma_i(x_i)$$

in the Fréchet topology on symbols of constant order.

Lemma 17 *Let $C\subset\mathbb{R}_+^k$ be a cone and let $\sigma_1,\cdots,\sigma_k$ be polyhomogeneous symbols in $\mathcal{P}$ whose orders $\alpha_1,\cdots,\alpha_k$ have non integer valued partial sums $\alpha_{j_1}+\cdots+\alpha_{j_i}$ for all subsets $\{j_1,\cdots,j_i\}\subset\{1,\cdots,k\}$. With the notations of (16.175) the sequence*

$$-\!\!\!\!\!\!\sum_{\vec{x}\in C\cap\mathbb{Z}_+^k}\prod_{i=1}^k(\sigma_i-\sigma_i^{(N)})(x_i) := \sum_{j_1=1}^{N-1}\cdots\sum_{j_k=1}^{N-1} c_{j_1}^1\cdots c_{j_I}^I -\!\!\!\!\!\!\sum_{C\cap\mathbb{Z}_+^k, x_i\neq 0}\otimes_{i=1}^k x_i^{\alpha_i-j_i}$$

converges as $N\to\infty$.

Proof: We first observe that $-\!\!\!\!\!\sum_{\mathcal{C}\cap\mathbb{Z}_+^k}\otimes_{i=1}^k x_i^{\alpha_i-j_i}$ is well defined under the assumptions on the orders. On the other hand, by Lemma 16 the sequence

$$\sum_{j_1=1}^{N-1}\cdots\sum_{j_k=1}^{N-1} c_{j_1}^1\cdots c_{j_I}^I -\!\!\!\!\!\!\sum_{\mathcal{C}\cap\mathbb{Z}_+^k}\otimes_{i=1}^k x_i^{\alpha_i-j_i}$$

is a Cauchy sequence which therefore converges as $N \to \infty$. $\square$

On the grounds of this lemma the cut-off conical sum extends to the tensor product by linearity and continuity and we set the following definition.

Definition 25 *Let $\mathcal{C} \subset \mathbb{R}_+^k$ be a cone and let $\sigma_1, \cdots, \sigma_k$ be symbols in $\mathcal{P}$ with orders $\alpha_1, \cdots, \alpha_k$ respectively whose partial sums $\alpha_{j_1} + \cdots + \alpha_{j_i}$ are non integer valued for all subsets $\{j_1, \cdots, j_i\} \subset \{1, \cdots, k\}$. With the notations of (16.175) we define the following conical sum:*

$$\begin{aligned}
-\!\!\!\!\!\!\sum_{\vec{x} \in \mathcal{C} \cap \mathbb{Z}_+^k} \prod_{i=1}^k \sigma_i(x_i) &:= \lim_{N \to \infty} -\!\!\!\!\!\!\sum_{\vec{x} \in \mathcal{C} \cap \mathbb{Z}^k} \prod_{i=1}^k (\sigma_i - \sigma_i^{(N)})(x_i) \\
&= \lim_{N \to \infty} \left(\sum_{j_1=1}^{N-1} \cdots \sum_{j_k=1}^{N-1} c_{j_1}^1 \cdots c_{j_I}^I -\!\!\!\!\!\!\sum_{\vec{x} \in \mathcal{C} \cap \mathbb{Z}_+^k} \prod_{i=1}^k x_i^{\alpha_i - j_i} \right).
\end{aligned}$$

In particular, this definition applies to symbols in $\mathcal{P}$ with irrational order.

We can now generalise the statement of Theorem 18 to symbols in $\mathcal{P}$.

Theorem 19 *Given a cone $C \subset \mathbb{R}_+^k$ and symbols $\sigma_1, \cdots, \sigma_k$ in $\mathcal{P}$ with orders $\alpha_1, \cdots, \alpha_k$, the map*

$$(z_1, \cdots, z_k) \mapsto \sum_{\vec{x} \in C \cap \mathbb{Z}_+^k} \sigma_1(x_1)\, x_1^{-s_1} \cdots \sigma_k(x_k)\, x_k^{-z_k}$$

is holomorphic on the intersection of half planes $\sum_{i=1}^k \mathrm{Re}(-\alpha_{\tau(i)} + z_{\tau(i)}) > r_i$, $i = 1, \cdots, k$, $\tau \in \Sigma_k$ where the r_i's are positive integers which depend on the shape of the cone.
It extends to a meromorphic map

$$(z_1, \cdots, z_k) \quad \mapsto \quad -\!\!\!\!\!\!\sum_{\vec{x} \in \mathcal{C} \cap \mathbb{Z}_+^k} \sigma_1(x_1)\, x_1^{-z_1} \cdots \sigma_k(x_k)\, x_k^{-z_k} \tag{16.176}$$

with poles $(z_1, \cdots, z_k) \in \mathbb{C}^k$ on a countable set of affine hyperplanes $z_{\tau(1)} + \cdots + z_{\tau(i)} - r_i \in \sum_{i=1}^k \alpha_i - \mathbb{N}_0$ with $i = 1, \cdots, k, \quad \tau \in \Sigma_k$.

Proof: As before we write$\sigma_i(x_1)\,|x_i|^{-z_i} = \lim_{N_i \to \infty} \sum_{j_i=0}^{N_i-1} c_{j_i}^i\, x_i^{\alpha_i - j_i - z_i}\, \chi(x_i)$ and

$$-\!\!\!\!\!\!\sum_{\vec{x} \in \mathcal{C} \cap \mathbb{Z}_+^k} \sigma_1(x_1)\,|x_1|^{-s_1} \cdots \sigma_k(x_k)\,|x_k|^{-z_k} = \lim_{N \to \infty} \sum_{j_1=0}^{N-1} \cdots \sum_{j_k=0}^{N-1} \prod_{i=1}^k c_{j_i}^i \left(-\!\!\!\!\!\!\sum_{\vec{x} \in \mathcal{C} \cap \mathbb{Z}_+^k} \prod_{i=1}^k x_i^{\alpha_i - j_i - z_i} \right).$$

By Theorem 18 each of the expressions $\sum_{\vec{x} \in \mathcal{C} \cap \mathbb{Z}_+^k} \prod_{i=1}^k x_i^{\alpha_i - j_i - z_i}$ which is holomorphic on the intersection of half planes $\sum_{i=1}^k \mathrm{Re}(-\alpha_{\tau(i)} + z_{\tau(i)}) > r_i$, $i = 1, \cdots, k$, is meromorphic with poles on a countable set of affine hyperplanes $z_{\tau(1)} + \cdots + z_{\tau(i)} - r_i \in \sum_{i=1}^k \alpha_i - \sum_{i=1}^k -j_i - \mathbb{N}_0$ which sits inside the countable set of affine hyperplanes $z_{\tau(1)} + \cdots + z_{\tau(i)} - r_i \in \sum_{i=1}^k \alpha_i - \mathbb{N}_0$. Since the limit is uniform on compact regions of $\mathbb{C}^k$, it defines a meromorphic function with the same properties. $\square$

16.4 Renormalised conical discrete sums of symbols

Let $\mathcal{LM}_0(\mathbb{C}^\infty) := \oplus_{k=1}^\infty \mathcal{LM}_0(\mathbb{C}^k) \subset \mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty)$ with $\mathcal{LM}_0(\mathbb{C}^k)$ defined as in (11.76) by:

$$\mathcal{LM}_0(\mathbb{C}^k) := \left\{ (z_1, \cdots, z_k) \mapsto \frac{h(z_1, \cdots, z_k)}{\prod_{L \in \mathcal{L}_k} (L(z_1, \cdots, z_k))^{m_L}}, \quad h \in \mathrm{Hol}_0\left(\mathbb{C}^k\right), \quad m_L \in \mathbb{N} \right\}, \tag{16.177}$$

equipped with the product (11.75)

$$\left((z_1,\cdots,z_k)\mapsto\prod_{i=1}^I f_i\circ L_i(z_1,\cdots,z_k)\right)\circledast\left((z_1,\cdots,z_l)\mapsto\prod_{j=1}^J f_{I+j}\circ L_{I+j}(z_1,\cdots,z_l)\right)$$
$$:=\quad (z_1,\cdots,z_{k+l})\mapsto\prod_{i=1}^I f_i\circ L_i(z_1,\cdots,z_k)\,\prod_{j=1}^J f_{I+j}\circ L_{I+j}(z_{k+1},\cdots,z_{k+l}).$$

Let $\mathcal{T}(\mathcal{P})$ denote the tensor algebra of the symbol algebra $\mathcal{P}$ equipped with the tensor product and $\mathcal{C}_+$ the set of cones in $\mathbb{R}_+^\infty$ ($\mathcal{C}_+(\mathbb{R}^k)$ the set of cones in $\mathbb{R}_+^k$) which is stable under concatenation that sends a cone $C\in\mathbb{R}_+^k$ and a cone $C'\in\mathbb{R}_+^{k'}$ to a cone $C\bullet C'\in\mathbb{R}_+^{k+k'}$. We equip the direct product $\mathcal{T}(\mathcal{P})\times\mathcal{C}_+$ with the induced product:

$$(\sigma,C)\bullet(\sigma',C'):=(\sigma\otimes\sigma',C\bullet C').$$

The above constructions give rise to a map on $\mathcal{T}(\mathcal{P})\times\mathcal{C}_+$ given by:

$$\begin{aligned}\Phi:\otimes^k\mathcal{P}\times\mathcal{C}_+(\mathbb{R}^k)&\rightarrow\mathcal{LM}_0(\mathbb{C}^\infty)\\ (\sigma,C)&\mapsto\left(\vec z\mapsto-\sum_{\vec x\in C\cap\mathbb{Z}_+^k}\sigma_1(x_1)\,x_1^{-z_1}\cdots\sigma_k(x_k)\,x_k^{-z_k}\right)\end{aligned}$$

with the following property:

$$\Phi\left((\sigma,C)\bullet(\sigma',C')\right)=\Phi((\sigma,C))\circledast\Phi\left((\sigma',C')\right).$$

This property which clearly holds for large $\mathrm{Re}(z_i)$ extends to an identity of meromorphic functions by analytic continuation [16]. Applying a renormalised evaluator (14) at zero Λ (which by definition is compatible with the product $\circledast$) on $\mathcal{LM}_0(\mathbb{C}^\infty)$ leads to a character on $\mathcal{T}(\mathcal{P})\times\mathcal{C}_+$ given by:

$$\begin{aligned}\phi^\Lambda:\otimes^k\mathcal{P}\times\mathcal{C}_+(\mathbb{R}^k)&\rightarrow\mathbb{C}\\ (\sigma,C)&\mapsto-\sum_{\vec x\in C\cap\mathbb{Z}_+^k}^{\Lambda}\sigma_1(x_1)\cdots\sigma_k(x_k):=\Lambda\left(\vec z\mapsto-\sum_{\vec x\in C\cap\mathbb{Z}_+^k}\sigma_1(x_1)\,x_1^{-z_1}\cdots\sigma_k(x_k)\,x_k^{-z_k}\right),\end{aligned}$$

ie

$$\phi^\Lambda\left((\sigma,C)\bullet(\sigma',C')\right)=\phi^\Lambda((\sigma,C))\cdot\phi^\Lambda\left((\sigma',C')\right),$$

which extends the ordinary sum: $\sum_{\vec x\in C\cap\mathbb{Z}_+^k}\sigma_1(x_1)\cdots\sigma_k(x_k)$ defined for symbols σ_i with negative enough orders.

Fixing the symbols $\sigma_i(x):=x^{-s_i}$ for some complex numbers $\vec s:=(s_1,\cdots,s_k,\cdots)$, induces a map $\Phi_{\vec s}^\Lambda$ defined on a cone C in $\mathcal{C}_+(\mathbb{R}^k)$ by

$$\begin{aligned}\Phi_{s_1,\cdots,s_k}:\mathcal{C}_+(\mathbb{R}^k)&\rightarrow\mathcal{LM}_0(\mathbb{C}^\infty)\\ C&\mapsto\left(\vec z\mapsto-\sum_{\vec x\in C\cap\mathbb{Z}_+^k}x_1^{-(s_1+z_1)}\cdots x_k^{-(s_k+z_k)}\right)\end{aligned}$$

and hence a character $\phi_{\vec s}^\Lambda$ on $\mathcal{C}_+$ defined on a cone C in $\mathcal{C}_+(\mathbb{R}^k)$ by

$$\begin{aligned}\phi^\Lambda_{s_1,\cdots,s_k}:\mathcal{C}_+&\rightarrow\mathbb{C}\\ C&\mapsto-\sum_{\vec x\in C\cap\mathbb{Z}_+^k}^{\Lambda}x_1^{-s_1}\cdots x_k^{-s_k}:=\Lambda\left(\vec z\mapsto-\sum_{\vec x\in C\cap\mathbb{Z}_+^k}x_1^{-(s_1+z_1)}\cdots x_k^{-(s_k+z_k)}\right),\end{aligned}$$

[16] For the necessary background in the theory of meromorphic functions in several variables, see for example [GR], in particular the Identity Theorem in Chapter 1, Section A, or [?].

i.e. it obeys the multiplicative property

$$\phi^{\Lambda}_{s_1,\cdots,s_{k+l}}\left(C\bullet C'\right)=\phi^{\Lambda}_{s_1,\cdots,s_k}(C)\cdot\phi^{\Lambda}_{s_{k+1},\cdots,s_l}\left(C'\right). \tag{16.178}$$

The map $\Phi_{\vec{s}}$ is additive on disjoint unions. Indeed, for any cones $C,C'\in\mathcal{C}_+(\mathbb{R}^k)$ such that $C\cap C'=\phi$ we have the following identity of meromorphic functions:

$$\begin{aligned}\Phi_{s_1,\cdots,s_k}\left(C\cup C'\right) &= -\sum_{\vec{x}\in C\cup C'\cap\mathbb{Z}^k_+} x_1^{-(s_1+z_1)}\cdots x_k^{-(s_k+z_k)}\\ &= -\sum_{\vec{x}\in C\cap\mathbb{Z}^k_+} x_1^{-(s_1+z_1)}\cdots x_k^{-(s_k+z_k)}+-\sum_{\vec{x}\in C'\cap\mathbb{Z}^k_+} x_1^{-(s_1+z_1)}\cdots x_k^{-(s_k+z_k)}\\ &= \Phi_{s_1,\cdots,s_k}(C)+\Phi_{s_1,\cdots,s_k}(C').\end{aligned}$$

Applying the evaluator Λ, we infer that the map $\phi^{\Lambda}_{\vec{s}}$ too is additive on disjoint unions (this corresponds to the valuation property of [BV]):

$$\phi^{\Lambda}_{\vec{s}}\left(C\cup C'\right)=\phi^{\Lambda}_{\vec{s}}(C)+\phi^{\Lambda}_{\vec{s}}(C')\quad\forall C,C'\in\mathcal{C}_+(\mathbb{R}^k),\quad\text{with}\quad C\cap C'=\phi \tag{16.179}$$

To the Chen cone $C_k=\langle e_1,e_1+e_2,\cdots,e_1+\cdots+e_k\rangle_+\in\mathcal{C}_+(\mathbb{R}^k)$ and comples numbers $(s_1,\cdots,s_k)$ we assign the value

$$\zeta^{\Lambda}_{s_1,\cdots,s_k}:=\phi^{\Lambda}_{s_1,\cdots,s_k}(C).$$

Combining (16.179) and (16.178) we derive the double stuffle relations. Indeed

$$C_1\bullet C_1=\langle e_1,e_1+e_2\rangle_+\cup\langle e_1+e_2,e_2\rangle_+\cup\langle e_1+e_2\rangle_+$$

implies that

$$\begin{aligned}\zeta^{\Lambda}_{s_1}\,\zeta^{\Lambda}_{s_2} &= \phi^{\Lambda}_{s_1}\left(C_1\right)\,\phi^{\Lambda}_{s_2}\left(C_1\right)\\ &= \phi^{\Lambda}_{s_1,s_2}\left(C_1\bullet C_1\right)\\ &= \phi^{\Lambda}_{s_1,s_2}\left(\langle e_1,e_1+e_2\rangle_+\cup\langle e_1+e_2,e_2\rangle_+\cup\langle e_1+e_2\rangle_+\right)\\ &= \phi^{\Lambda}_{s_1,s_2}\left(\langle e_1,e_1+e_2\rangle_+\right)+\phi^{\Lambda}_{(s_1,s_2)}\left(\langle e_1+e_2,e_2\rangle_+\right)+\phi^{\Lambda}_{(s_1,s_2)}\left(\langle e_1+e_2\rangle_+\right)\\ &= \zeta^{\Lambda}_{s_1,s_2}+\zeta^{\Lambda}_{s_2,s_1}+\zeta^{\Lambda}_{s_1+s_2}.\end{aligned}$$

17 Renormalised multiple integrals of symbols with linear constraints

In a similar manner to convergent nested sums of symbols we previously studied, convergent nested integrals of radial symbols $\sigma_i(\xi) = \tau_i(|\xi|)$ can be realised as multiple integrals with conical constraints writing:

$$\begin{aligned} & \int_{0\leq|\xi_k|\leq\cdots\leq|\xi_1|} \sigma_1(\xi_1)\cdots\sigma_k(\xi_k)\, d\xi_1\cdots, d\xi_k \\ & \int_{0\leq r_k\leq\cdots\leq r_1} \tilde\tau_1(r_1)\cdots\tilde\tau_k(r_k)\, dr_1\cdots, dr_k \\ = & \int_0^\infty dr_1\cdots\int_0^\infty dr_k \tilde\tau_1(r_1+\cdots+r_k)\,\tilde\tau_2(r_1+\cdots+r_{k-1})\cdots\tilde\tau_k(r_k)\, dr_1\cdots dr_k \\ = & \int_0^\infty dr_1\cdots\int_0^\infty dr_k(\tilde\tau_1\otimes\cdots\otimes\tilde\tau_k)\circ A(r_1,\cdots,r_k)\, dr_1\cdots dr_k, \end{aligned}$$

where $\tilde\tau_i(r) = r^{d-1}\,\tau_i(r)$ and A is the upper triangular $k\times k$ matrix

$$A := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

On the other hand, multiple integrals with linear constraints arise in Feynman type integrals such as, in dimension 4:

$$\begin{aligned} & \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} ((\sigma_1\otimes\sigma_2\otimes\sigma_3)\circ B)\,(\xi_1,\xi_2)\, d\xi_1\, d\xi_2 \\ = & \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \frac{1}{(m^2+|\xi_1|^2)^{s_1}}\,\frac{1}{(m^2+|\xi_2|^2)^{s_2}}\,\frac{1}{(m^2+|\xi_1+\xi_2|^2)^{s_3}}\, d\xi_1\, d\xi_2, \end{aligned}$$

with $\sigma_i(\xi) = \frac{1}{(m^2+|\xi|^2)^{s_i}}$ for some $m\in\mathbb{R}^*$ (which introduces a mass term) and B the 3 by 2 matrix

$$B := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},$$

an integral which converges for real numbers $s_i, i = 1, 2, 3$ chosen large enough.
Our aim in this section is to renormalise multiple integrals of the form

$$\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} (\sigma_1\otimes\cdots\otimes\sigma_I)\circ B)\,(\xi_1,\cdots,\xi_L)\, d\xi_1\cdots d\xi_L$$

for classical radial symbols $\sigma_i(\xi) = \tau_i(|\xi|), i = 1,\cdots, I$ on $\mathbb{R}^d$ and a matrix B of size I times L with maximal rank.
This chapter which is based on [Pa4], closely follows the pattern of the previous chapter devoted to renormalised discrete sums with conical constraints.

17.1 A linear extension of the tensor algebra of symbols

Let us first describe an abstract setup. Given

- a vector space V over $\mathbb{R}$ (or maybe a Z-module),
- a linear space $\mathcal{F}(V)$ over K of K-valued maps on V (K is a commutative field, $\mathbb{R}$ or $\mathbb{C}$ in practice),

- a linear form $\lambda : \mathcal{F}(V) \to K$,

uniquely extends to a character

$$\widetilde{\lambda} : \mathcal{T}(\mathcal{F}(V)) \to K$$

on the tensor algebra over $\mathcal{F}(V)$ (closed for the Grotendieck topology if required):

$$\mathcal{T}(\mathcal{F}(V)) = \bigoplus_{k=0}^{\infty} \otimes^k \mathcal{F}(V)$$

equipped with the tensor product $\otimes$:

$$\widetilde{\lambda}(f_1 \otimes \cdots \otimes f_k) = \prod_{i=1}^{k} \lambda(f_i).$$

We wish to extend it to a character on a linear extension

$$\mathcal{L}\mathcal{T}(\mathcal{F}(V)) := \oplus_{L=0}^{\infty} \mathcal{L}^L \mathcal{T}(\mathcal{F}(V)),$$

of the tensor algebra, where

$$\mathcal{L}^L \mathcal{T}(\mathcal{F}(V)) := \{\prod_{i\in I} f_i \circ L_i : V^L \to K, \quad L_i \in \left(V^L\right)^\star, \quad f_i \in \mathcal{F}(V), \quad I \subset \mathbb{N}\}, \tag{17.180}$$

equipped with the following product:

$$\left(\prod_{i=1}^{k} f_i \circ L_i\right) \circledast \left(\prod_{i=1}^{k'} f_{i+k} \circ L_{i+k}\right), \quad L_i \in \left(V^L\right)^*, \quad L_{i+k} \in \left(V^{L'}\right)^*.$$

A matrix $A \in \mathrm{gl}_k(V)$ induces a map

$$i_A : \mathcal{T}^k(\mathcal{F}(V)) \to \mathcal{L}^k \mathcal{T}(\mathcal{F}(V))$$
$$f_1 \otimes \cdots \otimes f_k \mapsto (f_1 \otimes \cdots \otimes f_k) \circ A,$$

and the space $\mathcal{T}^k(\mathcal{F}(V))$ is canonically embedded in $\mathcal{L}^k\mathcal{T}(\mathcal{F}(V))$ via the map $i = i_I$ associated with the identity matrix.

Remark 22 *An element in $\mathcal{L}^k\mathcal{T}(\mathcal{F}(V))$ can be written in different ways, for example if $\mathcal{F}(V) = K[X]$, $(X - Y)(X + Y) = X^2 - Y^2 \in \mathcal{T}^2(\mathcal{F}(V))$.*

In certain cases

$$\mathcal{T}^k\left(\mathcal{F}(V)\right) = \mathcal{F}(V^k).$$

This is the case for the algebra $\mathcal{F}(V) = \mathrm{Hol}_0(V)$ of holomorphic germs at zero.
In this example and in the case $\mathcal{F}(V) = K[X]$ of polynomials in one variable, the algebra $\mathcal{T}\left(\mathcal{F}(V)\right)$ is moreover stable under linear transformations in V, so that

$$\mathcal{L}\mathcal{T}(\mathcal{F}(V)) = \mathcal{T}(\mathcal{F}(V)).$$

In these examples,the linear form λ canonically extends to a character on $\mathcal{L}\mathcal{T}(\mathcal{F}(V))$.

Example 23 *Taking $V = \mathbb{C}$, $\mathcal{F}(V) = \mathrm{Mer}_0(\mathbb{C})$ leads to an algebra $\mathcal{L}\mathcal{M}_0(\mathbb{C}^\infty) := \oplus_{k=1}^{\infty} \mathcal{L}\mathcal{M}_0(\mathbb{C}^k) \subset \mathcal{L}\mathrm{Mer}_0(\mathbb{C}^\infty)$ with $\mathcal{L}\mathcal{M}_0(\mathbb{C}^k)$ defined as in (11.76) by:*

$$\mathcal{L}\mathcal{M}_0(\mathbb{C}^k) := \left\{(z_1, \cdots, z_k) \mapsto \frac{h(z_1, \cdots, z_k)}{\prod_{L\in\mathcal{L}_k}\left(L(z_1, \cdots, z_k)\right)^{m_L}}, \quad h \in \mathrm{Hol}_0\left(\mathbb{C}^k\right), \quad m_L \in \mathbb{N}\right\}, \tag{17.181}$$

equipped with the product (11.75)

$$\left((z_1, \cdots, z_k) \mapsto \prod_{i=1}^{I} f_i \circ L_i(z_1, \cdots, z_k)\right) \circledast \left((z_1, \cdots, z_l) \mapsto \prod_{j=1}^{J} f_j \circ L_j(z_1, \cdots, z_l)\right)$$
$$:= \quad (z_1, \cdots, z_{k+l}) \mapsto \prod_{i=1}^{I} f_i \circ L_i(z_1, \cdots, z_k) \prod_{j=1}^{J} f_j \circ L_J(z_{k+1}, \cdots, z_{k+l}).$$

We want to work with symbols which unfortunately enjoy neither a stability property under tensor products (a tensor product of symbols is not generally a symbol) nor a stability property under linear transformations [17] in the variables, hence the relevance of the defintion to come which combines tensor products and linear constraints.
Choosing $V = \mathbb{R}^d$, $\mathcal{F}(V) = CS_{\rm c.c}(\mathbb{R}^d)$, leads to the following definition where the subscript "max" stands for "maximal rank" of the matrix $(L_1, \cdots, L_I)$ formed by the line vectors L_i, with i varying from 1 to I.

Definition 26 *Let* $\mathcal{L}_{\rm max}\mathcal{T}(CS_{\rm c.c}(\mathbb{R}^d)) = \oplus_{L=1}^{\infty}\mathcal{L}_{\rm max}\mathcal{T}_L(CS_{\rm c.c}(\mathbb{R}^d))$

$$\mathcal{L}_{\rm max}\mathcal{T}_L(CS_{\rm c.c}(\mathbb{R}^d)) := \left\{(\xi_1, \cdots, \xi_L) \mapsto \prod_{i=1}^{I} \sigma_i \circ L_i(\xi_1, \cdots, \xi_L), \quad L_i \in \left(V^L\right)^*, {\rm rk}(L_1, \cdots, L_I) = L\right\}. \tag{17.182}$$

which is stable under the product:

$$\begin{aligned} &\left((\xi_1, \cdots, \xi_L) \mapsto \prod_{i=1}^{I} \sigma_i \circ L_i(\xi_1, \cdots, \xi_L)\right) \circledast \left((\xi_1, \cdots, \xi_M) \mapsto \prod_{j=1}^{J} \sigma_j \circ L_j(\xi_1, \cdots, \xi_M)\right) \\ := \quad &(\xi_1, \cdots, \xi_{L+M}) \mapsto \prod_{i=1}^{I} \sigma_i \circ L_i(\xi_1, \cdots, \xi_L) \prod_{j=1}^{J} \sigma_j \circ L_j(\xi_{k+1}, \cdots, \xi_{L+M}) \end{aligned}$$

since a Whitney sum of two matrices with maximal rank, also has maximal rank.

Remark 23 *When* $I = L$ *and* $L_i(\xi_1, \cdots, \xi_k) = \xi_i$*, the product* $\circledast$ *gives back the tensor product.*

We will need to restrict to classical radial symbols, i.e. to the algebra

$$CS_{\rm c.c}^{\rm rad}(\mathbb{R}^d) := \{\sigma(\xi) = \tau(|\xi|), \quad \tau \in CS_{\rm c.c}(\mathbb{R}_{\geq 0})\}$$

and the corresponding linearly extended tensor algebra: $\mathcal{L}_{\rm max}\mathcal{T}(CS_{\rm c.c}^{\rm rad}(\mathbb{R}^d)) = \oplus_{L=1}^{\infty}\mathcal{L}_{\rm max}\mathcal{T}_L(CS_{\rm c.c}^{\rm rad}(\mathbb{R}^d))$

$$\mathcal{L}_{\rm max}\mathcal{T}_L(CS_{\rm c.c}^{\rm rad}(\mathbb{R}^d)) := \left\{(\xi_1, \cdots, \xi_L) \mapsto \prod_{i=1}^{I} \sigma_i \circ L_i(\xi_1, \cdots, \xi_L), \quad L_i \in \left(V^L\right)^*, {\rm rk}(L_1, \cdots, L_I) = L\right\}. \tag{17.183}$$

Remark 24 *The notations* I *and* L *are chosen in coherence with the notations used by physicists, with* L *the numbers of loops and* I *the number of edges of a Feynman diagram.*

Notation: Letting B be the L times I matrix formed by the line vectors $(L_1, \cdots, L_I)$, we set for convenience:

$$(\sigma_1 \otimes \cdots \otimes \sigma_I) \circ B(\xi_1, \cdots, \xi_L) := \prod_{i=1}^{I} \sigma_i \circ L_i(\xi_1, \cdots, \xi_L) \quad \forall \xi_1, \cdots, \xi_L \in \mathbb{R}^d.$$

17.2 Multiple sums of symbols with linear constraints: meromorphy

Let us now show the existence of meromorphic extensions for integrals $\vec{z} \mapsto \int_{\left(\mathbb{R}^d\right)^L} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B$ built from more general matrices B, where as before $\widetilde{\mathcal{R}}$ is the multiplicative (for the tensor product) extension (14.122) to the tensor algebra $\mathcal{T}(CS(\mathbb{R}^d))$ of a holomorphic regularisation $\mathcal{R}$. For symbols $\sigma_1, \cdots, \sigma_I \in CS_{\rm c.c}(\mathbb{R}^d)$ we set $\tilde{\sigma} := \sigma_1 \otimes \cdots \otimes \sigma_I \in \mathcal{T}(CS(\mathbb{R}^d))$

The aim of this section is to prove the following result.

[17] If σ is a symbol, the map $(\xi_1, \xi_2) \mapsto \sigma(\xi_1 + \xi_2)$does not necessarily define a symbol in (ξ_1, ξ_2) since $\langle(\xi_1, \xi_2)\rangle \not\sim_\infty \langle \xi_1 + \xi_2 \rangle$ where we have set $\langle \eta \rangle = \sqrt{1 + |\eta|^2}$.

Theorem 20 *Let $\mathcal{R} : \sigma \mapsto \sigma(z)$ be a holomorphic regularisation procedure on $CS(\mathbb{R}_{\geq 0})$ and let $\xi \mapsto \sigma_i(\xi) := \tau_i(|\xi|) \in CS(\mathbb{R}^d), i = 1, \cdots, I$ be* radial *polyhomogeneous symbols of order a_i which are sent via $\mathcal{R}$ to $\xi \mapsto \sigma_i(z)(\xi) := \mathcal{R}(\tau)(z)(|\xi|)$ of non constant affine order $\alpha_i(z) = -qz_i + a_i$, for some positive real number q. For any matrix B of size $I \times L$ and rank L, the map*

$$\vec{z} \mapsto \int_{\left(\mathbb{R}^d\right)^L} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B$$

which is well defined and holomorphic on the domain $D = \{\vec{z} \in \mathbb{C}^I, \quad \mathrm{Re}(z_i) > -\frac{a_i+d}{\alpha_i'(0)}, \quad \forall i \in \{1, \cdots, I\}\}$ extends to a meromorphic map

$$\vec{z} \mapsto -\!\!\!\!\!\!\int_{\left(\mathbb{R}^d\right)^L} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B$$

on the whole complex plane with poles located on a countable set of affine hyperplanes

$$z_{\tau(1)} + \cdots + z_{\tau(i)} \in \frac{a_{\tau(1)} + \cdots + a_{\tau(i)} + d\, r_{\tau,i} - \mathbb{N}_0}{q}, i \in \{1, \cdots, I\}, \quad \tau \in \Sigma_I,$$

and where $r_{\tau,i} \in]0, i] \cap \mathbb{Z}$ depends on the matrix B.
In particular, the hyperplanes of poles passing through zero are of the form:

$$z_{\tau(1)} + \cdots + z_{\tau(i)} = 0, \quad i \in \{1, \cdots, I\}, \quad \tau \in \Sigma_I.$$

If none of the partial sums $a_{\tau(1)} + \cdots + a_{\tau(i)}, \quad i \in \{1, \cdots, I\}, \quad \tau \in \Sigma_I$ of the orders a_i are integers, then the hyperplanes of poles of the map $\vec{z} \mapsto -\!\!\!\!\!\int_{\left(\mathbb{R}^d\right)^L} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B$ do not contain 0 and the map is holomorphic in a neighborhood of 0.

Before going to the proof, let us illustrate this result by an example.

Example 24 *If we choose $I = 3, L = 2$, $\sigma_i, i = 1, 2, 3$, $\mathcal{R}(\sigma)(z)(\xi) = \sigma(\xi)\langle\xi\rangle^{-z}$ (here $q = 1$) with $\langle\xi\rangle := \sqrt{1 + |\xi|^2}$ and B as in (17.180), this yields back the known fact that the map*

$$(s_1, s_2, s_3) \mapsto \int_{\mathbb{R}^{d2}} \frac{1}{(|\xi_1|^2 + 1)^{-s_1}} \frac{1}{(|\xi_1 + \xi_2|^2 + 1)^{-s_2}} \frac{1}{(|\xi_2|^2 + 1)^{-s_3}}\, d\xi_1\, d\xi_2$$

has a meromorphic extension to the plane with poles on hyperplanes defined by equations involving partial sums of the s_i's. Whenever $s_1, s_2, s_3, s_1 + s_2, s_2 + s_3, s_1 + s_3, s_1 + s_2 + s_3$ are not half integers, the map is holomorphic in a neighborhood of 0.

Setting $z_i = z$ in the above theorem leads to the following result.

Corollary 9 *Let $\mathcal{R} : \sigma \mapsto \sigma(z)$ be a holomorphic regularisation procedure on $CS(\mathbb{R}_{\geq 0})$ and let $\xi \mapsto \sigma_i(\xi) := \tau_i(|\xi|) \in CS_{\mathrm{c.c}}(\mathbb{R}^d), i = 1, \cdots, I$ be* radial *polyhomogeneous symbols of order a_i which are sent via $\mathcal{R}$ to $\xi \mapsto \sigma_i(z)(\xi) := \mathcal{R}(\tau)(z)(|\xi|)$ of non constant affine order $\alpha_i(z) = -qz_i + a_i$, for some positive real number q. For any matrix B of siwe $I \times L$ and rank L, the map*

$$z \mapsto \int_{\left(\mathbb{R}^d\right)^L} (\mathcal{R}(\sigma_1)(z) \otimes \cdots \otimes \mathcal{R}(\sigma_I)(z)) \circ B$$

which is well defined and holomorphic on the domain $D = \{z \in \mathbb{C}, \quad \mathrm{Re}(z) > \frac{a_i+d}{q}, \quad \forall i \in \{1, \cdots, I\}\}$ extends to a meromorphic map

$$z \mapsto -\!\!\!\!\!\!\int_{\left(\mathbb{R}^d\right)^L} (\mathcal{R}(\sigma_1)(z) \otimes \cdots \otimes \mathcal{R}(\sigma_I)(z)) \circ B$$

on the whole complex plane with a countable set of poles with finite multiplicity

$$z \in \frac{a_{\tau(1)} + \cdots + a_{\tau(i)} + d\, r_{\tau,i} - \mathbb{N}_0}{q\, i}, \quad i \in \{1, \cdots, I\}, \quad \tau \in \Sigma_I,$$

where $r_{\tau,i} \in]0, i]$ is an integer depending on the matrix B.

Remark 25 *In a bosonic field theory with polynomial interaction and mass m, the σ_i's are all equal to a given symbol $\sigma(\xi) = \frac{1}{|\xi|^2+m^2}$ arising from the classical free action functional $\mathcal{A}(\phi) = \int_{\mathbb{R}^d} \langle (\Delta + m^2)\phi(\xi), \phi(\xi)\, d\xi$ via the n-point functions. As a "Gedanken" experiment, if instead we took all the symbols σ_i to be equal to $\sigma(\xi) = \frac{1}{(|\xi|^2+m^2)^s}$ for some* irrational number *s arising from a (non physical since* non local *because of the operator $(\Delta + m^2)^s$ being non differential) action $\mathcal{A}_s(\phi) = \int_{\mathbb{R}^d} \langle (\Delta + m^2)^s\phi(\xi), \phi(\xi)\, d\xi$, then the maps $z \mapsto \int_{\mathbb{R}^{nL}} (\mathcal{R}(\sigma_1)(z) \otimes \cdots \otimes \mathcal{R}(\sigma_I)(z)) \circ B$ would holomorphic around zero on the grounds of the above corollary in which case renormalisation is not necessary. This hints towards the fact that a field theory with non local free action $\mathcal{A}_s(\phi)$ and polynomial interaction should be renormalisable at every loop order as would follow from the result of the above corollary extended to affine constraints.*

To prove Theorem 20, we proceed in several steps, first reducing the problem to step matrices B, then to symbols of the type $\sigma_i : \xi \mapsto \langle \xi \rangle^{a_i}$ and finally proving the meromorphicity for such symbols and matrices.

Step 0: A holomorphy result

Proposition 48 *Let $\sigma_i \in CS_{\rm c.c}(\mathbb{R}^d)$ of order a_i. Let $\mathcal{R}$ be a holomorphic regularisation on $CS_{\rm c.c}(\mathbb{R}^d)$ and let for $i = 1, \cdots, I$, $\alpha_i(z)$ denote the order of $\sigma_i(z)$ which we assume is affine $\alpha_i(z) = \alpha_i'(0)z + a_i$ with real coefficients and such that $\alpha_i'(0) < 0$.*
If a matrix $B = (b_{il})$ of size $I \times L$ and rank L, the map

$$\vec{z} \mapsto \int_{\left(\mathbb{R}^d\right)^L} \widetilde{\mathcal{R}}\left(\tilde{\sigma}\right) \circ B(\vec{z})$$

is holomorphic on the domain $D = \{\vec{z} \in \mathbb{C}^I, \quad \mathrm{Re}(z_i) > -\frac{a_i+n}{\alpha_i'(0)}, \quad \forall i \in \{1, \cdots, I\}\}$.

Proof: The symbol property of each σ_i yields the existence of a constant C such that

$$\begin{aligned} |\tilde{\sigma}(\vec{z}) \circ B(\xi_1, \cdots, \xi_L)| &\leq C \prod_{i=1}^I \langle \sum_{l=1}^L b_{il}\xi_l \rangle^{\mathrm{Re}(\alpha_i(z_i))} \\ &\leq C \prod_{i=1}^I \langle \sum_{l=1}^L b_{il}\xi_l \rangle^{\alpha_i'(0)\mathrm{Re}(z_i)+a_i} \end{aligned}$$

where we have set $\langle \eta \rangle := \sqrt{1 + |\eta|^2}$.
We infer that for $\mathrm{Re}(z_i) \geq \beta_i > 0$

$$|\tilde{\sigma}(\vec{z}) \circ B(\xi_1, \cdots, \xi_L)| \leq \prod_{i=1}^I \langle \sum_{l=1}^L b_{il}\xi_l \rangle^{\alpha_i'(0)\beta_i+a_i}.$$

We claim that the map $(\xi_1, \cdots, \xi_L) \mapsto \langle \sum_{l=1}^L b_{il}\xi_l \rangle^{\alpha_i'(0)\beta_i+a_i}$ lies in $L^1\left(\left(\mathbb{R}^d\right)^L\right)$ if $\beta_i > -\frac{a_i+d}{\alpha_i'(0)}$. Indeed, the matrix B being of rank L by assumption, we can extract an invertible $L \times L$ matrix D. Assuming for simplicity (and without loss of generality, since this assumption holds up to permutation of the lines and columns) that it corresponds to the L first lines of B we write:

$$\begin{aligned} \prod_{i=1}^I \langle \sum_{l=1}^L b_{il}\xi_l \rangle^{\alpha_i'(0)\beta_i+a_i} &= \prod_{i=1}^I \rho_i \circ B(\xi_1, \cdots, \xi_L) \\ &\leq \prod_{i=1}^L \rho_i \circ D(\xi_1, \cdots, \xi_L) \end{aligned}$$

where we have set $\rho_i(\eta) := \langle \eta \rangle^{\alpha_i'(0)\beta_i+a_i}$ and used the fact that $\rho_i(\eta) \geq 1$ and $\alpha_i'(0)\beta_i + a_i < -d$.
But

$$\int_{\left(\mathbb{R}^d\right)^L} \otimes_{i=1}^L \rho_i \circ D = |\mathrm{det} D| \prod_{i=1}^L \int_{\mathbb{R}^d} \rho_i$$

converges as a product of integrals of symbols of order $< -d$ so that by dominated convergence, $\widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B$ lies in $L^1\left(\left(\mathbb{R}^d\right)^L\right)$ for any complex number $\vec{z} \in D$.
On the other hand, the derivative in z of holomorphic symbols have same order as the original symbols (see e.g. [PS]), the differentiation possibly introducing logarithmic terms. Replacing $\sigma_1(z_1), \cdots, \sigma_I(z_I)$ by $\partial_{z_1}^{\gamma_1}\sigma_1(z_1), \cdots, \partial_{z_I}^{\gamma_I}\sigma_I(z_I)$ in the above inequalities, we can infer by a similar procedure that for $\mathrm{Re}(z_i) \geq \beta_i > -\frac{a_i+d}{\alpha_i'(0)}$ the map $\vec{z} \mapsto \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B$ is uniformly bounded by an L^1 function. The holomorphicity of $\vec{z} \mapsto \int_{\left(\mathbb{R}^d\right)^L} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B$ then follows. □

Step 1: Reduction to step matrices

We consider $I \times J$ matrices B with real cwhich fulfill the following condition

$$\exists i_1 < \cdots < i_L \quad \text{in} \quad \{1, \cdots, I\} \quad \text{s.t} \quad b_{il} = 0 \quad \text{if} \quad i > i_l \quad \text{and} \quad b_{i_l,l} \neq 0, \tag{17.184}$$

as a consequence of which the matrix has rank $\geq L$. If $J = L$ then it has rank L; we call such an $I \times L$ matrix, a step matrix.

Proposition 49 *If Theorem 20 holds for step matrices then it holds for any $I \times L$ matrix B of rank L.*

Proof:

- Let us first observe that if the result holds for a matrix B then it holds for any matrix $P\,B\,Q$ where P and Q are permutation matrices i.e. after relabelling of the symbols and the variables. Indeed, a permutation $\tau \in \Sigma_I$ on the lines induced by the matrix P amounts to a relabelling of the symbols; since the statement should hold for all radial symbols, if it holds for $\tilde{\sigma} = \sigma_1 \otimes \cdots \otimes \sigma_I$ then it also holds for $\sigma_{\tau(1)} \otimes \cdots \otimes \sigma_{\tau(I)}$. Hence, if the statement of the theorem holds for a matrix B it also holds for the matrix $P\,B$.
  Assuming the statement of the theorem holds for a matrix B, then it also holds for the matrix $B\,Q$. Indeed, a permutation $\tau \in \Sigma_L$ on the columns induced by the matrix Q amounts to a relabelling of the variables ξ_l. By Proposition 48 we know that if B has rank L then both the maps $\vec{z} \mapsto \int_{\mathbb{R}^{nL}} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B$ and $\vec{z} \mapsto \int_{\mathbb{R}^{nL}} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B\,Q$ are well defined and holomorphic on the domain $D = \{\vec{z} \in \mathbb{C}^I, \quad \mathrm{Re}(z_i) > -\frac{a_i+n}{\alpha_i'(0)}, \quad \forall i \in \{1, \cdots, I\}\}$. By the Fubini property we further have that

  $$\int_{\mathbb{R}^{nL}} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B = |\mathrm{det} Q| \int_{\mathbb{R}^{nL}} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B\,Q \quad \forall \vec{z} \in D.$$

  If by assumption, the r.h.s has a meromorphic extension $\vec{z} \mapsto ⨍_{\mathbb{R}^{nL}} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B\,Q$ then so does the l.h.s. have a meromorphic extension

  $$⨍_{\mathbb{R}^{nL}} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B := |\mathrm{det} Q| ⨍_{\mathbb{R}^{nL}} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B\,Q$$

  which moreover has the same pole structure.

- Let B be a non zero matrix. Then there is an invertible matrix P and step matrix T such that $P\,B^t = T$ where B^t stands for the transpose of B. Hence the existence of an invertible matrix $Q = \left(P^t\right)^{-1}$ such that $B = T^t\,Q$. If B has rank L then so does the matrix T^t; along the same lines as above, one shows that if the statement of the theorem holds for T^t then it holds for B. On the other hand, there are permutation matrices P and Q such that $S := P\,T^t\,Q$ is a step matrix for the transpose of a step matrix can be turned into a step matrix by iterated permutations on its lines and columns. If the theorem holds for step matrices then by the first part of the proof, it also holds for T^t and hence for B.

□

Step 2: Reduction to symbols $\sigma_i : \xi \mapsto \langle\xi\rangle^{a_i}$

Let us first describe the asymptotic behaviour of classical radial symbols.

Lemma 18 *Given a* radial *polyhomogeneous symbol $\sigma : \xi \mapsto \tau(|\xi|)$ on $\mathbb{R}^d$, $\tau \in CS(\mathbb{R}_+)$ of order a there are real numbers $c_j, j \in \mathbb{N}_0$ such that*

$$\sigma(\xi) \sim \sum_{j=0}^{\infty} c_j \, \langle\xi\rangle^{a-j}$$

where $\sim$ stands for the equivalence of symbols modulo smoothing symbols. Here, as before we have set $\langle\xi\rangle = \sqrt{1+|\xi|^2}$.

Proof: A radial polyhomogeneous symbol σ on $\mathbb{R}^d$ of order a can be written

$$\sigma(\ \xi) = \sum_{j=0}^{N-1} \tau_{a-j}\,(|\xi|)\ \chi(|\xi|) + \tau^{(N)}(|\xi|)$$

where N is a positive integer, $\tau^{(N)}$ is a polyhomogeneous symbol the order of which has real part no larger than $\mathrm{Re}(a) - N$ and where τ_{a-j} are positively homogeneous functions of degree $a - j$. χ is a smooth cut-off function on $\mathbb{R}_0^+$ which vanishes in a small neighborhood of 1 and is identically 1 outside the unit interval. Setting $\gamma_{a-j} := \tau_{a-j}(1)$ we write

$$\begin{aligned}
\tau_{a-j}\,(|\xi|)\ \chi(|\xi|) &= \gamma_{a-j}\,|\xi|^{a-j}\,\chi(|\xi|)\\
&= \gamma_{a-j}\,(\langle\xi\rangle^2 - 1)^{\frac{a-j}{2}}\ \chi(|\xi|)\\
&= \gamma_{a-j}\,\langle\xi\rangle^{a-j}\,(1 - \langle\xi\rangle^{-2})^{\frac{a-j}{2}}\ \chi(|\xi|)\\
&\sim \gamma_{a-j}\,\langle\xi\rangle^{a-j}\,\chi(|\xi|)\ \sum_{k_j=0}^{\infty} b_{k_j}\,\langle\xi\rangle^{-2\,k_j}\\
&\sim \sum_{k_j=0}^{\infty} c_{k_j}\,\langle\xi\rangle^{a-j-2k_j}
\end{aligned}$$

where we have set $c_{k_j} := \gamma_{a-j}\, b_{k_j}$ for some sequence $b_{j_k}, k \in \mathbb{N}_0$ of real numbers depending on a and j and used the fact that $\chi \sim 1$. Applying this to each τ_{a-j} yields for any positive integer N, the existence of a symbol $\tilde{\tau}^{(N)}(|\xi|)$ the order of which has real part no larger than $\mathrm{Re}(a) - N$ and constants $\tilde{c}_j$ such that

$$\sigma(\xi) = \sum_{j=0}^{N-1} \tilde{c}_j\,\langle\xi\rangle^{a-j} + \tilde{\tau}^{(N)}(|\xi|)$$

which ends the proof of the lemma. $\square$

Let $\xi \mapsto \sigma_1(\xi) := \tau_1(|\xi|), \cdots, \xi \mapsto \sigma_I(\xi) := \tau_I(|\xi|)$ be radial polyhomogeneous symbol on $\mathbb{R}^d$ of order $a_1, \cdots, a_I$ respectively which we write

$$\begin{aligned}
\sigma_i(\xi) &= \sum_{j_i=0}^{N_i-1} \tau_{i,a_i-j_i}(|\xi|) + \tau_i^{(N_i)}(|\xi|)\,\chi(|\xi|)\\
&= \sum_{j_i=0}^{N_i-1} c^i_{j_i}\,\langle\xi_i\rangle^{a_i-j_i} + \widetilde{\tau}_i^{(N_i)}(|\xi|)
\end{aligned} \tag{17.185}$$

where $N_i, i = 1, \cdots, I$ are positive integers, $\tau_{i,a_i-j_i}, i = 1, \cdots, I$ are homogeneous functions of degree $a_i - j_i$, $\tau_i^{(N_i)}, \widetilde{\tau}_i^{(N_i)}, i = 1, \cdots, I$ polyhomogeneous symbols of order with real part no larger than $a_i - N_i$ and where we have set $c^i_{j_i} := \tau_{i,a_i-j_i}(1), i = 1, \cdots, I$.
It follows that

$$\prod_{i=1}^{I} \sigma_i(\xi_i) = \lim_{N\to\infty} \sum_{j_1=0}^{N-1} \cdots \sum_{j_I=0}^{N-1} c^1_{j_1} \cdots c^I_{j_I}\ \langle\xi_1\rangle^{a_1-j_1} \cdots \langle\xi_I\rangle^{a_I-j_I} \tag{17.186}$$

in the Fréchet topology on symbols of constant order.

Proposition 50 *If Theorem 20 holds for symbols $\sigma_i : \xi \mapsto \langle\xi\rangle^{a_i}$ then it holds for all classical radial symbols.*

Proof: Let B be an $L \times I$ matrix of rank L and let $\sigma_1, \cdots, \sigma_I$ be radial polyhomogeneous symbols in $CS_{\rm c.c}(\mathbb{R}^d)$ with orders $a_1, \cdots, a_I$ respectively. For each $j_i \in \mathbb{N}, i \in \{1, \cdots, I\}$ we set $\rho_i^{j_i}(\xi) := \langle\xi\rangle^{a_i - j_i}$ and for all multiindices $(j_1, \cdots, j_I)$ we set $\tilde{\rho}^{j_1\cdots j_I} := \otimes_{i=1}^I \rho_i^{j_i}$.
Let us first observe that since $\text{Re}(a_i) - j_i \leq \text{Re}(a_i)$, the maps

$$\vec{z} \mapsto \int_{\left(\mathbb{R}^d\right)^L} \widetilde{\mathcal{R}}(\tilde{\rho}^{j_1\cdots j_I})(\vec{z}) \circ B$$

are all well defined and holomorphic on the domain $D = \{\vec{z} \in \mathbb{C}^I, \quad \text{Re}(z_i) > -\frac{a_i+n}{\alpha_i'(0)}, \quad \forall i \in \{1, \cdots, I\}$.
Let us assume that the theorem holds for this specific class of symbols. Then using again the fact that $\rho_i^{j_i}$ has order $a_i - j_i$ which differs from a_i by a non negative integer, and replacing a_i by $\alpha_i(z_i)$, it follows that these maps extend to meromorphic maps

$$\vec{z} \mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^{nL}} \widetilde{\mathcal{R}}(\tilde{\rho}^{j_1\cdots j_I})(\vec{z}) \circ B$$

on the whole complex plane with poles $\vec{z} = (z_1, \cdots, z_I)$ on a countable set of affine hyperplanes

$$z_{\tau(1)} + \cdots + z_{\tau(i)} \in \frac{a_{\tau(1)} + \cdots + a_{\tau(i)} + d\, r_{\tau,i} - \mathbb{N}_0}{q}, \quad \tau \in \Sigma_I,$$

independent of the j_i's.
In the limit as $N \to \infty$ it follows from (17.186) that the map

$$\vec{z} \mapsto \int_{\mathbb{R}^{nL}} \widetilde{\mathcal{R}}(\tilde{\sigma})(\vec{z}) \circ B$$

extends to a meromorphic map on the complex plane:

$$\begin{aligned} \vec{z} \quad &\mapsto \quad -\!\!\!\!\!\!\int_{\left(\mathbb{R}^d\right)^L} \widetilde{\mathcal{R}}(\tilde{\rho}^{j_1\cdots j_I})(\vec{z}) \circ B \\ &:= \quad \lim_{N\to\infty} \sum_{j_1=0}^{N-1} \cdots \sum_{j_I=0}^{N-1} c_{j_1}^1 \cdots c_{j_I}^I -\!\!\!\!\!\!\int_{\left(\mathbb{R}^d\right)^L} \prod_{i=1}^I c_{j_1}^1 \cdots c_{j_I}^I \left(\mathcal{R}(\rho_1^{j_1})(z_1) \cdots \mathcal{R}(\rho_I^{j_I})(z_I)\right) \circ B \end{aligned}$$

with the same pole structure. □

Step 3: The case of symbols $\sigma_i : \xi \mapsto (|\xi|^2 + 1)^{a_i}$ and step matrices

We are therefore left to prove the statement of the theorem for an $I \times L$ matrix B with real coefficients which fulfills condition (17.184) and symbols $\sigma_i : \xi \mapsto (|\xi|^2 + 1)^{a_i}$. As previously observed, such a matrix has rank L.

Lemma 19 *Under assumption (17.184) on $B = (b_{il})$ the matrix B^*B is positive definite. Note that with the notations of (17.184), we have $i_l \geq l$.*

Proof: For $k \in \mathbb{R}^L$ in the kernel of B, we have $\sum_{l=1}^L b_{il}\xi_l = 0$ for any $i = 1, \cdots, I$, which applied to $i = i_L$ yields $\sum_{l=1}^L b_{i_L l}\xi_l = 0$. But since by assumption $b_{i_L l} = 0$ for $l < L$ only the term $b_{I_L L}\xi_l$ remains which shows that $\xi_l = 0$. Proceeding inductively yields the positivity of B^*B. □

Proposition 51 *Let $B := (b_{il})_{i=1,\cdots,I;l=1,\cdots,L}$ be a matrix with property (17.184). The map*

$$(a_1, \cdots, a_I) \mapsto \int_{\left(\mathbb{R}^d\right)^L} \prod_{i=1}^I \langle \sum_{l=1}^L b_{il}\, \xi_l \rangle^{a_i}\, d\xi_1 \cdots d\xi_L,$$

which is holomorphic on the domain $D := \{\underline{a} = (a_1, \cdots, a_I) \in \mathbb{C}^I, \mathrm{Re}(a_i) < -n, \forall i \in \{1, \cdots, I\}\}$*, has a meromorphic extension to the complex plane*

$$(a_1, \cdots, a_I) \mapsto -\!\!\!\!\!\!\int_{(\mathbb{R}^d)^L} \prod_{i=1}^{I} \langle \sum_{l=1}^{L} b_{il}\, \xi_L \rangle^{a_i}\, d\xi_1 \cdots d\xi_L \tag{17.187}$$

$$:= \frac{1}{\prod_{i=1}^{I} \Gamma(-a_i/2)} \sum_{\tau \in \Sigma_I} \frac{H_{\tau,\underline{m}}(a_1, \cdots, a_I)}{\prod_{i=1}^{I} \left[(a_{\tau(1)} + \cdots + a_{\tau(i)} + n\, s_{\tau,i}) \cdots (a_{\tau(1)} + \cdots + a_{\tau(i)} + n\, s_{\tau,i} - 2m_i) \right]}$$

for some holomorphic map $H_{\tau,\underline{m}}$ *on the domain* $\cap_{i=1}^{I} \{\mathrm{Re}(a_{\tau(1)} + \cdots + a_{\tau(i)}) + 2m_i < -n\, s_{\tau,i}\}$*, with* $\tau \in \Sigma_I$ *and* $\underline{m} := (m_1, \cdots, m_I)$ *a multiindex of non negative integers. The* $s_{\tau,i} \leq i$*'s are positive integers which depend on the permutation* τ*, on the size* $L \times I$ *and shape (i.e. on the* l_i*'s) of the matrix but not on the actual coefficients of the matrix.*
The poles of this meromorphic extension lie on a countable set of affine hyperplanes $a_{\tau(1)} + \cdots + a_{\tau(i)} \in -d\, r_{\tau,i} + \mathbb{N}_0$ *with* $\tau \in \Sigma_I, \quad i \in \{1, \cdots, I\}$*,* $r_{\tau,i} \in]0,1] \cap \mathbb{Z}$*.*

The proof, which is rather technical and lengthy is postponed to the Appendix at teh end of this section. It closely follows Speer's proof [Sp] which uses iterated Mellin transforms and integrations by parts.
This ends the proof of Theorem 20.

17.3 Renormalised multiple integrals with linear constraints

The above constructions give rise to a map on $\mathcal{L}_{\max}\mathcal{T}(CS^{\mathrm{rad}}_{\mathrm{c.c}}(\mathbb{R}^d))$ given by:

$$\begin{array}{rcl} \Psi : \mathcal{L}_{\max}\mathcal{T}(CS^{\mathrm{rad}}_{\mathrm{c.c}}(\mathbb{R}^d)) & \to & \mathcal{LM}_0(\mathbb{C}^\infty) \\ \tilde\sigma \circ B & \mapsto & \left(\vec{z} \mapsto -\!\!\!\!\!\!\int_{(\mathbb{R}^d)^L} \widetilde{\mathcal{R}}(\tilde\sigma)(\vec{z}) \circ B \right) \end{array}$$

with the following property:

$$\Psi\left((\tilde\sigma \circ B) \circledast (\tilde\sigma' \circ B') \right) = \Psi(\tilde\sigma \circ B) \circledast \Psi(\tilde\sigma' \circ B').$$

This property which clearly holds for large $\mathrm{Re}(z_i)$ extends to an identity of meromorphic functions by analytic continuation. Applying a renormalised evaluator (14) at zero Λ (which by definition is compatible with the product $\circledast$) on $\mathcal{LM}_0(\mathbb{C}^\infty)$ leads to a character on $\mathcal{L}_{\max}\mathcal{T}(CS^{\mathrm{rad}}_{\mathrm{c.c}}(\mathbb{R}^d))$ given by:

$$\begin{array}{rcl} \psi^\Lambda : \mathcal{L}_{\max}\mathcal{T}(CS^{\mathrm{rad}}_{\mathrm{c.c}}(\mathbb{R}^d)) & \to & \mathbb{C} \\ \tilde\sigma \circ B & \mapsto & -\!\!\!\!\!\!\int^{\Lambda}_{(\mathbb{R}^d)^L} \widetilde{\mathcal{R}}(\tilde\sigma) \circ B := \Lambda\left(\vec{z} \mapsto \int_{(\mathbb{R}^d)^L} \widetilde{\mathcal{R}}(\tilde\sigma)(\vec{z}) \circ B \right) \end{array}$$

ie

$$\psi^\Lambda\left((\tilde\sigma \circ B) \circledast (\tilde\sigma' \circ B') \right) = \psi^\Lambda(\tilde\sigma \circ B) \circledast \psi^\Lambda(\tilde\sigma' \circ B'),$$

which extends the ordinary multiple integral: $\int^{\Lambda}_{(\mathbb{R}^d)^L} \widetilde{\mathcal{R}}(\tilde\sigma) \circ B$ defined for symbols σ_i with negative enough orders.
Let $\mathcal{L}_L := \left(\mathbb{R}^L\right)^* \otimes \mathbb{R}^\infty = \cup_{I=1}^\infty \left(\mathbb{R}^L\right)^* \otimes \mathbb{R}^I$ denote the set of linear maps from $\mathbb{R}^L$ to $\mathbb{R}^I$ for some positive integer I. It is an algebra for the Whitney sum. Let $\mathcal{L}_L^{\max}$ be the subalgebra of linear maps with rank L. Specialising to symbols $\sigma_i(\xi) = \sigma(\xi) = \frac{1}{1+|\xi|^2}$ yields a map:

$$\begin{array}{rcl} \mathcal{L}_L^{\max} & \to & \mathbb{C} \\ B \in \left(\mathbb{R}^L\right)^* \otimes \mathbb{R}^I \in & \mapsto & -\!\!\!\!\!\!\int^{\Lambda}_{(\mathbb{R}^d)^L} \widetilde{\mathcal{R}}\left(\sigma^{\otimes I}\right) \circ B \end{array}$$

with the property that for any $B \in \left(\mathbb{R}^L\right)^* \otimes \mathbb{R}^I$ and any $C \in \left(\mathbb{R}^M\right)^* \otimes \mathbb{R}^J$

$$-\!\!\!\!\!\!\int^{\Lambda}_{(\mathbb{R}^d)^{L+M}} \widetilde{\mathcal{R}}\left(\sigma^{\otimes I+J}\right) \circ (B \oplus C) = \left(-\!\!\!\!\!\!\int^{\Lambda}_{(\mathbb{R}^d)^L} \widetilde{\mathcal{R}}\left(\sigma^{\otimes I}\right) \circ B \right) \left(-\!\!\!\!\!\!\int^{\Lambda}_{(\mathbb{R}^d)^M} \widetilde{\mathcal{R}}\left(\sigma^{\otimes J}\right) \circ C \right).$$

This multiplicative property corresponds for Feynman diagrams to the compatibility with concatenation of graphs.

Appendix : Proof of Proposition 51

To simplify notations, we set $q_i(\underline{\xi}) := \sum_{l=1}^L b_{il}\,\xi_l$ where $\underline{\xi} := (\xi_1, \cdots, \xi_L)$ and $b_i = -a_i$. For $\mathrm{Re}(b_i)$ chosen sufficiently large, we write

$$\begin{aligned}&\int_{\left(\mathbb{R}^d\right)^L}\prod_{i=1}^I \langle q_i(\underline{\xi})\rangle^{a_i}\,d\xi_1\cdots d\xi_L\\ =\;& \frac{1}{\Gamma(b_1/2)\cdots\Gamma(b_I/2)}\int_0^\infty \epsilon^{\frac{b_1}{2}-1}\cdots\epsilon^{\frac{b_I}{2}-1}\int_{\left(\mathbb{R}^d\right)^L} e^{-\sum_{i=1}^I \epsilon_i\,\langle q_i(\underline{\xi})\rangle^2}\,d\xi_1\cdots d\xi_L\end{aligned}$$

and

$$\sum_{i=1}^I \epsilon_i\langle q_i(\underline{\xi})\rangle^2 = \sum_{l,m=1}^L\sum_{i=1}^I \epsilon_i\, b_{i,l}b_{im}\xi_l\cdot\xi_m + \sum_{i=1}^I \epsilon_i = \sum_{l,m=1}^L \theta(\underline{\epsilon})_{lm}\xi_l\cdot\xi_m + \sum_{i=1}^I\epsilon_i,$$

where $\xi_l\cdot\xi_m$ stands for the inner product in $\mathbb{R}^d$ and where we have set

$$\theta(\underline{\epsilon})_{lm} := \sum_{i=1}^I \epsilon_i\, b_{il}b_{im}.$$

Since the ϵ_i are positive $\theta(\epsilon)$ is a non negative matrix, i.e. $\theta(\epsilon)(\underline{\xi})\cdot\underline{\xi}\geq 0$. It is actually positive definite since

$$\begin{aligned}&\sum_{l,m=1}^L \theta(\underline{\epsilon})_{l,m}\xi_l\cdot\xi_m = 0\\ \Rightarrow\;& \sum_{i=1}^I \epsilon_i|q_i(\underline{\xi})|^2 = 0 \Rightarrow q_i(\underline{\xi}) = 0 \quad \forall i\in\{1,\cdots,I\}\\ \Rightarrow\;& \sum_{i=1}^I |q_i(\underline{\xi})|^2 = |B\underline{\xi}|^2 = 0 \Rightarrow \underline{\xi} = 0,\end{aligned}$$

using the fact that B^*B is positive definite. The map $\xi \mapsto \sum_{l,m=1}^L \theta(\underline{\epsilon})_{lm}\,\xi_l\cdot\xi_m$ therefore defines a positive definite quadratic form of rank L.
A Gaussian integration yields $\int_{\left(\mathbb{R}^d\right)^L} e^{-\sum_{i=1}^I \epsilon_i|q_i(\underline{\xi})|^2}\,d\xi_1\cdots d\xi_L = \left(\det(\theta(\underline{\epsilon}))\right)^{-n/2}$. We want to perform the integration over $\underline{\epsilon}$:

$$\frac{1}{\Gamma(b_1/2)\cdots\Gamma(b_I/2)}\int_0^\infty d\epsilon_1\cdots\int_0^\infty d\epsilon_I\epsilon_1^{\frac{b_1}{2}-1}\cdots\epsilon_I^{\frac{b_I}{2}-1}\left(\det(\theta(\underline{\epsilon}))\right)^{-\frac{n}{2}}\; e^{-\sum_{i=1}^n\epsilon_i}.$$

Let us decompose the space $\mathbb{R}_+^k$ of parameters $(\epsilon_1,\cdots,\epsilon_I)$ in regions D_τ defined by $\epsilon_{\tau(1)}\leq\cdots\leq\epsilon_{\tau(I)}$ for permutations $\tau\in\Sigma_I$. This splits the integral $\int_0^\infty d\epsilon_1\cdots\int_0^\infty d\epsilon_I\epsilon_1^{\frac{b_1}{2}-1}\cdots\epsilon_I^{\frac{b_I}{2}-1}\left(\det(\theta(\underline{\epsilon}))\right)^{-\frac{n}{2}}\; e^{-\sum_{i=1}^n\epsilon_i}$ into a sum of integrals $\int_{D_\tau} d\epsilon_1\cdots d\epsilon_I\epsilon_1^{\frac{a_1}{2}-1}\cdots\epsilon_I^{\frac{a_I}{2}-1}\left(\det(\theta(\underline{\epsilon}))\right)^{-\frac{n}{2}}\; e^{-\sum_{i=1}^n\epsilon_i}$.
Let us focus on the integral over the domain D given by $\epsilon_1\leq\cdots\leq\epsilon_k$; the results can then be transposed to other domains applying a permutation $b_i\to a_{\tau(i)}$ on the b_i's. We write the domain of integration as a union of cones $0\leq\epsilon_{j_1}\leq\cdots\leq\epsilon_{j_I}$. For simplicity, we consider the region $0\leq\epsilon_1\leq\cdots\leq\epsilon_I$ on which we introduce new variables $t_1,\cdots,t_I$ setting $\epsilon_i = t_I t_{I-1}\cdots t_i$. These new variables vary in the domain $\Delta := \prod_{i=1}^{I-1}[0,1]\times[0,\infty)$. Let us assume that $b_{il}=0$ for $i>i_l$, then the l-th line of θ reads

$$\theta(\underline{\epsilon})_{lm} = \sum_{i=1}^I t_I\cdots t_i\, b_{il}b_{im} = \sum_{i=1}^{i_l} t_I\cdots t_i\, b_{il}b_{im} = t_I\cdots t_{i_l}\left(b_{i_l l}b_{i_l m} + \sum_{i=1}^{i_l-1} t_{i_l-1}\cdots t_i\, b_{il}b_{im}\right)$$

or equivalently the m-the column of θ reads

$$\theta(\underline{\epsilon})_{lm} = \sum_{i=1}^{I} t_I \cdots t_i \, b_{il} b_{im} = \sum_{i=1}^{i_m} t_I \cdots t_i \, b_{il} b_{im} = t_I \cdots t_{i_m} \left(b_{i_l l} b_{i_l m} + \sum_{i=1}^{i_m - 1} t_{i_m - 1} \cdots t_i \, b_{il} b_{im} \right).$$

Factorising out $\sqrt{t_I \cdots t_{i_l}}$ from the l-th row and $\sqrt{t_I \cdots t_{i_m}}$ from the m-th column for every $l, m \in [[1, L]]$ produces a symmetric matrix $\tilde{\theta}(\underline{t})$.
Following [Sp] we show that its determinant does not vanish on the domain of integration; if it did vanish at some point $\underline{\tau}$, $\tilde{\theta}(\underline{\tau})$ would define a non injective map $\tilde{\theta}(\underline{\tau}) : (x_1, \cdots, x_L) \mapsto \left(\sum_{l=1}^{L} \tilde{\theta}(\underline{\tau})_{1l} x_l, \cdots, \sum_{l=1}^{L} \tilde{\theta}(\underline{\tau})_{Ll} x_l \right)$, i.e. there would be some non zero L-tuple $\underline{w} := (x_1, \cdots, x_L) \in \mathbb{R}^L$ such that $\tilde{\theta}(\underline{\tau})(\underline{x}) = 0$ which would in turn imply that $\sum_{l=1}^{L} \sum_{m=1}^{L} x_l \left(\tilde{\theta}(\underline{\tau}) \right)_{lm} x_m = \underline{x} \cdot \tilde{\theta}(\underline{\tau})(\underline{x}) = 0$. From there we would infer that

$$\begin{aligned} & \sum_{l=1}^{L} \sum_{m=1}^{M} \sum_{i=1}^{I} \tau_I \cdots \tau_i \, b_{il} b_{im} x_l x_m = \sum_{i=1}^{I} \left(\sum_{l=1}^{L} \sqrt{\tau_I \cdots \tau_i} \, b_{il} x_l \right)^2 = 0 \\ \Longrightarrow \quad & \sum_{l=1}^{L} \sqrt{\tau_I \cdots \tau_i} \, b_{il} x_l = 0 \qquad (17.188) \\ \Longrightarrow \quad & \sum_{l=1}^{L} \left(b_{i_l l} x_l + \sqrt{\tau_{i_{l-1}} \cdots \tau_i} \, b_{il} x_l \right) = 0 \quad \forall i \in [[1, I]], \qquad (17.189) \end{aligned}$$

where we have factorised out $\tau_I \cdots \tau_{i_l}$ in the last expression. Let us as in [Sp] choose $M = \max\{l, x_l \neq 0\}$; in particular $l > M \Rightarrow x_l = 0$. On the other hand, since $l < M \Rightarrow i_l < i_M$ we have $l < M \Rightarrow b_{i_M l} = 0$. Choosing $i = i_M$ in (17.188) reduces the sum to one term $b_{i_M M} x_M$ which would therefore vanish, leading to a contradiction since neither $b_{i_M M}$ nor x_M vanish by assumption.
We thereby conclude that $\det \tilde{\theta}(\underline{t})$ does not vanish on the domain of integration.
Performing the change of variable $(\epsilon_1, \cdots, \epsilon_I) \mapsto (t_1, \cdots, t_I)$ in the integral, which introduces a jacobian determinant $\prod_{i=1}^{I} t_i^{i-1}$, we write the integral:

$$\begin{aligned} & \frac{1}{\Gamma(b_1/2) \cdots \Gamma(b_I/2)} \int_0^1 dt_1 \cdots \int_0^1 dt_{I-1} \int_0^\infty dt_I \prod_{i=1}^{I} t_i^{i-1} \prod_{l=1}^{L} (t_I \cdots t_{i_l})^{-\frac{d}{2}} \\ & \cdot \prod_{i=1}^{I} (t_I \cdots t_i)^{\frac{b_i}{2} - 1} e^{-\sum_{i=1}^{I} t_I \cdots t_i} \left(\det \tilde{\theta}(t) \right)^{-n/2} \\ = \quad & \frac{1}{\Gamma(b_1/2) \cdots \Gamma(b_I/2)} \int_0^\infty dt_I \int_0^1 dt_1 \cdots \int_0^1 dt_{I-1} \\ & \prod_{i=1}^{I} t_i^{\frac{b_1 + \cdots + b_i}{2} - 1} (t_I \cdots t_{i_L})^{-n \frac{L}{2}} (t_{i_L - 1} \cdots t_{i_{L-1}})^{-n \frac{L-1}{2}} \cdots (t_{i_2} \cdots t_{i_1})^{-\frac{n}{2}} h(\underline{t}) \\ = \quad & \frac{1}{\Gamma(b_1/2) \cdots \Gamma(b_I/2)} \int_\Delta dt_I \cdots dt_1 \prod_{i=1}^{I} t_i^{\frac{b_1 + \cdots + b_i - d\, r_i}{2} - 1} h(\underline{t}) \qquad (17.190) \end{aligned}$$

where the $r_i' s$ are positive integers depending on the size and shape of the matrix B (via the i_l's) [18] and where we have set

$$h(\underline{t}) := e^{-\sum_{i=1}^{I} t_I \cdots t_i} \left(\det \tilde{\theta}(t) \right)^{-d/2} = \left(\det \tilde{\theta}(t) \right)^{-d/2} \prod_{i=1}^{I} e^{-t_I \cdots t_i}.$$

Since $\det \tilde{\theta}(t)$ is polynomial in the t_i's, the convergence of the integral in t_I at infinity is taken care of by the function $e^{-t_I \cdots t_1}$ arising in h. On the other hand, h is smooth on the domain of integration

[18] The integers r_i's do not depend on the explicit coefficients of the matrix.We have $i_l \geq l$ so that $r_i \leq i$; in particular, $\mathrm{Re}(a_i) < -n \Rightarrow \mathrm{Re}(b_1) + \cdots + \mathrm{Re}(b_i) - d\, r_i \geq \mathrm{Re}(b_1) + \cdots + \mathrm{Re}(b_i) - d\, i > 0$ so that as expected, the above integral converges.

since it is clearly smooth outside the set of points for which $\det\tilde{\theta}(\underline{t})$ vanishes, which we saw is a void set. Thus, the various integrals converge at $t_i = 0$ for $\mathrm{Re}(b_i)$ sufficiently large.
Integrating by parts with respect to each $t_1, \cdots, t_I$ introduces factors $\frac{1}{b_1+\cdots+b_i-d\,r_i+2m_i}$, $m_i \in \mathbb{N}_0$ when taking primitives of $t_i^{\frac{b_1+\cdots+b_i-d\,r_i}{2}-1}$ and differentiating $h(\underline{t})$.
We thereby build a meromorphic extension $⨍_{\mathbb{R}^{nL}} \prod_{i=1}^k \langle \sum_{l=1}^L b_{il}\xi_l\rangle^{a_i}$ to the whole complex plane as a sum over permutations $\tau \in \Sigma_I$ of expressions:

$$\frac{1}{\prod_{i=1}^I \Gamma(b_i)} \left(\frac{\int_\Delta \prod_{i=1}^k t_i^{\frac{b_{\tau(1)}+\cdots+b_{\tau(i)}-n\,s_{i\tau(i)}}{2}+m_i}\, h_\tau^{(m_1+\cdots+m_I)}(\underline{t})}{\prod_{i=1}^I \left((b_{\tau(1)}+\cdots+b_{\tau(i)}-d\,r_{\tau,i})\cdots(b_{\tau(1)}+\cdots+b_{\tau(i)}-d\,r_{\tau,i}+2m_i)\right)} \right.$$
$$\left. + \quad \text{boundary} \quad \text{terms}\right),$$

where the boundary terms on the domain Δ are produced by the iterated m_i integrations by parts in each variable t_i. Here $r_{\tau,i} \leq i$ is a positive integer depending on τ and the shape of the matrix and we have chosen the m_i's sufficiently large for the term $\int_\Delta \prod_{i=1}^I t_i^{\frac{b_{\tau(1)}+\cdots+b_{\tau(i)}-d\,r_{\tau,i}}{2}+m_i} h_\tau^{(m_1+\cdots+m_I)}(\underline{t})$ to converge. The boundary terms are of the same type, namely they are proportional to

$$\frac{\int_{\Delta'} \prod_{i=1}^I t_i^{\frac{b_{\tau(1)}+\cdots+b_{\tau(i)}-ns_{\tau,i}}{2}+m'_i}\, h^{(m'_1+\cdots+m'_k)}(\underline{t})}{\prod_{i=1}^k \left((b_{\tau(1)}+\cdots+b_{\tau(i)}-d\,r_{\tau,i})\cdots(b_{\tau(1)}+\cdots+b_{\tau(i)}-d\,r_{\tau,i}+2m'_i)\right)}$$

for some domain $\Delta' = \prod_{i=1}^{I'-1}[0,1] \times [0,\infty[$ for some $I' < I$ or $\Delta' = \prod_{i=1}^{I'-1}[0,1]$ for some $I' \leq I$ and some non negative integers $m'_i \leq m_i$ with at least one $m'_{i_0} < m_{i_0}$.

This produces a meromorphic map which on the domain $\cap_{i=1}^I \{\mathrm{Re}(b_{\tau(1)}+\cdots+b_{\tau(i)}) + 2m_i > dr_{\tau,i}\}$ reads

$$\frac{1}{\prod_{i=1}^I \Gamma(b_i)} \sum_{\tau\in\Sigma_I} \frac{H_{\tau,\underline{m}}(b_1,\cdots,b_I)}{\prod_{i=1}^I \left((b_{\tau(1)}+\cdots+b_{\tau(i)}-d\,r_{\tau,i})\cdots(b_{\tau(1)}+\cdots+b_{\tau(i)}-d\,r_{\tau,i}+2m_i)\right)}$$

with $H_{\tau,\underline{m}}$ holomorphic on that domain. It therefore extends to a meromorphic map on the whole complex space with simple simple poles on a countable set of affine hyperplanes $\{a_{\tau(1)}+\cdots+a_{\tau(i)} + d\,r_{\tau,i} \in 2\,\mathbb{N}_0\}$, where as beofre, the $s_{\tau,i}$'s are integers which depend on the permutation τ and on the size $L \times I$ shape (i.e. on the l_i's) but not on the actual coefficients of the matrix.
Let us further observe that since $r_{\tau,i} \leq i$, if $\mathrm{Re}(a_i) < -d \Rightarrow \mathrm{Re}(b_i) > d$ for any $i \in \{1,\cdots,I\}$, then for any $\tau \in \Sigma_I$ we have $\mathrm{Re}(b_{\tau(1)}+\cdots+b_{\tau(i)}) - d\,r_{\tau,i} > 0$ so that we recover the fact that the map $(a_1,\cdots,a_I) \mapsto ⨍_{(\mathbb{R}^d)^L} \prod_{i=1}^k \langle \sum_{l=1}^L b_{il}\xi_l\rangle^{a_i}$ is holomorphic on the domain $D := \{\underline{a} = (a_1,\cdots,a_I) \in \mathbb{C}^I, \mathrm{Re}(a_i) < -d, \quad \forall i \in \{1,\cdots,I\}\}$. $\square$

PART III: Regularised traces

18 From symbols to pseudodifferential operators on manifolds

We extend the notion of pseudodifferential symbol on $\mathbb{R}^d$ to symbols with varying coefficients on an open subset of $\mathbb{R}^d$ which patched up using a partition of the unity, lead to pseudodifferential operators on a closed manifold.

18.1 Pseudodifferential operators on an open subset of $\mathbb{R}^d$

Definition 27 *A smooth function $\sigma \in C^\infty(U \times \mathbb{R}^d)$ is a* **scalar symbol on** U *whenever the following condition is statisfied. There is some real constant a such that for any multiindices $\gamma, \delta \in \mathbb{Z}_+^d$ for any compact subset $K \subset U$, there exists a constant $C_{\gamma,\delta,K} \in \mathbb{R}^+$ such that for any $x \in K, \xi \in \mathbb{R}^d$*

$$|\partial_x^\gamma \partial_\xi^\delta \sigma(x,\xi)| \leq C_{\gamma,\delta,K} \langle \xi \rangle^{a-|\delta|} \tag{18.191}$$

where we have set $\langle \xi \rangle := \sqrt{1+|\xi|^2}$.

Example 25 *The polynomial*

$$\sigma(x,\xi) = \sum_{|\alpha| \leq a} c_\alpha(x) \xi^\alpha$$

with $c_\alpha \in C^\infty(U)$ is a symbol since it satisfies condition (18.191).

Let $\mathcal{S}^a(U) \subset C^\infty(T^*U)$ denote the set of scalar valued symbols on U which fulfill condition (18.191). Let us further set $\mathcal{S}^a(U,V) := \mathcal{S}^a(U) \otimes_K \mathrm{End}(V)$ for any K-linear space V and

$$\mathcal{S}(U,V) := \bigcup_{a \in \mathbb{R}} \mathcal{S}^a(U,V).$$

With these notations we have $\mathcal{S}^a(U) = \mathcal{S}^a(U,K)$; $\mathcal{S}(U) = \mathcal{S}(U,K)$.

Remark 26 *For two real numbers a_1, a_2, we have*

$$a_1 \leq a_2 \Longrightarrow \mathcal{S}^{a_1}(U) \subset \mathcal{S}^{a_2}(U).$$

We call a symbol in the intersection

$$\mathcal{S}^{-\infty}(U) := \bigcap_{a \in \mathbb{R}} \mathcal{S}^a(U); \quad \mathcal{S}^{-\infty}(U,V) := \bigcap_{a \in \mathbb{R}} \mathcal{S}^a(U,V)$$

a *smoothing symbol*.

Example 26 *Symbols $\sigma(x,\xi)$ with compact support in ξ are smoothing symbols.*

Equality modulo smoothing symbols, i.e. the relation $\sigma \sim \sigma'$ defined for two symbols σ and σ' by $\sigma - \sigma' \in \mathcal{S}^{-\infty}(U,V)$ is an equivalence relation in $\mathcal{S}(U,V)$. For symbols $\sigma \in \mathcal{S}(U,V), \sigma_k \in \mathcal{S}(U,V), k \in \mathbb{N}_0$ we set

$$\sigma \sim \sum_{k \in \mathbb{N}_0} \sigma_k \Leftrightarrow \left(\forall \alpha \in \mathbb{R}, \quad \exists K(\alpha) \in \mathbb{N}, \text{ s.t.} \quad K \geq K(\alpha) \Rightarrow \sigma - \sum_{k \leq K} \sigma_k \in \mathcal{S}^\alpha(U,V) \right).$$

A symbol in $\mathcal{S}(U,V)$ is classical (resp. log-polyhomogeneous of type k) of complex order a if for any fixed x in U, the map $x \mapsto \sigma(x,\cdot)$ lies in $CS^a_{\mathrm{c.c}}(\mathbb{R}^d) \otimes \mathrm{End}(V)$ (resp. $CS^{a,k}_{\mathrm{c.c}}(\mathbb{R}^d) \otimes \mathrm{End}(V)$. More explicitely, we set the following defintion.

Definition 28 *Let a be a complex number.*

1. *A symbol $\sigma \in \mathcal{S}(U,V)$ is* **classical** *(or* **polyhomogeneous***) of* order a *if:*

$$\sigma(x,\xi) \sim \sum_{j=0}^\infty \chi(\xi)\, \sigma_{a-j}(x,\xi) \tag{18.192}$$

where

- *χ is some smooth function on $\mathbb{R}^d$ such that χ vanishes in a small neighborhood of 0 and χ is identically one outside the unit ball,*
- *$\sigma_{a-j} \in C^\infty(T^*U, \mathrm{End}(V))$ is positively homogeneous of order $a-j$, i.e.*

$$\sigma_{a-j}(x, t\xi) = t^{a-j}\sigma_{a-j}(x,\xi)$$

for any $t>0$, any $x \in U$ and any $\xi \in T_x^*U - \{0\}$.

2. *A symbol $\sigma \in \mathcal{S}(U,V)$ is* **log-polyhomogeneous of log-class** *k for some non integer integer k if*

$$\sigma(x,\xi) \sim \sum_{l=0}^{k}\sum_{j=0}^{\infty} \chi(\xi)\, \sigma_{a-j,l}(x,\xi)\, \log^l|\xi|$$

with $\sigma_{a-j,l}(x,\xi), l = 0,\cdots,k$ positively homogeneous functions of order $a-j$.

Let $CS^{a,k}(U,V) \subset S(U,V)$ denote the class of log-polyhomogeneous symbols of order a and log-class k and let us set $CS^{a,*}(U,V) := \bigcup_{k\in \mathbb{N}_0} CS^{a,k}(U,V)$. Then $CS^a(U,V) = CS^{a,0}(U,V)$ stands for the set of classical symbols of order a.
The set $\mathcal{S}(U,V)$ is stable under the following star-product of symbols which is defined modulo smoothing symbols by:

$$\sigma \star \tau \sim \sum_\gamma \frac{(-i)^{|\alpha|}}{\gamma!}\, \partial_\xi^\gamma \sigma(x,\xi) \,\circ\, \partial_x^\gamma \tau(x,\xi), \tag{18.193}$$

wher e$\circ$ stands for the composition in $\mathrm{End}(V)$. If σ is of order a and τ of order b then $\sigma\star\tau$ is of order $a+b$. The leading symbols σ_a and τ_b of σ and τ mutliply under this product:

$$(\sigma\star\tau)_{a+b} = \sigma_a\cdot\tau_b.$$

The star product preserves the set of classical (resp. of log-polyhomogeneous symbols); since these sets are not stable under summation, we consider the algebras

$$CS(U,V) := \langle \cup_{a\in\mathbb{C}} CS^a(U,V)\rangle; \quad CS^{*,*}(U,V) := \langle \cup_{a\in\mathbb{C}} CS^{a,*}(U,V)\rangle$$

generated by the corresponding unions of classical (resp. log-polyhomogeneous) symbols on U.
To a symbol $\sigma \in \mathcal{S}(U,V) \subset C^\infty(U\times\mathbb{R}^d)\otimes \mathrm{End}(V)$ corresponds a linear operator

$$\begin{array}{rcl} \mathrm{Op}(\sigma): C_c^\infty(U,V) & \to & C^\infty(U,V)\\ u & \mapsto & \mathrm{Op}(\sigma)(u)(x) := \mathcal{F}^{-1}\left(\sigma(x,\cdot)\,\hat{u}\right)\end{array}$$

called a **pseudodifferential operator** on U with coefficients in $\mathrm{End}(V)$. We call it classical (resp. log-polyhomogeneous) of order $a\in\mathbb{C}$ (resp. and of log-type $k\in\mathbb{N}_0$) if the symbol σ has order a and lies in the corresponding class of symbols.

Example 27 *A differential operator $A = \sum_{|\alpha|\le a} c_\alpha D_\xi^\alpha$ on U with coefficients c_α in $C^\infty(U,\mathrm{End}(V))$ and where we have set $D_\xi^\alpha = (-i)^\alpha\partial_x^\alpha$, is a pseudodifferential operator with coefficients in End(V) whose symbol reads $\sigma(x,\xi) = \sum_{|\alpha|\le a} c_\alpha \xi^\alpha$.*

For any symbol σ,

$$\begin{aligned} \mathrm{Op}(\sigma)u(x) &= \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\sigma(x,\xi)\hat{u}(\xi)đ\xi \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\langle x-y,\xi\rangle}\sigma(x,\xi)u(y)đy\, đ\xi \\ &= \int_{\mathbb{R}^d} K(x,y)u(y)\,dy, \end{aligned} \tag{18.194}$$

where

$$K(x,y) := \int_{\mathbb{R}^d} e^{i\langle x-y,\xi\rangle}\sigma(x,\xi)đ\xi$$

is the **kernel** of the operator $\mathrm{Op}(\sigma)$.
In general, this is a distribution kernel, whose singularities can be shown to be located on the diagonal. In particular, for two smooth functions $\chi, \tilde{\chi}$ with disjoint compact suports, the operator $\chi\,\mathrm{Op}(\sigma)\,\chi$ has smooth kernel.
A linear operator

$$\begin{aligned} A : C^\infty_{\mathrm{cpt}}(U, V) &\rightarrow C^\infty(U, V) \\ u &\mapsto Au(x) := \int_{\mathbb{R}^d} K(x, y)\, u(y)\, dy \end{aligned}$$

defined by a smooth kernel K is called a **smoothing operator**.
The operator $\mathrm{Op}(\sigma)$ does not generally take values in $C^\infty_{cpt}(U)$ but it does if σ has compact support in U. Let us denote by $\mathcal{S}_{\mathrm{cpt}}(U)$ the subset of compactly supported symbols on U.

Remark 27 *A pseudodifferential operator* $\mathrm{Op}(\sigma)$ *with compactly supported symbol* σ *in* U*, maps a smooth function with compact support in* U *to a function with compact support in* U*. It thereby defines a properly supported operator in* U *since the formal adjoint of its symbol, which can be expressed in terms of derivatives of the original symbol (see e.g. [Gi]), also has compact support. Indeed, a pseudodifferential operator* A *is properly supported (see e.g. [Sh], [Ta], [Tr]) in* U *if it sends a smooth function with compact support in* U *to a function who has support in some compact subset of* U*, and the same property holds for the formal adjoint of* A*.*

Unlike a differential operator A which is local in the sense that if u vanishes on U then Au also vanishes on U, pseudodifferential operators are not local since they are defined by Fourier transforms which smear out the support. However, a pseudodifferential operator A is *pseudo-local* in the following sense [Gi]. Given any open subset $V \subset U$ and $\chi, \tilde{\chi} \in C^\infty_{\mathrm{cpt}}(V)$, then

$$\chi u \in C^\infty_{\mathrm{cpt}}(V) \Longrightarrow \tilde{\chi}\, Au \in C^\infty_{\mathrm{cpt}}(V) \quad \forall u \in C^\infty(U).$$

18.2 Basic properties of pseudodifferential operators

We state some basic properties of pseudodifferential operators acting on functions with support in a compact set U. These properties easily extend to pseudodifferential operators on smooth maps $C^\infty(M, V)$ where V is some linear space.

Definition 29 *A pseudodifferential operator* A *on* U *is a linear operator* $A : C^\infty_{\mathrm{cpt}}(U) \to C^\infty(U)$ *of the form*

$$A = \mathrm{Op}(\sigma(A)) + R_A$$

for some smoothing operator R_A *and some compactly supported symbol* σ_A*, called the symbol of* A *which is determined up to a smoothing symbol.*

Thus, in contrast to a differential operator which has well-defined symbol, the symbol of a pseudodifferential operator only makes sense modulo smoothing symbols.
The **leading symbol** of a pseudodifferential operator A of order a on U is defined by

$$\sigma^L(A)(x, \xi) = \lim_{t \to \infty} t^{-a} \sigma_A(x, t\xi) \quad \forall (x, \xi) \in T^*U.$$

A pseudodifferential operator with a classical (resp. log-polyhomogeneous symbol of type k) symbol is called classical (resp. log-polyhomogeneous of type k). It is called elliptic if $\sigma^L(x, \xi)$ is invertible for any $x \in U, \xi \in T - x^*U - \{0\}$.
Since a compactly supported pseudodifferential operator sends functions with compact support to functions with compact support, we can compose two compactly supported operators.

Proposition 52 *The product of two pseudodifferential operators* $A = \mathrm{Op}(\sigma(A))$ *of order* a*,* $B = \mathrm{Op}(\sigma(B))$ *of order* b*, with compactly supported symbols* $\sigma(A)$ *and* $\sigma(B)$ *on* U*, is a pseudodifferential operator of order* $a + b$ *with symbol*

$$\sigma_{AB} \sim \sigma_A \star \sigma_B. \tag{18.195}$$

More precisely,

$$A\,B=\sum_{|\alpha|\le N-1}\frac{(-i)^{|\alpha|}}{\alpha!}\mathrm{Op}(\partial_\xi^\alpha\sigma_A\,\partial_x^\alpha\sigma_B)+R_N(A\,B), \tag{18.196}$$

for some operator $R_N(A,B)$ of order $\le a+b-N$ depending on A and B. In particular,

$$\sigma^L(AB)=\sigma^L(A)\,\sigma^L(B).$$

The composition of two pseudodifferental operators $\mathrm{Op}(\sigma(A))+R_A$ and $\mathrm{Op}(\sigma(B))+R_B$ on U then follows by (bi)linearity from combining the composition of two compactly supported operators $\mathrm{Op}(\sigma(A))$ and $\mathrm{Op}(\sigma(B))$ with composition with smoothing operators.

The symbol of a pseudodifferential operator does not transform covariantly under change of parametrisation.
For two open subsets U,U' of $\mathbb{R}^d$, a pseudodifferential operator A on U and a diffeomorphism $\kappa:U\to U'$, we set

$$\kappa_*A=\kappa_*\circ A\circ\kappa^*$$

where $\kappa^*u:=u\circ\kappa$ and $\kappa_*u=u\circ\kappa^{-1}$ for any $u\in C^\infty(U)$. Note that $d\kappa(x):T_xU\to T_{\kappa(x)}U'$ and $(d\kappa(x))^t:T^*_{\kappa(x)}U'\to T^*_xU$.
The following proposition tells us in how far $\kappa_*\mathrm{Op}(\sigma)$ differs from $\mathrm{Op}(\kappa_*\sigma)$ where

$$\kappa_*\sigma(x',\xi'):=\sigma(\kappa^{-1}(x'),d\kappa(\kappa^{-1}(x'))^t\,\xi'). \tag{18.197}$$

Proposition 53 *Let σ be a compactly supported symbol on U. Then $\kappa_*\mathrm{Op}(\sigma)$ has (compactly supported) symbol*

$$\widetilde{\kappa_*\sigma}(x',\xi'):=\kappa_*\sigma(x',\xi')+\sum_{|\alpha|>0}\frac{1}{\alpha!}\,\phi_\alpha(\kappa^{-1}(x),\eta)\,\partial_2^\alpha\sigma\left(\kappa^{-1}(x'),(d\kappa(\kappa^{-1}(x'))^t\,\eta\right). \tag{18.198}$$

Here $\phi_\alpha(x',\xi')$ is a polynomial in ξ of degree $\le\frac{|\alpha|}{2}$ with $\phi_0(x',\xi')=1$.

Remark 28 *In particular, $\widetilde{\kappa_*\sigma}(x',\xi')-\kappa_*\sigma(x',\xi')$ is of the form $\sum_{|\beta|\le\frac{|\alpha|}{2}}a_{\alpha,\beta}(x)\,\xi^\beta\,\partial_\xi^\alpha\sigma(\cdot,\xi)$, for some smooth functions $a_{\alpha,\beta}$ and where we have set $x'=\kappa(x)$, $\xi=d\kappa(\kappa^{-1}(x'))^t\,\xi'$, an observation which turns out to play a crucial role in the following.*

Idea of proof: We need to show that $\kappa_*\mathrm{Op}(\sigma)$ differs from $\mathrm{Op}\left(\widetilde{\kappa_*\sigma}\right)$ by a smoothing operator.
To motivate the result (we refer e.g. to [Gi] for a complete proof), let us first assume that κ is linear. We set $x'=\kappa(x),y'=\kappa(y)$ and $\xi=\kappa^t\xi'$, with κ^t the transpose matrix. With these notations we have

$$dx'\,d\xi'=|\mathrm{det}\kappa|\,dx\,|\mathrm{det}\kappa|^{-1}\,d\xi=dx\,d\xi$$

and

$$\langle x'-y',\xi'\rangle=\langle x'-y',\left(\kappa^t\right)^{-1}\xi'\rangle=\langle\kappa^{-1}(x'-y'),\xi'\rangle=\langle x-y,\xi\rangle.$$

With the above notations and $\kappa_*\sigma$ as in (18.197) we have

$$\kappa_*\sigma(x',\xi')=\sigma(x,\xi)$$

and

$$\begin{aligned}(\kappa_*\mathrm{Op}(\sigma))\,u'(x') &= (\kappa_*\circ A\circ\kappa^*)\;u'(\kappa(x))\\ &= \mathrm{Op}(\sigma)\,(\kappa^*u')\,(x)\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{i\,(x-y)\cdot\xi}\sigma(x,\xi)\,u'(\kappa(y))\,dy\,d\xi\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{i\,(x'-y')\cdot\xi'}\kappa_*\sigma(x',\xi')\,u'(y')\,dy'\,d\xi'\end{aligned}$$

so that in this case

$$\kappa_* \mathrm{Op}(\sigma) = \mathrm{Op}\left(\kappa_* \sigma\right).$$

When κ is non linear the situation is more complicated and requires introducing the notion of amplitude which we choose to avoid here. We write

$$\begin{aligned} x - y &= \kappa^{-1}(x') - \kappa^{-1}(y') \\ &= \int_0^1 \partial_t \kappa^{-1}(tx' + (1-t)y')\, dt \\ &= \left(\int_0^1 d\kappa^{-1}(tx' + (1-t)y')\, dt\right)(x' - y') \\ &= T(x', y')(x' - y') \end{aligned}$$

with $T(x', y') := \int_0^1 d\kappa^{-1}(tx' + (1-t)y')\, dt$ a smooth square matrix valued function of x' and y' with $T(x', x') = d(\kappa^{-1})(x')$ is invertible since κ is a diffeomorphism. Setting $\eta := T(x', y')^t \xi$ we write

$$\langle x - y, \xi \rangle = \langle T(x', y')(x' - y'), \xi \rangle = \langle x' - y', \eta \rangle.$$

Moreover, there is a neighborhood W of the diagonal on which T is invertible and on which we can write

$$d\, y\, d\xi = |\det(T(y', y'))\, |\det(T(x', y'))|^{-1}\, dy'\, d\eta.$$

Further setting $\sigma'(x', \eta) := \sigma(x, \xi)$ we can write similarly to the linear case:

$$\begin{aligned} (\kappa_* \mathrm{Op}(\sigma)) u'(x') &= (\kappa_* \circ \mathrm{Op}(\sigma) \circ \kappa^*) u'(\kappa(x)) \\ &= \mathrm{Op}(\sigma)\,(\kappa^* u')\,(x) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\,\langle x - y, \xi \rangle} \sigma(x, \xi)\, u'(\kappa(y))\, dy\, đ\xi \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\,\langle x' - y', \eta \rangle} \sigma'(x', \eta)\, u'(y')\, |\det(T(y', y'))\, |\det(T(x', y'))|^{-1}\, dy'\, đ\eta \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Xi(x', y')\, e^{i\,\langle x' - y', \eta \rangle} \sigma'(x', \eta)\, u'(y')\, |\det(T(y', y'))\, |\det(T(x', y'))|^{-1}\, dy'\, đ\eta \\ &+ (Ku')(x') \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\,\langle x' - y', \eta \rangle}\, a'(x', y', \eta)\, u'(y')\, dy'\, đ\eta + (Ku')(x'), \end{aligned}$$

where we have set [19]:

$$\begin{aligned} a'(x', y', \eta) &:= \sigma'(x', \eta)\, \Xi(x', y')\, |\det(T(y', y'))\, |\det(T(x', y'))|^{-1} \\ &= \sigma(\kappa^{-1}(x'),\, T^{-1}(x', y')^t\, \eta)\, \tilde{\Phi}(x', y'). \end{aligned} \tag{18.199}$$

Here $\tilde{\Phi}(x', x') := \Xi(x', y')\, |\det(T(y', y'))\, |\det(T(x', y'))|^{-1}$ where Ξ is a smooth function on $U \times U$ which is identically one in a neighborhood of the diagonal and with support contained in the open neighborhood W of the diagonal on which T is invertible. The remaining error term

$$(Ku')(x') := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 - \Xi(x', y'))\; e^{i\,\langle x' - y', \eta \rangle} \sigma'(x', \eta)\, u'(y')\, |\det(T(y', y'))\, |\det(T(x', y'))|^{-1}\, dy'\, d\eta$$

defines a smoothing operator.
On the other hand, one can show (see e.g. [Sh] Theorem 4.2) that $\mathrm{Op}(a') := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\,\langle x' - y', \eta \rangle}\, a'(x', y', \eta)\, u'(y')\, dy'\, đ\eta$

[19] The following expression defines a compactly supported amplitude.

defines an operator on $C^\infty_{\rm cpt}(U)$ with compactly supported symbol

$$\begin{aligned}
\widetilde{\kappa_*\sigma}(x',\eta) &\sim \sum_\alpha \frac{1}{\alpha!}\,\partial_2^\alpha\sigma\left(\kappa^{-1}(x'),(d\kappa(\kappa^{-1}(x'))^t\,\eta\right)\,\left(D_z e^{i\langle \kappa''_x(z),\eta\rangle}\right)_{|z=x}\\
&\sim \sum_\alpha \frac{1}{\alpha!}\,\phi_\alpha(\kappa^{-1}(x'),\eta)\,\partial_2^\alpha\sigma\left(\kappa^{-1}(x'),(d\kappa(\kappa^{-1}(x'))^t\,\eta\right),\\
&\sim \sigma\left(\kappa^{-1}(x'),(d\kappa(\kappa^{-1}(x'))^t\,\eta\right)+\sum_{|\alpha|>0}\frac{1}{\alpha!}\,\phi_\alpha(\kappa^{-1}(x'),\eta)\,\partial_2^\alpha\sigma\left(\kappa^{-1}(x'),(d\kappa(\kappa^{-1}(x'))^t\,\eta\right),\\
&\sim \kappa_*\sigma\,(x',\eta)+\sum_{|\alpha|>0}\frac{1}{\alpha!}\,\phi_\alpha(\kappa^{-1}(x'),\eta)\,\partial_2^\alpha\sigma\left(\kappa^{-1}(x'),(d\kappa(\kappa^{-1}(x'))^t\,\eta\right),
\end{aligned}$$

where $\kappa''_x(z):=\kappa(z)-\kappa(x)-d\kappa(x)(z-x)$ corresponds to the "non linear part" of κ and $\phi_\alpha(x,\eta):=D_z^\alpha e^{i\langle\kappa''_x(z),\eta\rangle}_{|z=x}$ is a polynomial of degree $\leq\frac{|\alpha|}{2}$ whose value is one for $\alpha=0$. $\square$

Even though the symbol of an operator is modified by a change of coordinates, some of its features are preserved.

Corollary 10 *Let σ be a compactly supported symbol on U and let $A:=\mathrm{Op}(\sigma)$.*

1. *κ_*A and A have same order given by the common order of $\widetilde{\kappa_*\sigma}$, $\kappa_*\sigma$ and σ.*
2. *If A is smoothing so is κ_*A.*
3. *If σ is a classical (resp. log-polyhomogeneous of type k) then so is $\widetilde{\kappa_*\sigma}$ so that the type of A is invariant under a coordinate change.*
4. *The leading symbol of A transforms covariantly*

$$\sigma_L(\kappa_*A)=\kappa_*\sigma_L(A).$$

18.3 Pseudodifferential operators on manifolds

Let M be a smooth d-dimensional closed manifold. Recall that a coordinate chart (U,ϕ) on M is an open subset of U of M and a diffeomorphism $\phi:U\to\phi(U)\subset\mathbb{R}^d$. An atlas is a collection $\{U_i,\phi_i,\chi_i,i\in I\}$ where the open subsets $U_i, i\in I$ form an open cover of M, for each $i\in I$, (U_i,ϕ_i) is a coordinate chart and $\{\chi_i,i\in I\}$ is a partition of unity subordinated to the covering.

Definition 30 *We call localisation subordinated to a chart (U,ϕ) around a point x in M of a linear operator $A:C^\infty(M)\mapsto C^\infty(M)$, any map $\chi\,A\,\tilde\chi$ where χ and $\tilde\chi$ are smooth functions with compact support in U which are identically one in a neighborhood of x.*

Remark 29 *Recall that if χ and $\tilde\chi$ had disjoint supports, then the operator $\chi\,A\,\tilde\chi$ would be smoothing.*

Definition 31 *A linear operator $A:C^\infty(M)\mapsto C^\infty(M)$ is called* pseudodifferential *on M if given any local chart (U,ϕ), any localisation $A_U:=\chi\,A\,\tilde\chi$ subordinated to this chart, the induced localised operator*

$$A_{\phi(U)}:f\mapsto\phi^*\circ A_U\circ\phi_*(f),$$

where $\phi^*f:=f\circ\phi$, is a pseudodifferential operator on $\phi(U)$.
The symbol $\sigma_\phi(A)(x,\cdot)$ of A in a given local chart (U,ϕ) around $x\in U$ is defined by the symbol of $A_{\phi(U)}$.

With these definitions at hand, we can write a pseudodifferential operator

$$A=\sum_{\mathrm{Supp}(\chi_i)\cap\mathrm{Supp}(\chi_j)=\phi}\chi_i\,A\,\chi_j+R(A)=\sum_{i,j\ \mathrm{s.t.}\ \mathrm{Supp}(\chi_i)\cap\mathrm{Supp}(\chi_j)=\phi}\mathrm{Op}(\sigma_{ij})+R(A)\qquad(18.200)$$

where σ_{ij} are compactly supported symbols with supports in $\phi_i(U_i)\cap\phi_j(U_j)$ and $R(A)$ is a smoothing operator.

Features of pseudodifferential operators of the type $\mathrm{Op}(\sigma)$ which are preserved under diffeomorphisms can be extended to psuedodifferential operators on manifolds. Let us make this statement more precise. If (U,ϕ) and (U',ϕ') are two local coordinate charts, then setting $\kappa := \phi' \circ \phi^{-1}$, on the intersection $U \cap U'$ we have

$$A_{\phi'(U\cap U')} = \kappa^* \circ A_{\phi(U\cap U')} \circ \kappa_* = \kappa_* A_{\phi(U\cap U')}. \tag{18.201}$$

By Proposition 53 and its Corollary 10, $A_{\phi'(U)}$ and $A_{\phi\prime(U)}$ are of the same type (classical, log-polyhomogeneous), have the same order and differ by a pseudodifferential operator of order strictly smaller. It therefore makes sense to set the following definitions.

Definition 32 *Let A be a pseudodifferential operator on M.*

1. *A is classical (resp. log-polyhomogeneous of type k) if if given any local chart (U,ϕ), any localisation A_U subordinated to this chart, the induced localised operator $A_{\phi(U)}$ is classical (resp. log-polyhomogeneous of type k).*
2. *A has order a if given any local chart (U,ϕ), any localisation A_U subordinated to this chart, the induced localised operator $A_{\phi(U)}$ has order a.*
3. *We call a linear operator smoothing if given any local chart (U,ϕ), any localisation A_U subordinated to this chart is smoothing.*
4. *The leading symbol $\sigma_L(A)(x)$ at a point $x \in U$ is given by $\sigma_L\left(A_{\phi(U)}\right)(\phi(x))$ for any local chart (U,ϕ) and any localisation $A_U := \tilde{\chi}\, A\, \chi$ subordinated to this chart where χ and $\tilde{\chi}$ are identically one in a neighborhood of x.*
5. *A pseudodifferential operator A with invertible leading symbol $\sigma_L(A)(x,\xi)$ for any $x \in M$ and any $\xi \neq 0$ is called an elliptic pseudodifferential operator.*

On the grounds of these definitions we can introduce the set $C\ell^a(M)$ (resp. $C\ell^{a,k}(M)$) of classical (resp. logpolyhomogeneous) pseudodifferential operators on M of order a (and log-type k).
It follows from Proposition 52 that the product AB of two classical (resp. log-polyhomogeneous) pseudodifferential operators A and B on M of order a and b (resp. and log types k and l) is a pseudodifferential operator of order $a+b$ (resp. and log-type $k+l$). Furthermore, the product is elliptic if A and B are elliptic as a result of the multiplicativity of leading symbols.

We can therefore define the algebras

$$C\ell(M) = \langle \bigcup_{a\in\mathbb{C}} C\ell^a(M)\rangle, \quad \text{resp.}\, C\ell^{*,*}(M) = \langle \bigcup_{a\in\mathbb{C}} C\ell^{a,*}(M)\rangle = \bigcup_{k\in\, \mathbb{N}_0} \langle \bigcup_{a\in\mathbb{C}} C\ell^{a,k}(M)\rangle$$

generated by all classical (resp. log-polyhomogeneous) pseudodifferential operators on M for the product of operators. Here $\langle S\rangle$ stands for the algebra generated by the set S. We also consider the algebra $C\ell^{-\infty}(M) := \bigcap_{a\in\mathbb{C}} C\ell^a(M)$ of smoothing operators which is a two-sided ideal in $C\ell(M)$ and $C\ell^{*,*}(M)$ since the product of two operators of orders a and b has order $a+b$.

These definitions extend to linear operators acting on smooth sections $C^\infty(M,E)$ of a smooth vector bundle E over M. If V is the model space for the vector bundle E, the above definitions and properties generalise replacing coordinate charts (U,ϕ) on M by local trivialisations (U,ϕ,Φ):

$$\begin{array}{rcl} E_{|_U} & \to & \phi(U)\times V \\ (x,v) & \mapsto & (\phi(x),\Phi(v))\,. \end{array}$$

This leads to algebras defined in a similar manner to the above algebras:

$$\begin{array}{rcl} C\ell(M,E) & = & \langle \bigcup_{a\in\mathbb{C}} C\ell^a(M,E)\rangle, \quad C\ell^{-\infty}(M,E) := \bigcap_{a\in\mathbb{C}} C\ell^a(M,E), \\ C\ell^{*,*}(M,E) & = & \langle \bigcup_{a\in\mathbb{C}} C\ell^{a,*}(M,E)\rangle = \bigcup_{k\in\, \mathbb{N}_0} \langle \bigcup_{a\in\mathbb{C}} C\ell^{a,k}(M,E)\rangle. \end{array}$$

As before, since the product of two operators with orders a and b has order $a+b$, the algebra $C\ell^{-\infty}(M,E)$ of smoothing operators is a two sided ideal in $C\ell(M,E)$ and $C\ell^{*,*}(M,E)$.

We now equip these infinite dimensional sets of symbols with the **Fréchet topology** of constant order symbols. For $a \in \mathbb{C}$ and any non negative integer k, the linear space $C\ell^{a,k}(M,E)$ of classical pseudodifferential operators of order a and log-type k can be equipped with a Fréchet topology. For this, one equips the set $CS^{a,k}(U,V) = CS^{a,k}(U) \otimes \mathrm{End}(V)$ of log-type k symbols of order a on an open subset U of $\mathbb{R}^d$ with values in a euclidean space V (with norm $\|\cdot\|$) with a Fréchet structure. The following semi-norms labelled by multiindices α, β and integers $j \geq 0$, N, $l \in \{0, \cdots, k\}$, give rise to a Fréchet topology on $CS^{a,k}(U,V)$ (see [H]):

$$\sup_{x\in K, \xi \in \mathbb{R}^d} (1+|\xi|)^{-\mathrm{Re}(a)+|\beta|} \, \|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\|;$$

$$\sup_{x\in K, \xi \in \mathbb{R}^d} |\xi|^{-\mathrm{Re}(a)+N+|\beta|} \left\| \partial_x^\alpha \partial_\xi^\beta \Big(\sigma - \sum_{j=0}^{N-1} \psi(\xi)\, \sigma_{a-j} \Big)(x,\xi) \right\|;$$

$$\sup_{x\in K, |\xi|=1} \|\partial_x^\alpha \partial_\xi^\beta \sigma_{a-j,l}(x,\xi)\|,$$

where K is any compact set in U and $\sigma_{a-j} = \sum_{l=0}^k \sigma_{a-j,l}$.
This Fréchet structure on $CS^{a,k}(U,V)$ induces one on $C\ell^{a,k}(M,E)$. Indeed, as in (18.200), given an atlas $(U_i, \phi_i)_{i\in I}$ on M and local trivialisations $\Phi_i : E_{|U_i} \simeq \phi_i(U_i) \times V, i \in I$ (for some finite set I) compatible with the charts, using a partition of the unity subordinated to the chosen atlas, we write an operator A in $C\ell^{a,k}(M,E)$ as follows:

$$A = \sum_{j\in J\subset I} A_j + R(A) = \sum_{j\in J\subset I} \mathrm{Op}(\sigma_j) + R(A), \quad R(A) \in C\ell^{-\infty}(M,E), \tag{18.202}$$

where the $A_j = \mathrm{Op}(\sigma_j)$'s are pseudodifferential operators in $C\ell^{a,k}(M,E)$ with compactly supported symbols in $CS^{a,k}(\phi_j(U_j), V)$.
The countable family of semi-norms built from

1. a countable family of semi-norms given by the supremum norm of the kernel of $R(A)$ and its derivative on a countable family of compact subsets,
2. the countable family of semi-norms on $\mathrm{Op}(\sigma_i)$ induced by the ones on the symbols σ_i described above,

provide a Fréchet topology on $C\ell^{a,k}(M,E)$.

19 Laplacians on closed manifolds

Laplacians on closed manifolds are useful to build invertible self-adjoint elliptic operators and from there Fredholm operators. let us recall the following fundamental result which we quote without proof, referring the reader to [Gi]. As before E is a hermitian finite rank vector bundle over a closed Riemannian manifold M.

Theorem 21 *An (essentially) self-adjoint elliptic operator $A : C^\infty(M,E) \to C^\infty(M,E)$ with positive order has finite dimensional kernel $Ker(A)$ and closed range $\mathrm{R}(A)$. There is a topological splitting*

$$C^\infty(M,E) = \mathrm{Ker}(A) \oplus \mathrm{R}(A)$$

which is orthogonal for the L^2-inner product on $C^\infty(M,E)$ induced by the hermiitan structure on E and the volume measure on M.

Consequently, the operator $Q := A \oplus \pi_A$ built from the orthogonal projection π_A onto the kernel of A, is an elliptic self-adjoint invertible operator in $C^\infty(M,E)$.

19.1 The Laplacian on the d-torus; a warmup

The Laplacian (19.209) on $\mathbb{R}^d$ induces a Laplacian on the d-torus. The constructions below provide a pedestrian description of this induced Laplacian.
Let us consider the d-dimensional torus $\mathbb{T}^d$ seen as the range of $(\mathbb{R}^d, +)$ under the morphism:

$$\begin{array}{rcl} \Phi_d : \mathbb{R}^d & \to & U(1)^d \\ (x_1, \cdots, x_d) & \mapsto & \left(e^{i\,x_1}, \cdots, e^{i\,x_d}\right) \end{array}$$

where $U(1)^d$ is equipped with coordinatewise multiplication. The map Φ_d has kernel $2\pi\mathbb{Z}^d$ so that its range $\mathbb{T}^d$ can be identified with a quotient space:

$$\mathbb{T}^d \simeq \mathbb{R}^d/2\pi\mathbb{Z}^d.$$

This amounts to identifying the algebra of smooth complex functions on $\mathbb{T}^d$ with an algebra of periodic functions on $\mathbb{R}^d$.

$$\begin{aligned} C^\infty(\mathbb{T}^d) &= \{f \in C^\infty(\mathbb{R}^d), f(x+2\pi k) = f(x) \quad \forall k \in \mathbb{Z}^d, \forall x \in \mathbb{R}^d\} \\ &= \{f \in C^\infty([0,2\pi]^d), f(x+2\pi e_i) = f(x) \quad \forall i = 1, \cdots, d, \forall x \in [0,2\pi]^d\}, \end{aligned}$$

where $\{e_i, i = 1, \cdots, d\}$ is the canonical basis of $\mathbb{R}^d$. To a periodic function f on $\mathbb{R}^d$ we associate the induced function $\bar{f}$ on $\mathbb{T}^d$.
The maps

$$\gamma_k(x) = e^{i\,k\cdot x}, \qquad k \in \mathbb{Z}^d \tag{19.203}$$

form an orthonormal basis for the L^2-scalar product:

$$\langle u, v \rangle_{\mathbb{T}^d} = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} u(x) \cdot \bar{v}(x)\, dx, \tag{19.204}$$

where $\bar{v}$ stands for the complex conjugate. Indeed, for any $k, l \in \mathbb{Z}^n$ we have

$$\langle \gamma_k, \gamma_l \rangle_{\mathbb{T}^d} = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} e^{i(k-l)\cdot x}\, dx = \delta_{k-l}.$$

Lemma 20 *A differential operator A on $\mathbb{R}^d$ with constant coefficients induces an operator $\bar{A}$ on $\mathbb{T}^d$ defined by*

$$\bar{A}\,\bar{f}(\bar{x}) := Af(x) \quad \forall x \in \pi^{-1}(\bar{x})$$

for any periodic smooth function f on $\mathbb{R}^d$.

Proof: Since f is periodic $t_{2k\pi}^* f = f$ for any $k \in \mathbb{Z}^d$. This combined with the translation invariance of A yields

$$t_{2k\pi}^*(Af) = t_{2k\pi}^* A t_{-2k\pi}^* \, (t_{2k\pi}^* f) = A\, f.$$

The function Af is therefore also periodic and A induces an operator $\bar{A}$ on $\mathbb{T}^d$ as defined in the lemma.
□

This applies to any differential operator $A = \sum_{\alpha \leq a} c_\alpha D^\alpha$ (where as before $D_x = -i\partial_x$) with constant coefficients c_α; its symbol reads

$$\sigma(A)(\xi) = \sum_{\alpha \leq a} c_\alpha \xi^\alpha. \tag{19.205}$$

The orthonormal family $\{\gamma_k, k \in \mathbb{Z}^d\}$ in $L^2(\mathbb{T}^d)$ yields a basis of eigenvectors for $\bar{A}$ since we have:

$$\begin{aligned} \bar{A}\gamma_k(x) &= A\,\gamma_k(x) \\ &= \sum_{|\alpha| \leq a} c_\alpha(x)\, \partial^\alpha \gamma_k(x) \\ &= \sigma_A(k)\, \gamma_k(x), \end{aligned}$$

so that $\bar{A}$ has purely discrete spectrum[20] given by

$$\mathrm{Spec}_{\bar{A}} = \{\sigma_A(k), \quad k \in \mathbb{Z}^n\}.$$

Remark 30 *This discreteness is a particular instance of the discreteness of the spectrum of elliptic operators on closed compact manifolds. Here it comes out as a consequence of the discreteness of the dual group* $\widehat{\mathbb{T}^d}$ *to* $\mathbb{T}^d$.

In particular the Laplacian $A = \Delta_{\mathbb{R}^d}$ induces a Laplacian $\Delta_{\mathbb{T}^d}$ on $\mathbb{T}^n$.
It has kernel

$$\mathrm{Ker}\Delta_{\mathbb{T}^n} = \{f \in C^\infty_{\mathrm{cpt}}(\mathbb{T}^n), \sum_{i=1}^n \partial_i^2 f = 0\} \simeq \mathbb{R}\, Id.$$

As a consequence of the above discussion, the operator $\Delta_{\mathbb{T}^d}$ has discrete spectrum

$$\mathrm{Spec}\,(\Delta_{\mathbb{T}^d}) := \{|k|^2, k \in \mathbb{Z}^d\} \tag{19.206}$$

with $|k|^2 = \sum_{i=1}^d k_i^2$ and

$$\Delta_{\Gamma^d}\gamma_k(x) = |k|^2\, \gamma_k(x) \quad \forall k in \mathbb{Z}^d$$

19.2 Laplace-Beltrami operator

The Laplacian on the d-dimensional torus is a particular instance of the more general concept of Laplace-Beltrami operator, since it corresponds to the Laplace-Beltrami operator on $\mathbb{T}^d$ for the metric induced by the canonical metric on $\mathbb{R}^d$.
Let (M, g) be a closed oriented d-dimensional Riemannian manifold. Let $g_{ij} = g\left(\partial_{x_i}, \partial_{x_j}\right)$ be the matrix representation of $g(x)$ in some local coordinate chart (U, ϕ) around a point x in M. Let $\left(g^{ij}\right)_{i,j=1,\cdots,d}$ stand for the inverse matrix. We set $\det g$ to be the determinant of the $d \times d$ matrix g_{ij}.
One can check that the form

$$d\mathrm{vol}_g(x) := \sqrt{\det g}(x)\, dx_1 \wedge \cdots \wedge dx_d$$

is invariant under a change of coordinate $x_i \to x_i'$. It induces a hermitian product $\langle f, f' \rangle_g = \int_M f(x)\, \bar{f}'(x)\, dx$ on smooth functions on M and hence one (denoted by the same symbol) on vector

[20] We refer to [?] for the notion of spectrum of an operator and related issues.

fields and on differential forms using musical isomorphisms. In local coordinates, writing $\alpha = \alpha_I dxi_I$, $\beta = \beta_J dx_J$ where $I = \{i_1, \cdots, i_p\}$, $J = \{j_1, \cdots, j_p\}$ are two multiindices of length p, we set

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha(x), \beta(x) \rangle_x \, \mathrm{dvol}_g$$

where

$$\langle \alpha, \beta \rangle_x := g^{i_1, j_1} \cdots g^{i_p j_p} \alpha_{i_1, \cdots, i_p} \beta_{j_1, \cdots, j_p}. \tag{19.207}$$

Following Einstein's conventions, one sums over repeated indices.

Using the inner product on forms, one can define the (formal) adjoint $d^* : \Omega^{p+1}(M) \to \Omega^p(M)$ of the exterior differentiation $d : \Omega^p(M) \to \Omega^{p+1}(M)$ setting:

$$\langle d\alpha, \beta \rangle := \langle \alpha, d^* \beta \rangle \quad \forall \alpha \in \Omega^p(M), \beta \in \Omega^{p+1}(M).$$

The divergence of a tangent vector field U on M is defined by

$$\langle U, \nabla f \rangle := \langle \mathrm{div} U, f \rangle \quad \forall f \in C^\infty(M),$$

where we have set $(\nabla U)_j := g_{ij} U^i$ and used the inner product

$$\langle U, V \rangle = \int_M g^{ij} \, U^i \, V^j \, d \, \mathrm{vol}_g.$$

The divergence can be expressed in local coordinates as follows:

$$\mathrm{div}(U)(x) := \frac{1}{\sqrt{\mathrm{detg}(x)}} \frac{\partial}{\partial x_i} \left(U^i \sqrt{\mathrm{det} g(x)} \right).$$

This expression is independent of the choice of local coordinates.

Definition 33 *The Laplace-Beltrami operator associated with the metric g is defined by*

$$\Delta_g := -\mathrm{div} \circ \nabla = -\frac{1}{\sqrt{\mathrm{det} g}} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_i} \sqrt{\mathrm{det} g} \, g^{ij} \frac{\partial}{\partial x_j} \tag{19.208}$$

Remark 31 *The r.h.s is independent of the choice of local chart as a result of the above discussion.*

The Laplace-Beltrami operator can be written

$$\Delta_g = -\sum_{i,j=1}^d \left(g^{ij}(x) \partial_i \partial_j - \sum_{k=1}^d \Gamma_{ij}^k \partial_k \right)$$

where Γ_{ij}^k are the Christoffel symbols defined in a local system of coordinates $(x_1, \cdots, x_d)$ by $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} := \sum_{k=1}^d \Gamma_{ij}^k \frac{\partial}{\partial x_i}$. The operator Δ_g is therefore a differential operator of order two on M with leading symbol

$$\sigma_L(\Delta_g)(x, \xi) = |\xi|^2,$$

and hence elliptic.

Example 28 *On $\mathbb{R}^d$ equipped with the canonical metric h given by the Euclidean scalar product, the Laplacian reads*

$$\Delta_h = -\sum_{i=1}^d \frac{\partial^2}{\partial x_i}. \tag{19.209}$$

19.3 Generalised Laplacians

Let $E \to M$ be a vector bundle based on a closed Riemannian manifold M and let it be equipped with a connection ∇^E. The Levi-Civita connection ∇ on M combined with ∇^E yields a connection $\nabla^{T^*M\otimes E}$ on $T^*M\otimes E$. Applied to a one form $\alpha \in \Omega(M,E)$ this connection reads $\nabla_X^{T^*M\otimes E}\alpha = \nabla_X^{Hom(TM,E)}\alpha = \nabla_X^E\alpha - \alpha(\nabla_X)$. Composed with ∇^E, this yields an operator $\nabla^{T^*M\otimes E}\nabla^E : C^\infty(M,E) \to C^\infty(M, T^*M\otimes T^*M\otimes E)$, or equivalently a bilinear form on $C^\infty(TM)$ with values in $C^\infty(\mathrm{End}(E))$:

$$\begin{aligned}\nabla^{T^*M\otimes E}\nabla^E\sigma(X,Y) &= \nabla_X^{T^*M\otimes E}\nabla_Y^E\sigma \\ &= \nabla_X^E\nabla_Y^E - \nabla_{\nabla_X Y}^E\sigma \quad \forall X,Y\in C^\infty(TM), \forall\sigma\in C^\infty(M,E).\end{aligned}$$

The trace of this bilinear form on TM yields a second-order differential operator

$$\begin{aligned}\Delta^E &:= -\mathrm{tr}(\nabla^{T^*M\otimes E}\nabla^E) \\ &= -\nabla_{e_i}^E\nabla_{e_i}^E - \nabla_{\nabla_{e_i}e_i}^E\end{aligned}$$

where $(e_i)_{i=1,\cdots,n}$ is an orthonormal basis of TM. This operator, called a (generalised) Laplacian on $C^\infty(M,E)$, is independent of the choice of basis.

Example 29 *When $E := M\times\mathbb{C}$, it yields back the Laplace-Beltrami operator on $C^\infty(M,\mathbb{C})$.*

In local coordinates, the generalised Laplacian Δ^E reads:

$$\Delta^E = -\sum_{i,j=1}^n g^{ij}(x)\left(\nabla_{\frac{\partial}{\partial x_i}}^E\nabla_{\frac{\partial}{\partial x_j}}^E - \sum_{k=1}^n \Gamma_{ij}^k\nabla_{\frac{\partial}{\partial x_k}}^E\right)$$

where as before, the $\left(\Gamma^E\right)^k$ defined by

$$\nabla_{\frac{\partial}{\partial x_i}}^E\frac{\partial}{\partial x_j} := \sum_{k=1}^n \left(\Gamma^E\right)_{ij}^k\frac{\partial}{\partial x_i}$$

are the Christoffel symbols corresponding to the connection ∇^E. Locally we have $\nabla_{\frac{\partial}{\partial x_j}}^E = \frac{\partial}{\partial x_j} + \theta(\frac{\partial}{\partial x_j})$ where θ is Hom(E)-valued a one form, so that the top order part of Δ^E coincides with the top order part of the Laplace-Beltrami operator. The leading symbol of a generalised Laplacian is therefore also given by:

$$\sigma_L(\Delta^E) = \sigma_L(\Delta_g) = |\xi|^2.$$

This motivates the following definition.

Definition 34 *A generalised Laplacian on a vector bundle $E\to M$ is a second order differential operator with scalar leading symbol given by $|\xi|^2$. It is therefore an elliptic differential operator.*

19.4 Laplacians on differential forms

We build two generalised Laplacians $\Delta^{\Lambda T^*M}$ and $(d+d^*)^2$ on differential forms on a closed oriented d-dimensional Riemannian manifold M.
Let

$$d\mathrm{vol}_g(x) = \sqrt{\det g(x)}\, dx_1\wedge\cdots\wedge dx_d$$

be the associated volume form.

The interior product is a bundle morphism:

$$\begin{aligned}TM\times\Lambda T^*M &\to \Lambda T^{*-1}M \\ (v,\alpha) &\mapsto \alpha(v,\cdot).\end{aligned}$$

It satisfies:

$$i(v)(\alpha \wedge \beta) = i(v)(\alpha) \wedge \beta + (-1)^p \alpha \wedge i(v)\beta, \forall v \in T_xM, \alpha \in \Omega^p(M).$$

In a local coordinate chart $(x_1, \cdots, x_n)$ on M we have:

$$i(v)(dx_1 \wedge \cdots \wedge dx_p) = \sum_{i=1}^{p} (-1)^{i+1} dx_i(v)\, dx_1 \wedge \cdots \wedge d\hat{x}_i \wedge \cdots dx_p, \quad \forall v \in T_xM.$$

Using the interior product, for any vector field V on M we set:

$$\nabla_V^E := i(V) \circ \nabla.$$

Combining the graded Leibniz rule for ∇^E extended to E-valued forms with the graded Leibniz rule for $i(V)$ we get for any $\alpha, \beta \in \Omega(M, E)$:

$$\nabla_V^E(\alpha \wedge \beta) = \nabla_V^E \alpha \wedge \beta + \alpha \wedge \nabla_V^E \beta.$$

The exterior product is a bundle morphism:

$$\begin{array}{rcl} TM \times \Lambda T^*M & \to & \Lambda T^{*+1}M \\ (v, \alpha) & \mapsto & v^* \wedge \alpha, \end{array}$$

where v^* is the dual linear form defined by $v^*(w) = \langle v, w \rangle_x$ using the scalar product $\langle \cdot, \cdot \rangle_x$ induced by the metric.
It is easy to check that $\epsilon(v)\epsilon(w) + \epsilon(w)\epsilon(v) = i(v)i(w) + i(w)i(v) = 0$ for all $v, w \in T_xM$.
Combining the interior and exterior products on $\Omega(M)$ yields a Clifford multiplication.

Lemma 21

- *$\epsilon(v)^* = i(v) \quad \forall v \in T_xM, x \in M$,*
- *$c = \epsilon - i$ defines a Clifford multiplication on $\Omega(M)$, i.e.*

  $$c(v)c(w) + c(w)c(v) = -2\langle v, w \rangle_x, \quad \forall v, w \in T_xM$$

  where $\langle \cdot, \cdot, \rangle_x$ is the inner product on T_xM induced by the metric structure.
- *$d = \sum_{i=1}^{d} \epsilon(e_j) \nabla_{e_j}$,*
- *$d^* = -\sum_{j=1}^{d} i(e_j) \nabla_{e_j}$, where $(e_1, \cdots, e_d)$ is an orthonormal basis of T_xM.*

(Partial) Proof: To avoid technicalities, we prove the results on one forms only.

- Given $v \in T_xM$, $f \in \Omega^0(M)$ and $\alpha \in \Omega^1(M)$ we have:

  $$\langle i(v)\alpha, f \rangle_x = \langle \alpha(v), f \rangle_x = \alpha(v)(x) f(x).$$

  On the other hand

  $$\langle \alpha, \epsilon(v) f \rangle_x = \langle \alpha, f v^* \rangle_x = \langle \alpha(x), v^*(x) \rangle_x f(x) = \alpha(v)(x) f(x).$$

  Hence $\epsilon^* = i$ on 1-forms.
- Let $v, w \in T_xM$. First observe that

  $$\epsilon(v)i(w) + i(w)\epsilon(v) = \langle v, w \rangle_x \quad \forall v, w \in T_xM.$$

  Here again, we check it on one a 1-form α.

  $$(\epsilon(v)i(w) + i(w)\epsilon(v))\, \alpha = \alpha(w)v^* + i(w)(v^* \wedge \alpha) = \alpha(w)v^* + v^*(w)\alpha - v^*\alpha(w) = v^*(w)\alpha = \langle v, w \rangle_x \alpha_x.$$

  As a consequence we have:

  $$c(v)c(w) + c(w)c(v) = \epsilon(v)\epsilon(w) + \epsilon(w)\epsilon(v) + i(v)i(w) + i(w)i(v) - 2\left(\epsilon(v)i(w) + i(w)\epsilon(v)\right) = -2\langle v, w \rangle_x,$$

  where we have used the fact that $\epsilon(v)\epsilon(w) + \epsilon(w)\epsilon(v) = i(v)i(w) + i(w)i(v) = 0$.

- Let us set $\tilde{d} = \sum_{i=1}^n \epsilon(e_j)\nabla_{e_j}$ and show that $\tilde{d}$ satisfies three requirements corresponding to the items i), ii), iii) below, which define d uniquely:

  i) Since ∇_{e_j} sends $\Omega^p(M)$ to $\Omega^p(M)$, and $\epsilon(e_j)$ increases the degree of the form by 1, $\tilde{d}$ sends $\Omega^p(M)$ to $\Omega^{p+1}(M)$.

  ii) $\tilde{d}\circ\tilde{d}(f) = 0 \quad \forall f \in C^\infty(M, \mathbb{C})$. We prove that $\tilde{d}^2 f = -\langle T, df\rangle$ where T is the torsion. Since the torsion of the Levi-Civita connection vanishes by definition, this will prove that $\tilde{d}^2 = 0$. To simplify notations we set $\nabla_j = \nabla_{\frac{\partial}{\partial x_j}} = \nabla_{e_j}$ where $e_j = \frac{\partial}{\partial x_j}$.

$$\begin{aligned}
\tilde{d}^2 f &= \tilde{d}(df) \\
&= \sum_{ij} \epsilon(dx_i)\nabla_i(\partial_j f dx_j) \\
&= \sum_{ij} \partial_i\partial_j f dx_i \wedge dx_j + \sum_{ij} \epsilon(dx_i)\partial_j f \nabla_i(dx_j) \\
&= \sum_{ij} \epsilon(dx_i)\partial_j f \nabla_i(dx_j).
\end{aligned}$$

  By Leibniz's rule:

$$0 = \frac{\partial}{\partial x_i}\langle dx_j, e_k\rangle - x = \langle \nabla_i dx_j, e_k\rangle + \langle dx_j, \nabla_i e_k\rangle_x$$

  so that

$$\begin{aligned}
\tilde{d}^2 f(x) &= -\sum_{ijk} \epsilon(dx_i)\frac{\partial}{\partial x_j} f \langle_x \, dx_j, \nabla_i e_k\rangle dx_k \\
&= -\sum_{i<k} \epsilon(dx_i) dx_k \langle_x \, , df, \nabla_i e_k - \nabla_k e_i\rangle \\
&= -\sum_{i<k} \epsilon(dx_i) dx_k \langle df, T(e_i, e_k)\rangle_x \\
&= -\langle T, df\rangle_x.
\end{aligned}$$

  iii) $\tilde{d}$ is a derivation. Indeed, the Levi-Civita connection on the tangent bundle TM extends to a connection on the exterior cotangent bundle ΛT^*M and satisfies the following rule:

$$\nabla_X(\alpha\wedge\beta) = \nabla_X\alpha\wedge\beta + \alpha\wedge\nabla_X\beta \quad \forall\alpha,\beta\in\Omega(M), \forall X \in C^\infty(TM).$$

  Hence $\tilde{d} = \sum_i \epsilon(e_i^*)\nabla_{e_i}$ satisfies a graded Leibniz rule:

$$\tilde{d}(\alpha\wedge\beta) = \tilde{d}\alpha\wedge\beta + (-1)^{|\alpha|}\,\alpha\wedge\tilde{d}\beta \quad \forall\alpha,\beta\in\Omega(M)$$

  and therefore yields a (graded) derivation.

- Given $\alpha\in\Omega^p(M)$ and $\beta\in\Omega^{p+1}(M)$ we want to check that $\langle\epsilon(dx_i)\nabla_i\alpha,\beta\rangle_x = \langle\alpha, i(dx_i)\nabla_i\beta\rangle_x$. Differentiating the one form defined on $v\in T_xM$ by $\alpha(v) = \langle\alpha, i(v)\beta\rangle_x$ and using Leibniz's rule yields:

$$\sum_i (e_i\alpha(e_i) - \alpha(\nabla_i e_i)) = \langle\nabla_i\alpha, i(e_i)\beta\rangle_x + \langle\alpha, \nabla_i i(e_i)\beta\rangle_x = \langle\epsilon(e_i)\nabla_i\alpha,\beta\rangle_x + \langle\alpha, i(e_i)\nabla_i\beta\rangle_x,$$

  where we have used the fact that $\epsilon^* = i$. On the other hand since the divergence is given by $d^*\alpha = -\mathrm{tr}(\nabla\alpha)$ for a one form α, it follows from Stokes's theorem that $\mathrm{tr}(\nabla\alpha) := \sum_{i=1}^n \nabla\alpha(e_i, e_i) = \sum_i (e_i\alpha(e_i) - \alpha(\nabla_i e_i))$ integrates to 0 on M, i.e.

$$\int_M \mathrm{tr}(\nabla\alpha)d\mathrm{vol}_g = -\int_M d^*\alpha = 0.$$

Thus

$$\langle \epsilon(e_i)\nabla_i\alpha, \beta\rangle_x + \langle \alpha, i(e_i)\nabla_i\beta\rangle_x = 0$$

so that $d^* = -i \circ \nabla$.

□

The **Bochner-Weitzenböck formula** relates the operator $(d+d^*)^2$ with $\Delta^{\Lambda T^*M}$.

Proposition 54 *Let $\alpha \in \Omega(M)$ then*

$$(d+d^*)^2(\alpha) = \Delta^{\Lambda T^*M}\alpha + \sum_{i<j} c(dx_i)c(dx_j)R(e_i,e_j)(\alpha)$$

where $R(u,v) := [\nabla_u, \nabla_v] - \nabla_{[u,v]}$ is the curvature tensor.
In particular, $(d+d^*)^2$ is a generalised Lapalacian.

Proof: Using the formula of the proposition, that $(d+d^*)^2$ is a generalised Laplacian follows from the fact that $\Delta^{\Lambda T^*M}$ is one.
To prove this formula, first notice that if ∇ denotes the Levi-Civita connection on TM with Christoffel coefficients given by Γ_{ij}^k then the induced dual connection ∇^* on T^*M reads $\nabla_i^* dx_j = -\Gamma_{ik}^j dx_k$. In the following, we drop the $\star$ in ∇^*. It follows that

$$\begin{aligned}
(d+d^*)^2\alpha &= \sum_{i,j=1}^n c(dx_i)\nabla_i(c(dx_j)\nabla_j\alpha) \\
&= \sum_{i,j=1}^n c(dx_i)c(\nabla_i dx_j)\nabla_j\alpha + \sum_{i,j=1}^n c(dx_i)c(dx_j)\nabla_i(\nabla_j\alpha) \\
&= -\sum_{i,j=1}^n c(dx_i)c(\Gamma_{ik}^j dx_k)\nabla_j\alpha \\
&- \sum_{i=1}^n \nabla_i\nabla_i\alpha + \sum_{i<j}^n c(dx_i)c(dx_j)\left(\nabla_i\nabla_j - \nabla_j\nabla_i\right)\alpha \\
&= -\sum_{i,j=1}^n c(dx_i)c(dx_k)\nabla_{\Gamma_{ik}^j \frac{d}{dx_j}}\alpha \\
&- \sum_{i=1}^n \nabla_i\nabla_i\alpha + \sum_{i<j} c(dx_i)c(dx_j)R(e_i,e_j)\alpha.
\end{aligned}$$

and hence

$$\begin{aligned}
(d+d^*)^2\alpha &= -\sum_{i<j}(c(dx_i)c(dx_k) + c(dx_k)c(dx_i))\nabla_{\Gamma_{ik}^j \frac{d}{dx_j}}\alpha - \sum_{i=1}^n c(dx_i)^2\nabla_{\nabla_i \frac{d}{dx_j}}\alpha \\
&- \sum_{i=1}^n \nabla_i\nabla_i\alpha + \sum_{i<j} c(dx_i)c(dx_j)R(e_i,e_j)\alpha \\
&= \Delta^{\Lambda T^*M}\alpha + \sum_{i<j} c(dx_i)c(dx_j)R(e_i,e_j)\alpha.
\end{aligned}$$

□

19.5 The Lichnerowicz formula

The operator $d+d^*$ studied in the previous paragraph is a perticular Dirac operator and the Bochner-Weitzenböck formula, a special case of the more general Lichnerowicz formule.
To state this result, we need the concept of spin manifold and bundle for which we refer the reader to

e.g. [LM], [BGV]. Let $E = S \otimes W$ be a twisted spinor bundle over an even d-dimensional closed spin manifold M with auxillary bundle W. Given a connection ∇^W on W, $\nabla^E = \nabla \otimes 1 + 1 \otimes \nabla^W$ defines a superconnection on E. The corresponding twisted Dirac operator acting on $C^\infty(M, E)$ is the first order differential oprator

$$D = \sum_{i=1}^{d} c(e_i)\, \nabla^E_{e_i},$$

where $\{e_i, i = 1, \cdots, d\}$ is some local orthonormal tangent frame. Then

$$D^2 = -\sum_{ij} g^{ij} \left(\nabla_{e_i} \nabla_{e_j} + \sum_k \Gamma^k_{ij} \nabla_{e_k} \right) + \sum_{i<j} c(dx^i) c(dx^j) [\nabla_{e_i}, \nabla_{e_j}]$$

defines a generalised Laplacian.
The Lichnerowicz formula (see e.g. Theorem 3.52 of [BGV]) or equivalently the general Bochner identity (see Theorem 8.2 of [LM]) relates the square D^2 of the Dirac operator D on a Clifford module E with the Laplace-Beltrami operator Δ^E associated with the superconnection ∇^E on E.

Proposition 55

$$\begin{aligned} D^2 &= \Delta^E + R^E \\ &= \Delta^E + R^W + \frac{r_M}{4}, \end{aligned} \tag{19.210}$$

where r_M stands for the scalar curvature on M and

$$R^E := \sum_{i<j} c(e_i)\, c(e_j)\, \Omega^E_{e_i, e_j}; \quad R^W := \sum_{i<j} c(e_i)\, c(e_j)\, \left(\nabla^W\right)^2_{e_i, e_j}.$$

In particular, for a flat auxillary bundle we have:

$$D^2 = \Delta_M + \frac{r_M}{4},$$

where Δ_M is the Laplace-Beltrami operator on the Riemannian manifold M and on manifold M with vanishing scalar curvature,we have:

$$D^2 = \Delta^E + R^W.$$

Proof: We can choose a local orthonormal tangent frame $\{e_i, i = 1, \cdots, d\}$ at point $x \in M$ such that $\left(\nabla^E_{e_i}\right)_x = 0$ for all $i \in \{1, \cdots, d\}$. Since $D_{\mathbb{A}} = \sum_{i=1}^n c(e_i)\, \nabla^E_{e_i}$, at that point x we have:

$$\begin{aligned} D^2 &= \sum_{i,j=1}^{d} c(e_i)\, \nabla^E_{e_i}\, c(e_j)\, \nabla^E_{e_j} \\ &= \sum_{i,j=1}^{d} c(e_i)\, c(e_j)\, \nabla^E_{e_i}\, \nabla^E_{e_j} \\ &= \sum_{i,j=1}^{d} c(e_i)\, c(e_j)\, \left[\left(\nabla^E\right)^2_{e_i, e_j} + \nabla^E_{\nabla_{e_i} e_j} \right] \\ &= \sum_{i,j=1}^{d} c(e_i)\, c(e_j)\, \left(\nabla^E\right)^2_{e_i, e_j} \\ &= -\sum_{i=1}^{n} \left(\nabla^E\right)^2_{e_i, e_i} + \sum_{i<j} c(e_i)\, c(e_j)\, \left[\left(\nabla^E\right)^2_{e_i, e_j} - \left(\nabla^E\right)^2_{e_j, e_i} \right] \\ &= \Delta^E + \sum_{i<j} c(e_i)\, c(e_j)\, \left(\nabla^E\right)^2_{e_i, e_j} \\ &= \Delta^E + R^E. \end{aligned}$$

The curvature term $\left(\nabla^E\right)^2 \in \Omega^2(M, \mathrm{End}(E))$ decomposes as $\left(\nabla^E\right)^2 = \Omega \otimes 1 + 1 \otimes \Omega^W$ so that

$$R^E = \sum_{i<j} c(e_i)\, c(e_j)\, \Omega_{e_i,e_j} + R^W.$$

A careful computation (see e.g. the proof of Theorem 3.52 in [BGV]) shows that $\sum_{i<j} c(e_i)\, c(e_j)\, \Omega_{e_i,e_j} = \frac{r_M}{4}$. □

20 From closed linear forms on symbols to traces on operators

We consider linear forms on pseudodifferential oprators of the form $\Lambda(A) = \int_M \lambda(\sigma(A)(x,\cdot)\, dx$, where λ is a linear form on symbols. We focus on two examples, the noncommutative residue and the canonical trace.

20.1 Closed linear forms on symbols

Let U be a connected open subset of $\mathbb{R}^d$. Let $\mathcal{D}(U)$ be a subset of $CS^{*;*}_{\rm cpt}(U)$. For any complex number a we set $\mathcal{D}^a(U) = \mathcal{D}(U) \cap CS^{a;*}_{\rm cpt}(U)$.
We borrow from [MMP] (see also [LP]) the following notations and some of the subsequent definitions.

Definition 35 *For any non negative integer k anc complex number a, let*

$$\begin{aligned}\Omega^k\, \mathcal{D}^a_{\rm cpt}(U) &= \{\alpha \in \Omega^k(T^*U), \quad \alpha = \sum_{I,J\subset\{1,\cdots,n\}, |I|+|J|=k} \alpha_{IJ}(x,\xi)\, d\xi_I \wedge dx_J \\ &\text{with} \qquad \alpha_{IJ} \in \mathcal{D}^{a-|I|}_{\rm cpt}(U)\}\end{aligned}$$

denote the set of order a classical symbol valued forms on U with compact support. Let

$$\begin{aligned}\Omega^k\, \mathcal{D}(U) &= \{\alpha \in \Omega^k(T^*U), \quad \alpha = \sum_{I,J\subset\{1,\cdots,n\}, |I|+|J|=k} \alpha_{IJ}(x,\xi)\, d\xi_I \wedge dx_J \\ &\text{with} \qquad \alpha_{IJ} \in \mathcal{D}(U)\}\end{aligned}$$

denote the set of classical symbol valued k-forms on U of all orders with compact support.

Provided $\mathcal{D}(U)$ equipped with the star product is an alegbra, exterior product on forms combined with the star product on symbols induces a product $\Omega^k\mathcal{D}(U) \times \Omega^l\mathcal{D}(U) \to \Omega^{k+l}\mathcal{D}(U)$; let

$$\Omega\mathcal{D}(U) := \bigoplus_{k=0}^{\infty} \Omega^k\mathcal{D}(U)$$

stand for the $\mathbb{Z}_+$ graded algebra (also filtered by the symbol order) of $\mathcal{D}(U)$- valued forms on U.

Example 30 *In particular, we consider the algebras $\Omega CS_{\rm cpt}(U)$ of classical symbols valued forms, the algebra $\Omega CS^{\mathbb{Z}}_{\rm cpt}(U) := \bigcup_{a\in\mathbb{Z}} \Omega\, CS^a_{\rm cpt}(U)$ of integer order classical symbols valued forms, the algebras $\Omega CS^{\rm odd}_{\rm cpt}(U)$, resp. $\Omega CS^{\rm even}_{\rm cpt}(U)$ of odd- (resp. even-) classical symbol valued forms.*

Even though $CS^{\notin\mathbb{Z}}(U)$ is not an algebra, we can still build the set $\Omega^k CS^{\notin\mathbb{Z}}(U) := \bigcup_{a\notin\mathbb{Z}} \Omega^k\, CS^a_{\rm cpt}(U)$ of non integer order classical symbol valued k-forms and the set $\Omega CS^{\notin\mathbb{Z}}(U) = \oplus_{k=1}^{\infty} \Omega^k CS^{\notin\mathbb{Z}}(U)$ of non integer order classical symbol valued forms.

Whenever $\mathcal{D}(U)$ is stable under partial differentiation, exterior differentiation on forms extends to $\mathcal{D}(U)$-valued forms (see (5.14) in [LP]):

$$\begin{aligned} d : \Omega^k\mathcal{D}(U) &\to \Omega^{k+1}\mathcal{D}(U) \\ \alpha_{IJ}(x,\xi)\, d\xi_I \wedge dx_J &\mapsto \sum_{i=1}^{2n} \partial_i \alpha_{IJ}(\xi)\, du_i \wedge d\xi_I \wedge dx_J, \end{aligned}$$

where $u_i = \xi_i, \partial_i = \partial_{\xi_i}$ with the index i varying from 1 to d and $u_i = x_i, \partial_i = \partial_{x_i}$ with the index i varying from $d+1$ to $2d$.
As before, we call a symbol valued form α closed if $d\alpha = 0$ and exact if $\alpha = d\,\beta$ where β is a symbol valued form; this gives rise to the following cohomology groups

$$H^k\mathcal{D}(U) := \{\alpha \in \Omega^k\mathcal{D}(U), \quad d\alpha = 0\} \,/\, \{d\,\beta, \quad \beta \in \Omega^{k-1}\mathcal{D}(U)\}.$$

A linear form $\rho : \mathcal{D}(U) \to \mathbb{C}$ extends to a linear form[21] $\tilde\rho : \Omega\mathcal{D}(U) \to \mathbb{C}$ defined by

$$\tilde\rho\left(\alpha_{IJ}(x,\xi)\, d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_{|I|}} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{|J|}}\right) := \rho(\alpha_{IJ})\, \delta_{|I|+|J|-2d},$$

with $i_1 < \cdots < i_{|I|}$, $j_1 < \cdots < j_{|J|}$.
We quote without proof an obvious but nevertheless useful result. Here $\mathcal{D}(U)$ is a subset of $CS^{*;*}_{\rm cpt}(U)$.

Lemma 22 *Let $\rho : \mathcal{D}(U) \to \mathbb{C}$ be a linear form. The following two conditions are equivalent:*

$$\left(\exists i, j \in \{1, \cdots, n\}, \quad \exists \tau \in \mathcal{D}(U), \quad \text{s.t.} \quad \sigma = \partial_{\xi_i}\tau \in \mathcal{D}(U) \quad \text{or} \quad \sigma = \partial_{x_j}\tau \in \mathcal{D}(U)\right) \implies \rho(\sigma) = 0$$
$$\left(\exists \beta \in \Omega^{d-1}\mathcal{D}(U)\right), \quad \alpha = d\,\beta \in \Omega^d\mathcal{D}(U)\Big) \implies \widetilde{\rho}(\alpha) = 0$$

As before we call *closed* a linear form $\tilde\rho$ obeying the second condition and by extension ρ is then also said to be closed. We also say that ρ satisfies Stokes' condition.

Remark 32 *A closed linear form $\tilde\rho$ on $\Omega\mathcal{D}(U)$ induces a linear form $\bar\rho : H^\bullet\mathcal{D}(U) \to \mathbb{C}$.*

Proposition 56 *A linear form $\rho : \mathcal{D}(U) \subset CS^{*;*}_{\rm cpt}(U) \to \mathbb{C}$ is closed whenever it vanishes on truncated Poisson brackets in $\mathcal{D}(U)$, i.e whenever, for any non negative integer N*

$$\rho\left(\{\sigma, \tau\}_\star^{(N)}\right) = 0 \quad \forall \sigma, \tau \in \mathcal{D}(U), \quad \text{s.t.} \quad \{\sigma, \tau\}_\star^{(N)} \in \mathcal{D}(U)$$

where we have set:

$$\{\sigma, \tau\}_\star^{(N)} := \sum_{|\alpha| \le N-1} \frac{(-i)^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha \sigma \partial_x^\alpha \tau - \partial_\xi^\alpha \tau \partial_x^\alpha \sigma\right). \tag{20.211}$$

Proof: If the linear form is closed, we can perform iterated "integration by parts" in the variables x and ξ and write:

$$\begin{aligned} \rho\left(\{\sigma, \tau\}_\star^{(N)}\right) &= \sum_{|\alpha| \le N-1} \frac{(-i)^{|\alpha|}}{\alpha!} \rho\left(\partial_\xi^\alpha \sigma \partial_x^\alpha \tau - \partial_x^\alpha \sigma \partial_\xi^\alpha \tau\right) \\ &= \sum_{|\alpha| \le N-1} \frac{(-i)^{|\alpha|}}{\alpha!} \rho\left(\partial_x^\alpha \sigma \partial_\xi^\alpha \tau - \partial_x^\alpha \sigma \partial_\xi^\alpha \tau\right) \\ &= 0. \end{aligned}$$

Conversely, if the linear form vanishes on truncated brackets $\{\cdot, \cdot\}_\star^{(N)}$ contained in $\mathcal{D}(U)$ then for any $\sigma \in \mathcal{D}(U)$ such that $\partial_{x_i}\sigma \in \mathcal{D}(U)$ we have

$$\rho(\partial_{x_i}\sigma) = i\,\rho\left(\{\xi_i, \sigma\}_\star^{(N)}\right) = 0$$

and similarly for any $\sigma \in CS_{\rm cpt}(U)$ such that $\partial_{\xi_i}\sigma \in \mathcal{D}(U)$ we have

$$\rho(\partial_{\xi_i}\sigma) = i\,\rho\left(\{x_i, \sigma\}_\star^{(N)}\right) = 0.$$

□

Definition 36
- *If $\mathcal{D}(U)$ is closed in $CS^{*,*}_{\rm cpt}(U)$ for the Fréchet topology on symbols of constant order, we call a linear form $\rho : \mathcal{D}(U) \to \mathbb{C}$ (resp. $\tilde\rho : \Omega\mathcal{D}(U) \to \mathbb{C}$) continuous if it restricts to a continuous map on $\mathcal{D}(U)^a(U) = \mathcal{D}(U) \cap CS^{a,*}_{\rm cpt}(U)$ (resp. $\Omega\mathcal{D}^a(U) \to \mathbb{C}$) for any complex number a.*
- *If $\mathcal{D}(U)$ contains the algebra $CS^{-\infty}(U)$ of smoothing symbols, we call a linear form $\rho : \mathcal{D}(U) \to \mathbb{C}$ (resp. $\tilde\rho : \Omega\mathcal{D}(U) \to \mathbb{C}$) singular if it vanishes on $CS^{-\infty}(U)$ (resp. $\Omega CS^{-\infty}_{\rm cpt}(U)$).*

[21] In the case of $\mathcal{D}(U) = CS^{\notin\mathbb{Z}}(U)$, we need to weaken this assumption; in this case we say a map $\lambda : CS^{\notin\mathbb{Z}}(U) \to \mathbb{C}$ is linear if it takes linear combinations to linear combinations.

Example 31 *Let $S^*U \subset T^*U$ denote the cotangent unit sphere of U. Using Stokes' theorem, one checks that the singular continuous linear forms indexed by non negative integers k*

$$\mathrm{res}_k(\sigma) := \int_{S^*U} \sigma_{-d,k}(x,\xi)\, dx\, đ_S\xi$$

define closed linear forms on $CS^{*,k}_{\mathrm{cpt}}(U)$ called the higher k-th noncommutative residue of σ. When $k = 0$, then $\mathrm{res} = \mathrm{res}_0$ defines a closed linear form on $CS_{\mathrm{cpt}}(U)$.
Indeed, for a symbol $\sigma \in CS^{*,k}(U)$ we have $(\partial_{\xi_i}\sigma)_{-d,k}(x,\xi) = \partial_{\xi_i}\sigma_{-d+1,k}(x,\xi)$ so that by Stokes' theorem applied to the boundaryless manifold corresponding to the cotangent unit sphere S^*_xU, we have

$$\int_{S^*U} (\partial_{\xi_i}\sigma)_{-d,k}(x,\xi)\, dx\, d_S\xi = \int_U \left(\int_{S^*_xU} \partial_{\xi_i}\sigma_{-d+1,k}(x,\xi) d_S\xi \right) dx = 0.$$

On the other hand, since σ has compact support in U, again by Stokes' theorem we have:

$$\int_{S^*U} (\partial_{x_i}\sigma)_{-d,k}(x,\xi)\, dx\, d_S\xi = \int_U \partial_{x_i} \left(\int_{S^*_xU} \sigma_{-d,k}(x,\xi) d_S\xi \right) dx = 0.$$

Example 32 *Since the map $⨍_{\mathbb{R}^d} : CS^{\notin \mathbb{Z}}_{\mathrm{c.c}}(\mathbb{R}^d) \to \mathbb{C}$ is closed, in a similar manner, one shows that the map*

$$\begin{array}{rcl} CS^{\notin \mathbb{Z}}_{\mathrm{cpt}}(U) & \to & \mathbb{C} \\ \sigma & \mapsto & ⨍_{T^*U} \sigma(x,\xi)\, đ\xi\, dx, \end{array}$$

is closed.

20.2 From closed linear forms on symbols to linear forms on operators

Let U be an open subset of $\mathbb{R}^d$ and $\mathcal{D}(U)$ a subset of $CS^{*,*}(U)$ stable under partial derivatives $\partial_{x_i}, \partial_{\xi_i}, i = 1, \cdots, d$ and under multiplication by functions in $C^\infty_{\mathrm{cpt}}(U)$. If $\mathcal{D}(U)$ is a linear subset [22] of $CS^{*,*}(U)$, this makes it a $C^\infty_{\mathrm{cpt}}(U)$-submodule of $CS^{*,*}(U)$.
Let

$$\mathcal{S} := \{\sigma \in CS_{\mathrm{c.c}}(\mathbb{R}^d), f \otimes \sigma \in \mathcal{D}(U)\},$$

and let us assume that $C^\infty_{\mathrm{cpt}}(U) \otimes \mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{D}_{\mathrm{cpt}}(U)$ for the Fréchet topology of symbols of constant order.
$\mathcal{S}$ is stable under partial differentiation ∂_{ξ_i} since $\mathcal{D}(U)$ is.

Example 33 *Corresponding to the algebras*

$$\mathcal{D}_1(U) = CS^{*,*}_{\mathrm{cpt}}(U), \quad \mathcal{D}_2(U) = CS_{\mathrm{cpt}}(U), \quad \mathcal{D}_3(U) = CS^{\mathbb{Z}}_{\mathrm{cpt}}(U), \quad \mathcal{D}_4(U) = CS^{\mathrm{odd}}_{\mathrm{cpt}}(U), \quad \mathcal{D}_5(U) = CS^{\mathrm{even}}_{\mathrm{cpt}}(U),$$

we have the following sets of symbols with constant coefficients

$$\mathcal{S}_1 = CS^{*,*}_{\mathrm{c.c}}(\mathbb{R}^d), \quad \mathcal{S}_2 = CS_{\mathrm{cc}}(\mathbb{R}^d), \quad \mathcal{S}_3 = CS^{\mathbb{Z}}_{\mathrm{c.c}}(\mathbb{R}^d), \quad \mathcal{S}_4 = CS^{\mathrm{odd}}_{\mathrm{c.c}}(\mathbb{R}^d), \quad \mathcal{S}_5 = CS^{\mathrm{even}}_{\mathrm{c.c}}(\mathbb{R}^d).$$

Corresponding to the sets

$$\mathcal{D}_6(U) = CS^{\notin \mathbb{Z}}_{\mathrm{cpt}}(U); \quad \mathcal{D}_7(U) = CS^{\notin \mathbb{Z},*}_{\mathrm{cpt}}(U); \mathcal{D}_8(U) = CS^{*,k}_{\mathrm{cpt}}(U); \quad \mathcal{D}_9(U) = CS^a_{\mathrm{cpt}}(U),$$

the latter corresponding to symbols of order a, we have the following set of symbols with constant coefficients

$$\mathcal{S}_6 = CS^{\notin \mathbb{Z}}(\mathbb{R}^d); \quad \mathcal{S}_7 = CS^{\notin \mathbb{Z},*}_{\mathrm{c.c}}(\mathbb{R}^d), \quad \mathcal{S}_8 = CS^{*,k}_{\mathrm{c.c}}(\mathbb{R}^d); \quad \mathcal{S}_7 = CS^a_{\mathrm{c.c}}(\mathbb{R}^d).$$

[22] We shall also consider subsets such as $CS^{\notin \mathbb{Z},*}(U) := \cup_{a \notin \mathbb{Z}} CS^{a,*}(U)$ which are not linear spaces.

Since $C^\infty_{\rm cpt}(U)\otimes \mathcal{S}$ is dense in $\mathcal{D}(U)$, a continuous linear form $\lambda:\mathcal{S}\to\mathbb{C}$ induces a linear map [23] :

$$\begin{array}{rcl}\mathcal{D}(U) & \to & C^\infty_{\rm cpt}(U)\\ \sigma & \mapsto & (x\mapsto \lambda(\sigma(x,\cdot)))\,.\end{array}$$

Integrating it along M yields a linear form:

$$\begin{array}{rcl}\lambda^U:\mathcal{D}(U) & \to & \mathbb{C}\\ \sigma & \mapsto & \int_U \lambda(\sigma(x,\cdot))\,dx.\end{array}$$

Example 34 *1. If λ is the closed linear form* res_k *on $CS^{*,k}_{\rm c.c}(\mathbb{R}^d)$, then λ^U is the k-th higher order noncommutative residue on $CS^{*,k}_{\rm pt}(U)$:*

$$\mathrm{res}_k(\sigma)=\int_{S^*U}\sigma_{-d,k}(x,\xi)\,đ_S\xi\,dx.$$

2. If λ is the closed linear form $⨍_{\mathbb{R}^d}$ on $CS^{\notin\mathbb{Z},*}_{\rm c.c}(\mathbb{R}^d)$, then λ^U is the cut-off integral on $CS^{*,k}_{\rm pt}(U)$:

$$⨍_{T^*U}\sigma=\int_{T^*U}\sigma(x,\xi)\,đ_S\xi\,dx.$$

Proposition 57 *With the above notations, we assume that $\mathcal{S}$ is stable under partial differentiations and is invariant under the action $\sigma\mapsto\sigma\circ T$ of the linear group* $\mathrm{Gl}(\mathbb{R}^d)$*. Then for any diffeomorphism $\kappa:U\to U'$,*

$$\sigma\in\mathcal{D}(U)\Longrightarrow \kappa_\sigma\in\mathcal{D}(U')\quad\text{and}\quad \widetilde{\kappa_*\sigma}\in\mathcal{D}(U')$$

with the notations of Proposition 53.
Let $\lambda:\mathcal{S}\to\mathbb{C}$ be a linear form. Provided λ is covariant i.e:

$$|\det(T)|\,\lambda(\sigma\circ T)=\lambda(\sigma)\quad\forall T\in \mathrm{Gl}(\mathbb{R}^d)\quad\forall\sigma\in\mathcal{S},\tag{20.212}$$

and

$$\omega_\lambda\left(\mathrm{Op}(\sigma)\right):=\lambda(\sigma(x,\cdot))\,dx_1\wedge\cdots\wedge dx_d$$

transforms covariantly under a diffeomorphism $\kappa:U\to U'$, i.e. for any σ in $\mathcal{D}(U)$:

$$\kappa^*\omega_\lambda\left(\kappa_*\mathrm{Op}(\sigma)\right)=\omega_\lambda\left(\mathrm{Op}(\sigma)\right)\quad\forall\sigma\in\mathcal{D}(U).$$

Proof: Recall that the symbol $\widetilde{\kappa_*\sigma}$ of $\kappa_*\mathrm{Op}(\sigma)$ differs from $\kappa_*\sigma$ by a polynomial expression in ξ,

$$\phi_\alpha(x,\xi)=\sum_{0<|\beta|\leq\frac{|\alpha|}{2}}a_{\alpha,\beta}(x)\,\xi^\beta\,\partial_\xi^\alpha\sigma(\cdot,\xi).$$

Since $\mathcal{D}(U)$ is stable under partial differentiation and multiplication by smooth functions with compact support, ϕ_α lies in $\mathcal{D}(U)$. Since $\mathcal{S}$ is invariant under linear transormations, $\kappa_*\sigma$ also lies in $\mathcal{D}(U)$. It follows that $\widetilde{\kappa_*\sigma}$ lies in $\mathcal{D}(U)$.
We now need to check that $\lambda(\phi_\alpha(x,\cdot))=0$. Since the sum in the expression for ϕ_α runs over $|\beta|\leq\frac{|\alpha|}{2}<|\alpha|$ and λ is closed, it follows from (2.19) that

$$\lambda(\phi_\alpha(x,\cdot))=0\quad\forall x\in U.$$

Combining this with $dx'=|\det\left(d\kappa(x)\right)|\,dx$ and the covariance property (20.212) of λ applied to $T=(d\kappa(x))^t$, we infer that at point $x'=\kappa(x)$:

$$\begin{array}{rcl}\omega_\lambda\left(\kappa_*(\mathrm{Op}(\sigma))\right)(x') & := & \lambda\left(\widetilde{\kappa_*\sigma}(x',\cdot)\right)\,dx'_1\wedge\cdots\wedge dx'_d\\ & = & \lambda\left(\widetilde{\kappa_*\sigma}(x',\cdot)\right)\,dx'_1\wedge\cdots\wedge dx'_d\\ & = & \lambda\left(\sigma\left(\kappa^{-1}(x'),\left(d\kappa(\kappa^{-1}(x'))\right)^t\,\cdot\right)\right)\,dx'_1\wedge\cdots\wedge dx'_d\\ & = & \lambda\left(\sigma\left(\kappa^{-1}(x'),\cdot\right)\right)\,dx_1\wedge\cdots\wedge dx_d\\ & = & \lambda\left(\sigma(x,\cdot)\,\right)\,dx_1\wedge\cdots\wedge dx_d\\ & = & \omega_\lambda\left(\mathrm{Op}(\sigma)\right)(x).\end{array}$$

[23] As before, if $\mathcal{S}$ is not linear, we assume that λ, resp. λ^U only preserve linear combinations lying in $\mathcal{S}$, resp.$\mathcal{D}(U)$.

□

Remark 33 *For future reference, let us observe that the two essential ingredients in the proof are the closedness and covariance of the linear form* λ.

In view of this covariance property, $\omega_\lambda\left(\mathrm{Op}(\sigma)\right)(x)$ defines a global density on the closed manifold M. Integrating it over M yields a map

$$\Lambda\left(\mathrm{Op}(\sigma)\right) = \int_M \omega_\lambda\left(\mathrm{Op}(\sigma)\right)(x).$$

This map is defined on a set $\mathcal{D}(M)$ (stable under multiplication by functions in $C^\infty_{\mathrm{cpt}}(U)$) of operators in $C\ell^{*,*}(M)$ determined by a purely symbolic condition given by the set $\mathcal{S}$. By this we mean that the localised symbol of an operator in $\mathcal{D}(M)$ in any coordinate chart (U,ϕ) lies in $\mathcal{D}(\phi(U)) \subset CS_{\mathrm{cpt}}(\phi(U))$ with the property that

$$\mathcal{S} := \{\sigma \in CS^{*;*}_{\mathrm{c.c}}(\mathbb{R}^d), \quad f\otimes\sigma \in \mathcal{D}(U)\}$$

is independent of the set U.

Corollary 11 *With these notations, let us assume that* $\mathcal{S}$ *is invariant under the action of the linear group* $\mathrm{Gl}(\mathbb{R}^d)$ *and under partial differentiation.*
Any closed and covariant linear form λ *on* $\mathcal{S}$ *yields a linear form*

$$\begin{aligned} \Lambda : \mathcal{D}(M) &\to \mathbb{C} \\ A &\mapsto \int_M \lambda(\sigma(A)(x,\cdot))\,dx. \end{aligned} \tag{20.213}$$

Remark 34 *If* λ *is continuous in the Fréchet topology of constant order symbols it induces a uniformly continuous map* $\sigma \mapsto \lambda(\sigma(x,\cdot))$ *in* x *on compact sets so that* Λ *is continuous in the Fréchet topology of constant order operators.*

□

Example 35 *The higher noncommutative residue* $\lambda = \mathrm{res}_k$ *on* $\mathcal{S} = CS^{*;k}_{\mathrm{c.c}}(\mathbb{R}^d)$ *gives rise to the higher noncommutative residue* res_k *on* $\mathcal{D}(M) = C\ell^{*,k}(M)$:

$$\mathrm{res}_k(A) = \int_M \mathrm{res}_k(\sigma(A)(x,\cdot))\,dx = \int_{S^*M} \sigma_{-d,k}(A)(x,\xi)\,đ_S\xi\,dx, \tag{20.214}$$

where $S^*M \subset T^*M$ *stands for the cotangent unit sphere of* M. *When* $k=0$, *this yields the ordinary noncommutative residue on* $C\ell(M)$:

$$\mathrm{res}(A) = \int_M \mathrm{res}(\sigma(A)(x,\cdot))\,dx = \int_{S^*M} \sigma_{-d}(A)(x,\xi)\,đ_S\xi\,dx. \tag{20.215}$$

it is continuous in the Fréchet topology of operators of cosntant order.

One can read off this definition various straightforward but important properties of the noncommutative residue:

- res_k vanishes on operators of order $< -d$ and hence on smoothing operators,
- res_k vanishes on operators of non integer order since the $-d$-th positively homogeneous component of their symbol vanishes,
- res_k vanishes on differential operators for the same reason,
- res_k is continuous for the Fréchet topology on operators of constant order.

Example 36 *The cut-off integral* $\lambda = -\!\!\!\!\!\!\int_{\mathbb{R}^d}$ *on* $CS^{*;*}_{\mathrm{c.c}}(\mathbb{R}^d)$ *whose restriction to* $\mathcal{S} = CS^{\notin\mathbb{Z},k}_{\mathrm{c.c}}(\mathbb{R}^d) \subset \mathrm{Ker}(\mathrm{res}_k)$ *is closed and covariant, gives rise to the canonical trace* TR *on* $\mathcal{D}(M) = C\ell^{\notin\mathbb{Z},*}_{\mathrm{c.c}}(\mathbb{R}^d) \subset \mathrm{Ker}(\mathrm{res}_k)$:

$$\mathrm{TR}(A) = \int_M \int_{T^*_xM} \sigma(A)(x,\cdot))\,dx = \int_{T^*M} \sigma(A)(x,\xi)\,đ_S\xi\,dx. \tag{20.216}$$

Remark 35 *The canonical trace can also be expressed in terms of the amplitude $a(x,y,\xi)$ of the operator A:*

$$Au(x) = \int_{T^*U} a(x,y,\xi)\, u(y)\, e^{i\langle x-y,\xi\rangle}\, dy\, đ\xi$$

by

$$\mathrm{TR}(A) = \int_{T^*U} a(x,x,\xi)\, dx\, đ\xi.$$

Since the symbol

$$\sigma(A)(x,\xi) \simeq \sum_{\alpha\in\mathbb{Z}_+^d} \frac{(-i)^{|\alpha|}}{\alpha!}\, \partial_\xi^\alpha \partial_y^\alpha (x,y,\xi)_{|y=x}$$

differs from $a(x,x,\xi)$ by partial derivatives, the canonical integral being closed on non integer order amplitudes, we have:

$$-\!\!\!\!\!\!\int_{T^*U} \sigma(A)(x,\xi)\, dx\, đ\xi = -\!\!\!\!\!\!\int_{T^*U} a(x,x,\xi)\, dx\, đ\xi \quad \forall A \in C\ell^{\notin\mathbb{Z},*}(M,E).$$

Since the cut-off integral $-\!\!\!\!\!\int_{\mathbb{R}^d}$ is continuous on $CS^{*,*}(\mathbb{R}^d)$ in the Fréchet topology, so is the canonical trace TR continuous in the Fréchet topology.
A further striking property of linear forms of the type (20.213) is that they vanish on operator brackets in $\mathcal{D}(M)$.

Proposition 58 *A linear form Λ on $\mathcal{D}(m)$ built from a continuous, closed and covariant linear form λ as in (20.213) vanishes on brackets:*

$$[A,B] \in \mathcal{D}(M) \Longrightarrow \Lambda([A,B]) = 0 \quad \forall A,B \in \mathcal{D}(M).$$

Proof: In view of (18.196), the symbol of the bracket $[A,B]$ reads

$$\{\sigma(A),\sigma(B)\}_* :\sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha\sigma(A)\partial_x^\alpha\sigma(B) - \partial_\xi^\alpha\sigma(B)\partial_x^\alpha\sigma(A)\right), \tag{20.217}$$

which is of order $a+b$ where a is the order of A, b the order of B.
Combining the closedness of λ which yields "integration by parts formulae" in ξ and Stokes' theorem on M which yields "integration by parts formulae" in x with the continuity of the linear form λ on symbols of constant order, we infer that:

$$\begin{aligned}
\Lambda([A,B]) &= \int_M \lambda\left(\mathrm{Op}(\{\sigma(A),\sigma(B)\}_\star)\right) dx \\
&= \lim_{N\to\infty} \int_M \lambda\left(\mathrm{Op}(\{\sigma(A),\sigma(B)\}_\star^{(N)})\right) dx \\
&= \lim_{N\to\infty} \sum_{|\alpha|\le N-1} \frac{(-i)^{|\alpha|}}{\alpha!} \int_M \lambda\left(\partial_\xi^\alpha\sigma(A)\partial_x^\alpha\sigma(B) - \partial_\xi^\alpha\sigma(B)\partial_x^\alpha\sigma(A)\right) dx \\
&= \lim_{N\to\infty} \sum_{|\alpha|\le N-1} \frac{(-i)^{|\alpha|}}{\alpha!} \int_M \lambda\left(\partial_\xi^\alpha\sigma(A)\partial_x^\alpha\sigma(B) - \partial_x^\alpha\sigma(B)\partial_\xi^\alpha\sigma(A)\right) dx \\
&= 0.
\end{aligned}$$

It follows that $\Lambda([A,B]) = 0$.

Definition 37
- *A trace on a Lie algebra $\mathcal{L}$ is a linear form τ on $\mathcal{L}$ which vanishes on brackets of operators in $\mathcal{L}$:*

$$\tau([a,b]) = 0 \quad \forall a,b \in \mathcal{L}.$$

- *A graded trace on a graded Lie algebra $\mathrm{Gr}\mathcal{L} = \oplus_{k=0}^\infty \mathrm{Gr}_k\mathcal{L}$ is a family parametrised by $k \in \mathbb{Z}_+$ of linear forms τ_k on $\mathrm{Gr}_k\mathcal{L}$, which vanishes on graded brackets:*

$$\tau_{k+l}([a,b]) = 0 \quad \forall a \in \mathcal{L}_k, b \in \mathcal{L}_l.$$

Example 37 *The higher order residues* $\mathrm{res}_k, k \in \mathbb{Z}_+$ *define a graded trace on* $\mathrm{Gr}C\ell^{*,*}(M) = \sum_{k=0}^{\infty} \mathrm{Gr}_k C\ell^{*,*}(M)$, *where* $\mathrm{Gr}_k C\ell^{*,*}(M) = C\ell^{*,k}(M)/C\ell^{*,k-1}(M)$. *since [L1]*

$$A \in C\ell^{*,k}(M), B \in C\ell^{*,l}(M) \Longrightarrow \mathrm{res}_{k+l}([A,B]) = 0,$$

resp. a trace on $C\ell(M)$ *since for* $k = l = 0$ *this implies*

$$A \in C\ell(M), B \in C\ell(M) \Longrightarrow \mathrm{res}([A,B]) = 0,$$

as a consequence of Proposition 58 applied to $\lambda = \mathrm{res}_k$ *on* $\mathcal{S} = CS^{*,k}_{\mathrm{c.c}}(\mathbb{R}^d)$, *resp.* $\lambda = \mathrm{res}$ *on* $\mathcal{S} = CS_{\mathrm{c.c}}(\mathbb{R}^d)$.

□

Example 38 *The canonical trace TR on* $C\ell^{\notin \mathbb{Z},*}(M)$ *deserves this name in so far as it vanishes on brackets of operators in* $C\ell^{\notin \mathbb{Z},*}(M)$

$$[A,B] \in C\ell^{\notin \mathbb{Z},*}(M) \Longrightarrow \mathrm{TR}([A,B]) = 0 \quad \forall A, B \in C\ell^{\notin \mathbb{Z},*}(M),$$

resp. in $C\ell^{\notin \mathbb{Z}}(M)$

$$[A,B] \in C\ell^{\notin \mathbb{Z}}(M) \Longrightarrow \mathrm{TR}\,[A,B]) = 0 \quad \forall A, B \in C\ell^{\notin \mathbb{Z}}(M),$$

as a consequence of Proposition 58 applied to $\lambda = -\!\!\!\!\!\int_{\mathbb{R}^d}$ *on* $\mathcal{S} = CS^{\notin \mathbb{Z},*}_{\mathrm{c.c}}(\mathbb{R}^d)$, *resp.* $\mathcal{S} = CS^{\notin \mathbb{Z}}_{\mathrm{c.c}}(\mathbb{R}^d)$.

21 A characterisation of the noncommutative residue and the canonical trace

We provide a characterisation of the noncommutative residue [24] on the algebra of classical operators and the canonical trace on non integer order operators.

21.1 The noncommutative residue: a first characterisation

Proposition 59 *Any closed singular continuous linear form $\rho : CS_{\rm cpt}(U) \to \mathbb{C}$ is proportional to the noncommutative residue.*
Equivalently, any closed singular continuous linear form $\tilde\rho : \Omega CS_{\rm cpt}(U) \to \mathbb{C}$ is proportional to the noncommutative residue $\widetilde{\rm res}$ extended to forms.

Proof: By Lemma 22, the two statements are equivalent. Let us prove the first one. Let ρ be a closed singular continuous linear form on $CS_{\rm cpt}(U)$. For any fixed $f \in C^\infty_{\rm cpt}(U)$ the map $\rho_f : \tau \mapsto \rho(f\,\tau)$ defines a singular linear form on $CS_{\rm c.c}(\mathbb{R}^d)$ which vanishes on derivatives in ξ since we have $\rho(f\,\partial_{\xi_j}\tau) = \rho(\partial_{\xi_j}(f\,\tau)) = 0$. By Theorem 1 which characterises the noncommutative residue in terms of closed singular linear forms on classical symbols, it follows that there is a constant $c(f)$ such that $\rho(f\otimes\tau) = c(f)\,{\rm res}(\tau)$ for any $\tau \in CS_{\rm c.c}(\mathbb{R}^d)$. Since $f \mapsto \rho(f\otimes\tau)$ is continuous, $\rho(f\otimes\tau) = F({\rm res}(f\otimes\tau)) = F(f){\rm res}(\tau)$ for some continuous distribution F in $\left(C^\infty_{\rm cpt}(U)\right)'$. A general symbol $\sigma \in CS_{\rm cpt}(U)$ can be approximated in the Fréchet topology of symbols of constant order by linear combinations of tensor products $f\otimes\tau$ with $f \in C^\infty_{\rm cpt}(U)$, $\tau \in CS_{\rm c.c}(\mathbb{R}^d)$. It follows from the continuity of ρ that there is a distribution $F \in \left(C^\infty_{\rm cpt}(U)\right)'$ such that $\rho(\sigma) = F(x \mapsto {\rm res}(\sigma(x,\cdot))$ for any $\sigma \in CS_{\rm cpt}(U)$. This distribution being continuous, it reads $F(f) = \int_U \psi(x)\,f(x)\,dx$ for some $\psi \in C^\infty(U)$ so that

$$\rho(\sigma) = \int_U \psi(x)\,{\rm res}(\sigma(x,\cdot))\,dx.$$

But since ρ is closed by assumption, for any $\sigma = f\otimes\tau$ with $\tau \in CS_{\rm c.c}(\mathbb{R}^d)$ and $f \in C^\infty(U)$ we have

$$0 = \rho\left(\partial_{x_i}\left(f\otimes\tau\right)\right) = \rho\left(\partial_{x_i}f\otimes\tau\right).$$

Choosing τ with non vanishing residue and integrating by parts implies that

$$\int_U \partial_{x_i}\psi(x)\,f(x)\,dx = \int_U \psi(x)\,\partial_{x_i}f(x)\,dx = 0 \quad \forall f \in C^\infty_{\rm cpt}(U).$$

Hence $\partial_{x_i}\psi = 0$ for any $i \in \{1,\cdots,d\}$ so that ψ is a constant c and $\rho(\sigma) = c\int_U {\rm res}(\sigma(x,\cdot))\,dx$ is proportional to the noncommutative residue. $\square$

Definition 38 *A linear form on a subspace $\mathcal{D}(M) \subset C\ell(M)$ which contains smoothing operators is singular provided it vanishes on these operators.*

The following result provides a first characterisation of the noncommutative residue. The proof, which uses the characterisation of the noncommutative residue on $CS_{\rm cpt}(U)$ proved in Proposition 59, is inspired by methods of [MSS] whose authors characterised the canonical trace.

Theorem 22 *Any continuous singular trace on $C\ell(M)$ is proportional to the noncommutative residue.*

Proof: Let $L : C\ell(M) \to \mathbb{C}$ be a continuous singular trace. Given a local chart (U,ϕ) on M, the map

$$\rho_\phi := L \circ \phi^* \circ {\rm Op}$$

clearly defines a singular linear form on $CS_{\rm cpt}(\phi(U))$.
For any $\sigma \in CS_{\rm cpt}(\phi(U))$ and for any $x_j, j = 1,\cdots,n$ corresponding to the coordinates in the local chart (U,ϕ) we have[25]

$$\left({\rm Op}(\partial_{\xi_j}\sigma)u\right)(x) = \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\partial_{\xi_j}\sigma(x,\xi)\hat u(\xi)d\,\xi = -i\left(\left[x_j, {\rm Op}(\sigma)\right]\,u\right)(x) \quad \forall u \in C^\infty_{\rm cpt}(U).$$

[24] This is only a first yet non optimal characterisation which will be refined later in these notes.
[25] We borrow this observation from [MSS] who use it to prove the uniqueness of the extension of the ordinary trace on trace-class operators to non integer order operators.

Furthermore,

$$
\begin{aligned}
\rho_\phi\left(\partial_{\xi_j}\sigma\right) &= L\circ\phi^*\circ \mathrm{Op}\circ\partial_{\xi_j}(\sigma)\\
&= -i\,L\circ\phi^*\circ \mathrm{ad}_{x_j}\circ \mathrm{Op}(\sigma)\\
&= -i\,L\circ \mathrm{ad}_{x_j}\circ\phi^*\circ \mathrm{Op}(\sigma)\\
&= -i\,L\left([x_j,\phi^*\circ \mathrm{Op}(\sigma)]\right).
\end{aligned}
$$

Since L vanishes on brackets, ρ_ϕ vanishes on derivatives $\partial_{\xi_j}\sigma$. Similarly, for any $u\in C^\infty_{\mathrm{cpt}}(U)$

$$
\begin{aligned}
\left(\mathrm{Op}(\partial_{x_j}\sigma)u\right)(x) &= \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\partial_{x_j}\sigma(x,\xi)\hat{u}(\xi)d\,\xi\\
&= \partial_{x_j}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\sigma(x,\xi)\hat{u}(\xi)d\,\xi - i\int_{\mathbb{R}^n}\xi_j e^{i\langle x,\xi\rangle}\,\sigma(x,\xi)\hat{u}(\xi)d\,\xi\\
&= \partial_{x_j}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\sigma(x,\xi)\hat{u}(\xi)d\,\xi - \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\,\sigma(x,\xi)\widehat{\partial_{x_j}u(\xi)}d\,\xi\\
&= \left[\partial_{x_j},\mathrm{Op}(\sigma)\right]u(x).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\rho_\phi\left(\partial_{x_j}\sigma\right) &= L\circ\phi^*\circ \mathrm{Op}\circ\partial_{x_j}(\sigma)\\
&= L\circ\phi^*\circ \mathrm{ad}_{\partial_{x_j}}\circ \mathrm{Op}(\sigma)\\
&= L\left(\left[\partial_{x_j},\phi^*\circ \mathrm{Op}(\sigma)\right]\right).
\end{aligned}
$$

Since L vanishes on brackets it follows that ρ_ϕ vanishes on derivatives $\partial_{x_i}\sigma$. It therefore satisfies Stokes' property and defines a closed singular linear form.
By Proposition 59, ρ_ϕ which is continuous as a result of the continuity of L, is proportional to the noncommutative residue so that there is a constant c_ϕ such that

$$\rho_\phi(\sigma) = L(\phi^*\mathrm{Op}(\sigma)) = c_\phi\cdot \mathrm{res}(\sigma)\quad \forall\sigma\in CS_{\mathrm{cpt}}(\phi(U)).$$

We now use a partition of unity $(U_i,\chi_i)_{i\in I}$ subordinated to an atlas $(U_i,\phi_i)_{i\in I}$ on M to write any operator $A\in C\ell(M)$ according to (18.200) as $A=\sum_{i\in I}\mathrm{Op}(\sigma_i)+R(A)$ with $\sigma_i\in CS_{\mathrm{cpt}}(\phi_i(U_i))$ and $R(A)$ some smoothng operator. Applying these results to each U_i, we infer the existence of constants $c_i, i\in I$ such that

$$L(A_i) = \rho_{\phi_i}(\sigma_i) = c_i\cdot \mathrm{res}(\sigma_i) = c_i\cdot\mathrm{res}(A_i).$$

By linearity of L and since it vanishes on smoothing operators,we infer that

$$L(A) = \sum_{i\in I} c_i\cdot \mathrm{res}(A_i).$$

But since the l.h.s is globally defined, the r.h.s is independent of the local chart and the constants are independent of i. Hence, there is a constant $c=c_i$ such that

$$L(A) = c\cdot\sum_{i\in I}\mathrm{res}(\sigma_i) = c\cdot\mathrm{res}(A).$$

□

Remark 36 *Similarly, any continuous singular graded linear form on* $\mathrm{Gr}C\ell^{*,*}(M,E)$ *is a graded trace proportional to the higher order residues* res_k, $k\in\mathbb{Z}_+$ *[L1].*

21.2 A characterisation of the canonical trace

We provide a characterisation of the canonical trace on non integer order operators. Our approach, similar in spirit to the one adopted in [MSS], stresses the role of Stokes' property of the canonical integral on non integer order symbols.
Recall from Definition 6 that a subset $\mathcal{S}$ of $CS_{\mathrm{c.c}}(\mathbb{R}^d)$ is admissible if it satifies the following conditions:

1. it is stable under partial differentiation

$$\sigma \in \mathcal{S} \Longrightarrow \partial_i \sigma \in \mathcal{S} \quad \forall i \in \{1, \cdots, d\},$$

2. for σ in $\mathcal{S}$, the symbols $\tau_{i,a-j+1}\,\chi$ with $\tau_i \sim \sum_{j=0}^{\infty} \tau_{i,a-j}\,\chi$ arising in the asymptotic expansion (2.22) can be chosen in $\mathcal{S}$.

Definition 39 *Let us call a subset $\mathcal{D}_{\rm cpt}(U)$ of $CS_{\rm cpt}(U)$ admissible whenever*

1. *$\mathcal{D}_{\rm cpt}(U)$ is stable under multiplication by functions in $C^\infty_{\rm cpt}(U)$,*
2. *the set*

$$\mathcal{S} := \{\tau \in CS_{\rm c.c}(\mathbb{R}^d), \quad f \otimes \tau \in \mathcal{D}_{\rm cpt}(U) \quad \forall f \in C^\infty_{\rm cpt}(U)\} \tag{21.218}$$

is admissible.

Remark 37 *Note that under these assumptions,*

$$(\mathcal{D}_{\rm cpt}(U) \subset {\rm Ker(res)}) \Longrightarrow (\mathcal{S} \subset {\rm Ker(res)})$$

since ${\rm res}(f \otimes \tau) = 0$ *for any f in $C^\infty_{\rm cpt}(U)$ and τ in $\mathcal{S}$ implies that* ${\rm res}(\tau)$ *vanishes.*

Example 39 *The sets $CS^{\notin \mathbb{Z}}_{\rm cpt}(U) = \cup_{a \notin \mathbb{Z}} CS^a_{\rm copt}(U)$,*

$$CS^{\rm odd}_{\rm cpt}(U) = \{\sigma \in CS_{\rm cpt}(U), \quad \sigma_{a-j}(x,-\xi) = (-1)^{a-j}\,\sigma_{a-j}(x,\xi) \quad \forall x \in U \quad \text{with} \quad a = {\rm ord}(\sigma)\}$$

and

$$CS^{even}_{\rm cpt}(U) := \{\sigma \in CS_{\rm cpt}(U), \quad \sigma_{a-j}(x,-\xi) = (-1)^{a-j+1}\,\sigma_{a-j}(x,\xi) \quad \forall x \in U \quad \text{with} \quad a = {\rm ord}(\sigma)\}$$

are admissible since we saw in the examples following Definition 6 that $CS^{\notin \mathbb{Z}}_{\rm c.c}(\mathbb{R}^d)$, $CS^{\rm odd}_{\rm c.c}(\mathbb{R}^d)$ and $CS^{\rm even}_{\rm c.c}(\mathbb{R}^d)$ are admissible sets.

Here comes a local version of Theorem 2.

Proposition 60 *Let $\mathcal{D}_{\rm cpt}(U)$ be an admissible subset of $CS_{\rm cpt}(U)$ such that:*

$$CS^{-\infty}_{\rm cpt}(U) \subset \mathcal{D}_{\rm cpt}(U) \subset {\rm Ker(res)}.$$

If $-\!\!\!\!\!\!\int_{T^*U}$ defines a closed form (i.e satisfies Stokes' property) on $\mathcal{D}_{\rm cpt}(U)$, then any other closed *continuous*[26] *linear form $\rho : \mathcal{D}_{\rm cpt}(U) \to \mathbb{C}$ is proportional to the cut-off regularised integral:*

$$\rho = c \cdot -\!\!\!\!\!\!\int_{T^*U}.$$

Proof: Since $\mathcal{D}_{\rm cpt}(U)$ is admissible, the set $\mathcal{S}$ defined by (21.218) is admissible; since $\mathcal{D}_{\rm cpt}(U)$ lies in Ker(res), $\mathcal{S}$ lies in Ker (res) so that the set $\mathcal{S}$ satisfies the assumptions of Theorem 2.
Using the density of $C^\infty_{\rm cpt}(U) \otimes \mathcal{S}^a$ in $\mathcal{D}_{\rm cpt}(U) \cap CS^a_{\rm cpt}(U)$, where we have set $\mathcal{S}^a = \mathcal{S} \cap CS^a_{\rm c.c.}(\mathbb{R}^d)$, the problem boils down to finding bilinear linear forms $(f,\tau) \mapsto b(f,\tau)$ on $C^\infty_{\rm cpt}(U) \times \mathcal{S}$ that satisfy Stokes' property in each variable and which restrict to continuous bilinear forms on $C^\infty_{\rm cpt}(U) \times \mathcal{S}^a$ for all orders a. For fixed $f \in C^\infty_{\rm cpt}(U)$, such a bilinear form induces a closed linear form $\rho_f : \tau \mapsto b(f,\tau)$ on $\mathcal{S}$ which by Theorem 2 is proportional to the cut-off regularised integral $\rho_f(\tau) = c_f -\!\!\!\!\!\!\int_{\mathbb{R}^d} \tau$ for some constant c_f.
On the other hand, since the map $f \mapsto b(f,\tau)$ is continuous on $C^\infty_{\rm cpt}(U)$ and satisfies Stokes' property, for fixed $\tau \in \mathcal{S}$ we have $b(f,\tau) = c_\tau \int_U f(x)\,dx$ for some constant c_τ.
Combining these two results shows that $b(f,\tau) = c\left(\int_U f(x)\,dx\right)\left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \tau(\xi)\,d\xi\right)$ for some constant c. With the help of the continuity and density assumptions, we infer that

$$\rho(\sigma) = c \cdot \int_U dx \left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \sigma(x,\xi)\,d\xi\right) = c \cdot -\!\!\!\!\!\!\int_{T^*U} \sigma(x,\xi)\,dx\,d\xi \quad \forall \sigma \in \mathcal{D}_{\rm cpt}(U).$$

[26] i.e. its restriction to symbols of constant order is continuous.

□

We are now ready to determine all closed linear forms on admissible subsets of $C\ell(M)$.
We call a subset $\mathcal{D}(M)$ of $C\ell(M)$ admissible if in any local chart (U,ϕ) of M, the set

$$\mathcal{D}_{\rm cpt}(\phi(U)) := \{\sigma \in CS_{\rm cpt}(\phi(U)), \quad \phi^\sharp \,{\rm Op}(\sigma) \in \mathcal{D}(M)\} \tag{21.219}$$

is an admissible subset of $CS_{\rm cpt}(\phi(U))$.

Example 40 *The sets $C\ell^{\notin \mathbb{Z}}(M)$ of non integer order classical pseudodifferential operators on M,*

$$C\ell^{\rm odd}(M) = \{A \in C\ell^{\mathbb{Z}}(M), \quad \sigma(A) \sim \sum_{j=0}^\infty \chi\, \sigma_{a-j}(A), \quad \sigma_{a-j}(A)(x,-\xi) = (-1)^{a-j}\sigma_{a-j}(A)(x,\xi)\}$$

of odd-class operators on odd dimensional manifolds M introduced in [KV] (see also [?] where such operators are called even-even) and

$$C\ell^{\rm even}(M) = \{A \in C\ell^{\mathbb{Z}}(M), \quad \sigma(A) \sim \sum_{j=0}^\infty \chi\, \sigma_{a-j}(A), \quad \sigma_{a-j}(A)(x,-\xi) = (-1)^{a-j+1}\sigma_{a-j}(A)(x,\xi)\}$$

of even-class operators on even dimensional manifolds M (see [?] where such operators are called even-odd) are all admissible subsets of $C\ell(M)$.

Theorem 23 *Let $\mathcal{D}(M)$ be an admissible susbset of $C\ell(M)$. We further assume that*

$$C\ell^{-\infty}(M) \subset \mathcal{D}(M) \subset {\rm Ker}\,({\rm res})\,.$$

Provided the canonical trace is well-defined on $\mathcal{D}(M)$ and vanishes on brackets in $\mathcal{D}(M)$ then any continuous [27] linear form [28] $L : \mathcal{D}(M) \to \mathbb{C}$ which vanishes on brackets:

$$L\left([A,B]\right) = 0 \quad \forall A, B \in C\ell(M) \quad \text{s.t.} \quad [A,B] \in \mathcal{D}(M) \tag{21.220}$$

is proportional to the canonical trace:

$$\exists c \in \mathbb{C}, \quad L(A) = c \cdot {\rm TR}(A) \quad \forall A \in \mathcal{D}(M).$$

Proof: Given any local chart (U,ϕ) on M, the set $\mathcal{D}(\phi(U))$ defined by (21.219) fulfills the assumptions of Proposition 60 with U replaced by $\phi(U)$; in particular $\mathcal{D}(\phi(U)) \subset {\rm Ker(res)}$ for if the operator residue ${\rm res}\left(\phi^\sharp {\rm Op}(\sigma)\right)$ vanishes then so does the symbol residue ${\rm res}(\sigma)$.
From a linear form L on $\mathcal{D}(M)$ which obeys the requirements of the theorem, we build the linear form $\rho_\phi := L \circ \phi^\sharp{\rm Op}$ on $\mathcal{D}(\phi(U))$ which obeys the requirements of Proposition 60, from which we infer that ρ_ϕ is proportional to the cut-off regularised integral. Hence, there is a constant c_ϕ such that

$$\rho_\phi(\sigma) = L(\phi^\sharp{\rm Op}(\sigma)) = c_\phi \cdot -\!\!\!\!\!\!\int_{T^*\phi(U)} \sigma \quad \forall \sigma \in \mathcal{D}_{\rm cpt}(\phi(U)).$$

As before, we use a partition of unity to write an operator $P \in C\ell(M)$ as a finite sum $P = \sum_{i\in I} \phi_i^\sharp {\rm Op}(p_i) + R(P)$ for some symbols $p_i, i \in I$ in $CS_{\rm cpt}(\phi_i(U_i))$ and a smoothing operator $R(P)$. Since L restricts to a trace on the algebra of smoothing operators, by [?] (see Appendix), its restriction to $C\ell^{-\infty}(M)$ is proportional to the L^2-trace tr.
It follows by linearity of L that there are constants $c_{\phi_i}, i \in I$ and c such that :

$$L(P) = \sum_{i\in I} c_{\phi_i} \cdot -\!\!\!\!\!\!\int_{T^*\phi_i(U_i)} p_i + c\,{\rm tr}(R(P)).$$

But the constants ϕ_i coincide for the l.h.s being globally defined, the r.h.s is independent of the local chart. They further coincide with c for a smooth perturbation of the symbol induces an extra contribution to the trace on the smoothing operator. We conclude the existence of some constant c such

[27] i.e. which restricts to a continuous map on $D(M) \cap C\ell^a(M)$ for any $a \in \mathbb{C}$.
[28] By linear we mean here that $L(\alpha A + \beta B) = \alpha L(A) + \beta L(B)$ whenever A, B and $\alpha A + \beta B \in \mathcal{D}(M)$

that $L(P) = c \cdot \int\!\!\!\!\!-_{T^*M} \sigma(P) = c \cdot \mathrm{TR}(P)$. □

Here are a few known examples of sets $\mathcal{D}(M)$ which obey Assumptions 1 and 2 of the above theorem. In particular, they lie in Ker(res). Applying Theorem 23 to $\mathcal{D}(M) = C\ell^{\notin \mathbb{Z}}(M)$, resp. $\mathcal{D}(M) = C\ell^{\mathrm{odd}}(M)$ in odd dimensions, resp. $\mathcal{D}(M) = C\ell^{\mathrm{even}}(M)$ in even dimensions, leads to the following uniqueness result.

Corollary 12 *The canonical trace is (up to a multiplicative constant) the unique linear form on $C\ell^{\notin \mathbb{Z}}(M)$, resp. $C\ell^{\mathrm{odd}}(M)$ in odd dimensions, resp. $C\ell^{\mathrm{even}}(M)$ in even dimensions which is continuous on operators of constant order and which vanishes on brackets that lie in $C\ell^{\notin \mathbb{Z}}(M)$, resp. $C\ell^{\mathrm{odd}}(M)$ in odd dimensions, resp. $C\ell^{\mathrm{even}}(M)$.*

Remark 38 *In the course of the proof we showed that the vanishing of L on brackets (21.220) implies Stokes' property for ρ_ϕ. Conversely, Stokes' property for ρ_ϕ implies that $L(\phi^\sharp \mathrm{Op}(\sigma)) := \rho_\phi(\sigma)$ vanishes on brackets $[x_i, \cdot]$ and $[\partial_{x_i}, \cdot]$ contained in $\mathcal{D}(M)$. But this implies that L vanishes on brackets $[P_U, \cdot] \in \mathcal{D}(M)$ where P_U is the localisation of any classical pseudodifferential operator. Stokes' property on symbols and the vanishing on brackets of operators are therefore equivalent.*

22 Complex powers and logarithms

22.1 Complex powers

We review the construction and properties of complex powers and logarithms of elliptic operators.

An operator $A \in C\ell(M,E)$ has principal angle θ if for every $(x,\xi) \in T^*M - \{0\}$, the leading symbol $\sigma^L(A)(x,\xi)$ has no eigenvalue on the ray $L_\theta = \{re^{i\theta}, r \geq 0\}$; in that case A is elliptic.

Definition 40 *We call an operator $A \in C\ell(M,E)$ admissible with spectral cut θ if A has principal angle θ and the spectrum of A does not meet $L_\theta = \{re^{i\theta}, r \geq 0\}$. In particular such an operator is invertible and elliptic. Since the spectrum of A does not meet L_θ, θ is called an Agmon angle of A.*

Remark 39 *In applications, an invertible operator A is usually obtained from an essentially self-adjoint elliptic operator $B \in C\ell(M,E)$ by setting $A = B + \pi_B$ using the orthogonal projection π_B onto the kernel $Ker(B)$ of B corresponding to the orthogonal splitting $L^2(M,E) = \mathrm{Ker}(B) \oplus \mathrm{R}(B)$ where $R(B)$ is the (closed) range of B. Here $L^2(M,E)$ denotes the closure of $C^\infty(M,E)$ w.r. to a Hermitian structure on E combined with a Riemannian structure on M.*

Let $A \in C\ell(M,E)$ be admissible with spectral cut θ and positive order a. For $\mathrm{Re}(z) < 0$, the complex power A_θ^z of A is defined by the Cauchy integral [Se1]

$$A_\theta^z = \frac{i}{2\pi} \int_{\Gamma_{r,\theta}} \lambda_\theta^z (A-\lambda)^{-1}\, d\lambda$$

where $\lambda_\theta^z = |\lambda|^z e^{iz(\arg\lambda)}$ with $\theta \leq \arg\lambda < \theta + 2\pi$. In particular, for $z = 0$, we have $A_\theta^0 = I$.
Here

$$\Gamma_{r,\theta} = \Gamma_{r,\theta}^1 \cup \Gamma_{r,\theta}^2 \cup \Gamma_{r,\theta}^3 \tag{22.221}$$

where

$$\Gamma_{r,\theta}^1 = \{\rho\, e^{i\theta}, \infty > \rho \geq r\}$$
$$\Gamma_{r,\theta}^2 = \{\rho\, e^{i(\theta - 2\pi)}, \infty > \rho \geq r\}$$
$$\Gamma_{r,\theta}^3 = \{r\, e^{it}, \theta - 2\pi \leq t < \theta\},$$

is a contour along the ray L_θ around the non zero spectrum of A. Here r is any small positive real number such that $\Gamma_{r,\theta} \cap Sp(A) = \emptyset$.
The operator A_θ^z is a classical pseudodifferential operator of order az with homogeneous components of the symbol of A_θ^z given by

$$\sigma_{az-j}(A_\theta^z)(x,\xi) = \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda_\theta^z\, b_{-a-j}(x,\xi,\lambda)\, d\lambda$$

where the components b_{-a-j} are the positive homogeneous components of the resolvent $(A - \lambda I)^{-1}$ in $(\xi, \lambda^{\frac{1}{a}})$). In particular, its leading symbol is given by $\left(\sigma(A_\theta^z)(x,\xi)\right)^L = \left((\sigma(A)(x,\xi))^L\right)_\theta^z$ and hence A_θ^z is elliptic.
The definition of complex powers can be extended to the whole complex plane by setting $A_\theta^z := A^k A_\theta^{z-k}$ for $k \in \mathbb{N}$ and $\mathrm{Re}(z) < k$; this definition is independent of the choice of k in $\mathbb{N}$ and preserves the usual properties, i.e. $A_\theta^{z_1} A_\theta^{z_2} = A_\theta^{z_1+z_2}$, $A_\theta^k = A^k$, for $k \in \mathbb{Z}$.
Complex powers of operators depend on the choice of spectral cut. Wodzicki [Wo1] (Ponge in [Po1], see Proposition 4.1, further quotes an unpublished paper by Wodzicki [Wo2]) established the following result.

Proposition 61 *[Wo1, Wo2, Po1] Let θ and ϕ be two spectral cuts for an admissible operator A in $C\ell(M,E)$ such that $0 \leq \theta < \phi < 2\pi$. The complex powers for these two spectral cuts are related by*

$$A_\theta^z - A_\phi^z = \left(1 - e^{2i\pi z}\right) \Pi_{\theta,\phi}(A) A_\theta^z, \tag{22.222}$$

where we have set $\Pi_{\theta,\phi}(A) = A\left(\frac{1}{2i\pi}\int_{\Gamma_{\theta,\phi}} \lambda^{-1}(A-\lambda)^{-1}\, d\lambda\right)$ where $\Gamma_{\theta,\phi}$ is a contour around the cone

$$\Lambda_{\theta,\phi} := \{\rho\, e^{it}, \infty > \rho \geq r, \quad \theta < t < \phi\}. \tag{22.223}$$

Remark 40 Formula (22.222) generalises to spectral cuts θ and ϕ such that $0 \leq \theta < \phi + 2k\pi < (2k+1)\pi$ for some non negative integer k by

$$A_\theta^z - A_\phi^z = e^{2ik\,\pi z}\, I + \left(1 - e^{2i\pi z}\right) \Pi_{\theta,\phi}(A) A_\theta^z. \tag{22.224}$$

If the cone $\Lambda_{\theta,\phi}$ defined by (22.223) delimited by the angles θ and ϕ does not intersect the spectrum of the leading symbol of A, it only contains a finite number of eigenvalues of A and $\Pi_{\theta,\phi}(A)$ is a finite rank projection and hence a smoothing operator. In general (see Propositions 3.1 and 3.2 in [Po1]), $\Pi_{\theta,\phi}(A)$, which is a pseudodifferential projection, is a zero order operator with leading symbol given by $\pi_{\theta,\phi}(\sigma^L(A))$ defined similarly to $\Pi_{\theta,\phi}$ replacing A by the leading symbol $\sigma^L(A)$ of A so that:

$$\sigma^L\left(\Pi_{\theta,\phi}(A)\right) = \pi_{\theta,\phi}(\sigma^L(A)) := \sigma^L(A) \left(\frac{1}{2i\pi} \int_{\Gamma_{\theta,\phi}} \lambda^{-1} (\sigma^L(A) - \lambda)^{-1}\, d\lambda\right),$$

where we have set $\sigma^L(B)(x,\xi) = \sigma^L(B)(x,\xi)$ for any $(x,\xi) \in T^*M$ and any $B \in Cl(M,E)$.

22.2 The logarithm of an admissible operator

The logarithm of an admissible operator A with spectral cut θ is defined in terms of the derivative at $z = 0$ of this complex power:

$$\log_\theta(A) = \partial_z A^z_{\theta}{}_{|_{z=0}}.$$

Remark 41 For a real number t, A and A_θ^t have spectral cuts θ and $t\theta$; for t close to one, $(A_\theta^t)^z_{t\theta} = (A_\theta^t)^z_\theta$ and hence, $(A_\theta^t)^z_{t\theta} = A_\theta^{tz}$ so that

$$\log_\theta(A^t) = \partial_z (A_\theta^t)^z_{t\theta}{}_{|_{z=0}} = \partial_z (A_\theta^{tz})_{|_{z=0}} = t \log_\theta A.$$

Just as complex powers, the logarithm depends on the choice of spectral cut [O1]. Indeed, differentiating (22.222) w.r. to z at $z = 0$ yields for spectral cuts θ, ϕ such that $0 \leq \theta < \phi < 2\pi$ (compare with formula (1.4) in [O1]):

$$\log_\theta A - \log_\phi A = -2i\pi \Pi_{\theta,\phi}(A). \tag{22.225}$$

Formula (22.225) generalises to spectral cuts θ and ϕ such that $0 \leq \theta < \phi + 2k\pi < (2k+1)\pi$ for some non negative integer k by

$$\log_\theta A - \log_\phi A = 2ik\,\pi\, I - 2i\pi \Pi_{\theta,\phi}(A). \tag{22.226}$$

As a result of the above discussion and as already observed in [O1], when the leading symbol $\sigma^L(A)$ has no eigenvalue inside the cone $\Lambda_{\theta,\phi}$ delimited by $\Gamma_{\theta,\phi}$ then $\Pi_{\theta,\phi}$ which is a finite rank projection, is smoothing.

Logarithms of classical pseudodifferential operators are not classical anymore since their symbols involve a logarithmic term $\log|\xi|$. As the following elementary result shows, they are of log-polyhomogeneous of log-type 1.

Proposition 62 *Let $A \in C\ell(M,E)$ be an admissible operator with spectral cut θ. In a local trivialisation, the symbol of $\log_\theta(A)$ reads:*

$$\sigma_{\log_\theta(A)}(x,\xi) = a\, \log|\xi| I + \sigma_0(A)(x,\xi) \tag{22.227}$$

where a denotes the order of A and $\sigma_0(A)$ a symbol of order zero.
Moreover, the leading symbol of $\sigma_0(A)$ is given by

$$\sigma_0^L(A)(x,\xi) = \log_\theta\left(\sigma^L(A)(x, \frac{\xi}{|\xi|})\right) \quad \forall (x,\xi) \in T^*M - \{0\}. \tag{22.228}$$

In particular, if $\sigma(A)$ has scalar leading symbol then so have $\sigma_\theta(A)$ and $\sigma\left(\Pi_{\theta,\phi}(A)\right)$ for any other spectral cut ϕ.

Proof: Given a local trivialisation over some local chart, the symbol of A_θ^z has the formal expansion $\sigma(A_\theta^z) \sim \sum_{j\geq 0} b_{az-j}^{(z)}$ where a is the order of A and $b_{az-j}^{(z)}$ is a positively homogeneous function of degree $az-j$. Since $\log_\theta A = \partial_z A^z_{\theta|_{z=0}}$, the formal expansion of the symbol of $\log_\theta A$ is $\sigma_{\log_\theta A} \sim \sum_{j\geq 0} \partial_z b^{(z)}_{az-j|_{z=0}}$
Since $A^z_{\theta|_{z=0}} = I$, we have $\sigma(A_\theta^z)_{|_{z=0}} \sim I$ where now I stands for the identity on matrices. Thus $b_{az-j}^{(z)}(x,\xi)_{|_{z=0}} = \delta_{0,j} I$.
Suppose that $\xi \neq 0$; using the positive homogeneity of the components, we have: $b_{az}^{(z)}(x,\xi) = |\xi|^{az} b_{az}^{(z)}\left(x, \frac{\xi}{|\xi|}\right)$; hence

$$\partial_z b_{az}^{(z)}(x,\xi) = a \log|\xi| |\xi|^{az} b_{az}^{(z)}\left(x, \frac{\xi}{|\xi|}\right) + |\xi|^{az} \partial_z b_{az}^{(z)}\left(x, \frac{\xi}{|\xi|}\right).$$

It follows that

$$\partial_z b_{az}^{(z)}(x,\xi)_{|_{z=0}} = a\log|\xi| I + \partial_z b_{az}^{(z)}\left(x, \frac{\xi}{|\xi|}\right)_{|_{z=0}}.$$

Similarly for $j>0$, we have $\quad b_{az-j}^{(z)}(x,\xi) = |\xi|^{az-j} b_{az-j}^{(z)}\left(x, \frac{\xi}{|\xi|}\right)$ so that

$$\partial_z b_{az-j}^{(z)}(x,\xi) = a\log|\xi||\xi|^{az-j} b_{az-j}^{(z)}\left(x, \frac{\xi}{|\xi|}\right) + |\xi|^{az-j} \partial_z b_{az-j}^{(z)}\left(x, \frac{\xi}{|\xi|}\right).$$

Consequently,

$$\partial_z b_{az-j}^{(z)}(x,\xi)_{|_{z=0}} = |\xi|^{-j} \partial_z b_{az-j}^{(z)}\left(x, \frac{\xi}{|\xi|}\right)_{|_{z=0}}.$$

The terms $\partial_z b_{az}^{(z)}(x, \frac{\xi}{|\xi|})$ and $\partial_z b_{az-j}^{(z)}(x, \frac{\xi}{|\xi|})$ are homogeneous functions of degree 0 in ξ. Summing up, we obtain

$$\sigma(\log_\theta(A))(x,\xi) = a\ \log|\xi| I + \sigma_\theta^A(x,\xi)$$

where $\sigma_0(A)(x,\xi) = \partial_z b_{az}^{(z)}(x, \frac{\xi}{|\xi|})_{|_{z=0}} + \sum_{j>0} |\xi|^{-j} \partial_z b_{az-j}^{(z)}(x, \frac{\xi}{|\xi|})_{|_{z=0}}$. $\sigma_0(A)$ is a symbol of order 0. Its leading symbol reads $(\sigma_0(A)(x,\xi))^L = \partial_z b_{az}^{(z)}(x, \frac{\xi}{|\xi|})_{|_{z=0}} = \partial_z \left(\sigma^L(A)(x, \frac{\xi}{|\xi|}\right)^z_{\theta_{|_{z=0}}} = \log_\theta \sigma^L(A)(x, \frac{\xi}{|\xi|})$ for any (x,ξ) in $T^*M - \{0\}$. $\square$

Remark 42 *Powers of the logarithm of a given admissible operator Q combined with all classical pseudodifferential operators generate the algebra of log-polyhomogenous operators [Ou]:*

$$C\ell^{*,*}(M,E) = \bigoplus_{k=0}^{\infty} C\ell(M,E)\ \log^k Q = \bigoplus_{k=0}^{\infty} \log^k Q\, C\ell(M,E).$$

22.3 Classical operators built from logarithms

Useful classical operators can be built from logarithms of operators, the first instances being the difference of two logarithms and the bracket of a classical operator with a logarithm.

Proposition 63 *1. Let A and B be two admissible operators in $C\ell(M,E)$ with spectral cuts θ and ϕ and positive orders a and b. Then*

$$\frac{\log_\theta A}{a} - \frac{\log_\theta B}{b} \in C\ell^0(M,E). \tag{22.229}$$

2. Let Q admissible operator in $C\ell(M,E)$ with spectral cut θ and and positive order q.Then

$$A \in C\ell(M,E) \Longrightarrow [\log_\theta Q, A] \in C\ell(M,E). \tag{22.230}$$

Proof: This follows from (22.227).

1. Indeed, with the notations of (22.227), the operator $\frac{\log_\theta A}{a}$ has symbol $\log|\xi| + \frac{\sigma_0(A)(x,\xi)}{a}$ and the operator $\frac{\log_\phi B}{b}$ has symbol $\log|\xi| + \frac{\sigma_0(B)(x,\xi)}{b}$, so that their difference has symbol $\frac{\sigma_0(A)(x,\xi)}{a} - \frac{\sigma_0(B)(x,\xi)}{b}$ which is classical of order zero.

2. By (20.217), the symbol of $[\log Q, A]$ is given by the star bracket

$$\{\sigma(\log_\theta Q), \sigma(A)\}_* \sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha \sigma(\log_\theta Q)\partial_x^\alpha \sigma(A) - \partial_\xi^\alpha \sigma(A)\partial_x^\alpha \sigma(\log_\theta Q)\right),$$

in which the logarithm $\log|\xi|$ only arises once, namely for $\alpha = 0$ in the expression $[q \log|\xi|, \sigma(A)]$ which clearly vanishes. Thus $[\log_\theta Q, A]$ is classical.

□

Logarithms of operators naturally arise in determinants since their logarithms are expected to be traces of logarithms. In that context we shall come across the following operator.

Proposition 64 *Given admissible classical operators A with spectral cut θ, B with spectral cut ϕ and their product AB with spectral cut ψ, the expression*

$$L(A,B) := \log_\psi(AB) - \log_\theta A - \log_\phi B$$

lies in $C\ell^0(M,E)$ and has leading symbol $L(\sigma^L(A), \sigma^L(B))\left(x, \frac{\xi}{|\xi|}\right)$.

Proof: By formula (22.225), another choice of spectral cut only changes the logarithms by adding an operator in $C\ell^0(M,E)$ so that it will not affect the statement. As usual, we drop the explicit mention of spectral cut assuming the operators have common spectral cuts.
If A has order a and B has order b then AB has order $a+b$, we have

$$\begin{aligned}\sigma(L(A,B)) &= \sigma(\log AB)(x,\xi) - \sigma(\log A)(x,\xi) - \sigma(\log B)(x,\xi)\\ &= (a+b)\,\log|\xi|\,I + \sigma_0(AB)(x,\xi) - a\,\log|\xi| I\\ &\quad -\sigma_0(A)(x,\xi) - b\,\log|\xi|\,I - \sigma_0(B)(x,\xi)\\ &\sim \sigma_0(AB)(x,\xi) - \sigma_0(A)(x,\xi) - \sigma_0(B)(x,\xi)\end{aligned} \tag{22.231}$$

so that the operator $L(A,B)$ is indeed classical of order 0 and by (22.228) it has leading symbol given for any $(x,\xi) \in T^*M/M$ by

$$\begin{aligned}\sigma_0^L\left(L(A,B)\right)(x,\xi) &= \log\sigma^L(AB)(x,\frac{\xi}{|\xi|}) - \log\sigma^L(A)(x,\frac{\xi}{|\xi|}) - \log\sigma^L(B)(x,\frac{\xi}{|\xi|})\\ &=: L(\sigma^L(A),\sigma^L(B))\left(x,\frac{\xi}{|\xi|}\right).\end{aligned}$$

□

One can also build classical operators from squares of logarithms.

Proposition 65 *Let A, B be admissible operators in $C\ell(M,E)$ with positive orders a, b and spectral cuts θ and ϕ respectively and such that AB (which is elliptic) is also admissible with spectral cut ψ. Then*

$$K(A,B) := \frac{1}{2(a+b)}\log_\psi^2 A\,B - \frac{1}{2a}\log_\theta^2 A - \frac{1}{2b}\log_\phi^2 B$$

is log-polyhomogeneous of log-type one but

$$L(A,B)\,\frac{\log A}{a} - K(A,B) \in C\ell^0(M,E), \quad L(A,B)\,\frac{\log B}{b} - K(A,B) \in C\ell^0(M,E).$$

Proof: By formula (22.225), another choice of spectral cut only changes the logarithms by adding an operator in $C\ell^0(M,E)$ so that it will not affect the statement. As usual, we drop the explicit mention of spectral cut assuming the operators have common spectral cuts.
An explicit computation on symbols shows the result. Indeed, since $\sigma(\log A)(x,\xi) \sim a \log|\xi| + \sigma_0(A)(x,\xi)$, we have

$$\begin{aligned}
\sigma(\log^2 A)(x,\xi) &= \sigma(\log A) \star \sigma(\log A)(x,\xi) \\
&\sim a^2 \log^2|\xi| I + 2a \log|\xi|\, \sigma_0(A)(x,\xi) + \sigma_0(A)(x,\xi)\cdot\sigma_0(A)(x,\xi) \\
&\quad + \sum_{\alpha\neq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_0(A)(x,\xi)\, \partial_x^\alpha \sigma_0(A)(x,\xi).
\end{aligned}$$

This yields:

$$\begin{aligned}
\sigma_K(x,\xi) \sim\ & \log|\xi| \left(\sigma_0(AB) - \sigma_0(A) - \sigma_0(B)\right)(x,\xi) \\
& + \frac{1}{2(a+b)} \sigma_0(AB)(x,\xi)\sigma_0(AB)(x,\xi) + \sum_{\alpha\neq 0} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_0(AB)(x,\xi) D_x^\alpha \sigma_0(AB)(x,\xi) \\
& - \frac{1}{2a} \sigma_0(A)(x,\xi)\sigma_0(A)(x,\xi) - \sum_{\alpha\neq 0} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_0(A)(x,\xi) D_x^\alpha \sigma_0(A)(x,\xi) \\
& - \frac{1}{2b} \sigma_0(B)(x,\xi)\sigma_0(B)(x,\xi) - \sum_{\alpha\neq 0} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_0(B)(x,\xi) D_x^\alpha \sigma_0(B)(x,\xi)
\end{aligned}$$

from which we infer that $K(A,B)$ has a symbol of the form

$$\sigma(K(A,B)) \sim \log|\xi|(\sigma_0(AB) - \sigma_0(A) - \sigma_0(B)) + \sigma_0(K).$$

It is therefore log-polyhomogeneous of log-type one.
On the other hand, (22.231) combined with (22.227) shows that the symbols of the operators $L(A,B)\,\frac{\log A}{a}$ and $L(A,B)\,\frac{\log B}{b}$ differ from $\ln|\xi|\left(\sigma_0(AB) - \sigma_0(A) - \sigma_0(B)\right)(x,\xi)$ by a classical symbol of order zero, from which we infer the second part of the statement. □

23 The noncommutative residue as a complex residue

Integrating over the underlying manifold the formula which relates the noncommutative residue on symbols to a complex residue, expresses the noncommutative residue on operators in terms of complex residues.

23.1 The noncommutative residue and canonical trace extended to matrix valued operators

Given a closed manifold M, we saw that the (higher) noncommutative residue, resp. the canonical trace on operators in $C\ell^{*,*}(M)$ are of the type

$$\Lambda(A) = \int_M \lambda(\sigma(A)(x,\cdot))\, dx,$$

where λ is the (higher) noncommutative residue, resp. the canonical integral on symbols. This can be extended to operators in $C\ell^{*,*}(M,E)$ where $\pi : E \to M$ is some finite rank vector bundle, by inserting the matrix trace tr:

$$\Lambda(A) = \int_M \lambda(\mathrm{tr}(\sigma(A))(x,\cdot))\, dx. \tag{23.232}$$

Applying (23.232) to $\lambda = \mathrm{res}_k$ and $\lambda = \mathrm{res}$ yields the higher and ordinary noncommutative residue on matrix valued operators.

Definition 41 *The higher noncommutative residue of an operator $A \in C\ell^{*,k}(M,E)$ is defined by*

$$\mathrm{res}_k(A) = \int_{S^*M} \mathrm{tr}\left(\sigma_{-d,k}(A)\right)(x,\xi)\, đ_S\xi\, dx, \tag{23.233}$$

where $S^*M \subset T^*M$ stands for the cotangent unit sphere of M. When $k = 0$, this yields the ordinary noncommutative residue on $C\ell(M,E)$:

$$\mathrm{res}(A) = \int_{S^*M} \mathrm{tr}\left(\sigma_{-d}(A)\right)(x,\xi)\, đ_S\xi\, dx. \tag{23.234}$$

As in the case of scalar valued operators, higher order residues $\mathrm{res}_k, k \in \mathbb{Z}_+$ define a graded trace on $\mathrm{Gr}C\ell^{*,*}(M,E) = \sum_{k=0}^\infty \mathrm{Gr}_k C\ell^{*,*}(M,E)$, where $\mathrm{Gr}_k C\ell^{*,*}(M,E) = C\ell^{*,k}(M,E)/C\ell^{*,k-1}(M,E)$. since [L1]

$$A \in C\ell^{*,k}(M,E), B \in C\ell^{*,l}(M,E) \Longrightarrow \mathrm{res}_{k+l}\,[A,B]) = 0,$$

resp. a trace on $C\ell(M,E)$ since for $k = l = 0$ this implies

$$A \in C\ell(M,E), B \in C\ell(M,E) \Rrightarrow \mathrm{res}\,[A,B]) = 0.$$

Applying (23.232) to $\lambda = -\!\!\!\!\!\!\int_{\mathbb{R}^d}$ gives rise to the canonical trace on non integer order matrix valued operators.

Definition 42 *The canonical trace of an operator $A \in C\ell^{\notin\mathbb{Z},*}(M,E)$ is defined by*

$$\mathrm{TR}(A) = -\!\!\!\!\!\!\int_{T^*M} \mathrm{tr}\left(\sigma(A)\right)(x,\xi)\, đ\xi\, dx. \tag{23.235}$$

As in the case of scalar valued operators, the canonical trace vanishes on non integer order brackets

$$[A,B] \in C\ell^{\notin\mathbb{Z},*}(M,E) \Longrightarrow \mathrm{TR}\left([A,B]\right) = 0.$$

23.2 A fundamental formula

Given an operator $A \in C\ell(M,E)$ and an admissible elliptic operator $Q \in C\ell(M,E)$ with positive order and spectral cut θ, the family

$$z \mapsto A\, Q_\theta^{-z}$$

yields a holomorphic family of pseudodifferential operators in the following sense.

Definition 43 *Let $(A(z))_{z\in\Omega}$ be a family of classical pseudodifferential operators in $C\ell(M,E)$ with distribution kernels $(x,y)\mapsto K_{A(z)}(x,y)$. The family is holomorphic if*

1. *the order $\alpha(z)$ of $A(z)$ is holomorphic in z,*

2. *in any local trivialisation of E, we can write $A(z)$ in the form $A(z) = Op(\sigma_z) + R(z)$, for some holomorphic family of $End(V)$-valued symbols $(\sigma(z))_{z\in\Omega}$ where V is the model space of the fibres of E, and some holomorphic family $(R(z))_{z\in\Omega}$ of smoothing operators i.e. given by a holomorphic family of smooth Schwartz kernels,*

3. *the (smooth) restrictions of the distribution kernels $K_{A(z)}$ to the complement of the diagonal $\Delta \subset M\times M$, form a holomorphic family with respect to the topology given by the uniform convergence in all derivatives on compact subsets of $M \times M - \Delta$.*

A holomorphic family of log-polyhomogeneous operators of holomorphic order $\alpha(z)$ parametrised by Ω has integer order $\geq -d$ on the set $\Omega \cap \alpha^{-1}(\mathbb{Z}\cap[-d,\infty[)$. Outside that set, the canonical trace $\mathrm{TR}\,(A(z))$ defines a holomorphic map in the complex variable z.

Theorem 24 *[KV][L1] Let for any non positive integer k, $z\mapsto A(z)\in C\ell^{\alpha(z),k}(M,E)$ be a holomorphic family of log-polyhomogeneous symbols on a domain $\Omega\subset\mathbb{C}$. Then the map*

$$z \mapsto \mathrm{TR}\,(A(z))$$

is meromorphic with poles at points $z_j \in \Omega\cap\alpha^{-1}([-d,+\infty[\cap\mathbb{Z})$ of order $\leq k+1$.
For any $A\in C\ell^{*,k}(M,E)$ and any holomorphic family $A(z)\in C\ell^{*,k}(M,E)$ with order $\alpha(z)$ such that $A(0)=A$ and $\alpha'(0)\neq 0$, the following expression defined in terms of the complex residue of order $k+1$ at 0:

$$\mathrm{res}_k(A) := (-1)^{k+1}\,(\alpha'(0))^{k+1}\,\mathrm{Res}^{k+1}_{z=0}\mathrm{TR}\,(A(z))\,, \tag{23.236}$$

is independent of the family $A(z)$.
When $k=0$, i.e. for $A\in C\ell^{\alpha(z)}(M,E)$ there are only poles of order 1 whose residues are given by

$$\mathrm{Res}_{z=z_0}\mathrm{TR}\,(A(z_0)) = -\frac{1}{\alpha'(z_0)}\mathrm{res}(A(z_0)). \tag{23.237}$$

Remark 43 *These formulae provide an alternative way to define the (higher order) residue from the canonical trace, namely via complex residues in terms of the canonical trace.*

Proof: Applying formula (7.53) relative to the residue of cut-off regularised integrals of holomorphic families of symbols to the holomorphic family of symbols $\mathrm{tr}\,(\sigma(A(z)))$, yields the meromorphy of the map $z\mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}\mathrm{tr}\,(\sigma(A))\,(z)(x,\xi)\,đ\xi$ with poles at most of order $k+1$. Moreover, at a pole z_0 we have

$$\mathrm{Res}^{k+1}_{z_0} -\!\!\!\!\!\!\int_{\mathbb{R}^d}\mathrm{tr}\,(\sigma\,(A(z)))\,(x,\xi)\,đ\xi = \frac{(-1)^{k+1}}{(\alpha'(z_0))^{k+1}}\mathrm{res}_{x,k}\,(\sigma\,(A(z_0)(x,\cdot)))\,. \tag{23.238}$$

Outside the poles, the operators have non integer order so that $-\!\!\!\!\!\!\int_{\mathbb{R}^d}\mathrm{tr}\,(\sigma\,(A(z)))\,(x,\xi)\,đ\xi$ integrates over M to $\mathrm{TR}(A(z))$. Since $z\mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^d}\mathrm{tr}\,(\sigma\,(A(z)))\,(x,\xi)\,đ\xi$ defines a meromorphic map with poles of order $\leq k+1$ so does $z\mapsto\mathrm{TR}(A(z))$. Integrating (23.238) over M leads to (23.237). □

Remark 44 *This formula applied to $z_0=0$ and $A(0)=A$ gives back known properties of the non-commutative residue.*

1. *When A has order $< -d$ or non integer order, the same holds for the family $A(z)$ in a small neighborhood of 0 so that $\mathrm{Tr}(A(z))$ has no pole in that neighborhood and defines a holomorphic map. Consequently, $\mathrm{res}_k(A)$ which is proportional to the residue at 0, vanishes.*

2. *Given two log-polyhomogeneous operators A, B of log-types k, l respectively, and two holomorphic families $A(z)$ and $B(z)$ of log-types k, l respectively such that $A(0) = A$, $B(0) = B$ and $\alpha'(0) \neq 0$, $\beta'(0) \neq 0$ and $\alpha'(0) + \beta'(0) \neq 0$, the bracket $C(z) := [A(z), B(z)]$ defines a holomorphic family of order $\gamma(z) := \alpha(z) + \beta(z)$ of log-tye $k + l$ such that $C(0) = [A, B]$ and $\gamma'(0) \neq 0$.*
   By (23.236) applied to $C(z)$ and $z_0 = 0$ we get:

$$\mathrm{res}_k([A, B]) = (-1)^{k+1} \left(\alpha'(z_0)\right)^{k+1} \mathrm{Res}_{z_0}^{k+1} \mathrm{TR}\left([A(z), B(z)]\right) = 0.$$

   Here we have used the property that the canonical trace vanishes on brackets of non integer order operators.

Theorem 24 applied to $A(z) = A\, Q_\theta^{-z}$ where Q is an admissible operator in $C\ell(M, E)$ with spectral cut θ and positive order q, yields a meromorphic extension

$$\zeta_\theta^{\mathrm{mer}}(A, Q)(z) := \mathrm{TR}\left(A\, Q_\theta^{-z}\right), \tag{23.239}$$

of the generalised ζ-function

$$\zeta(A, Q)(z) := \mathrm{Tr}\left(A\, Q_\theta^{-z}\right), \tag{23.240}$$

which is holomorphic on the half plane $\mathrm{Re}(z) < \frac{a+d}{q}$. Its poles and finite part are given by the following formula.

Proposition 66 *For any $A \in C\ell^{*,k}(M, E)$ and any admissible operator $Q \in C\ell(M, E)$ with spectral cut θ and with positive order q, $\zeta_\theta^{\mathrm{mer}}(A, Q)(z)$ has poles of order $\leq k+1$ in the discrete set $\{\frac{-a+d-k}{q}, k \in \mathbb{N}_0\}$ expressed in terms of higher order residues:*

$$\mathrm{Res}_0^{k+1} \zeta_\theta^{\mathrm{mer}}(A, Q)(z) = q^{k+1}\, \mathrm{res}_k(A). \tag{23.241}$$

When A is classical, the pole is simple and

$$\mathrm{Res}_0 \zeta_\theta^{\mathrm{mer}}(A, Q)(z) = q\, \mathrm{res}(A). \tag{23.242}$$

□

24 Tracial anomalies/discrepancies

One way to extend the canonical trace TR beyond non integer order operators, namely to an integer order operator A, is to pick the constant term in the Laurent expansion of the canonical trace $\mathrm{TR}(A(z))$ of some holomorphic perturbation $A(z)$ of A around zero. The choice of constant term depends on the choice of regularised evaluator. In spite of their name, the resulting linear forms called regularised traces, are not traces in so far that they do not generally vanish on brackets, thus leading to anomalies or discrepancies.

24.1 Holomorphic regularisation schemes

Let

$$\begin{aligned}\mathrm{Hol}_0\left(C\ell(M,E)\right) &= \langle z\mapsto A(z)\in C\ell(M,E);\quad A(z)\quad \text{holomorphic} \\ &\quad \text{and}\quad \exists q\in \mathbb{R}^+\quad \text{s.t.}\quad A(z)\quad \text{has}\quad \text{order}\quad \alpha(z)\quad \text{with}\quad \mathrm{Re}(\alpha'(0))\leq -q\rangle,\end{aligned} \tag{24.243}$$

respectively,

$$\begin{aligned}\mathrm{Hol}_0\left(C\ell^{*,*}(M,E)\right) &= \langle z\mapsto A(z)\in C\ell^{*,*}(M,E);\quad A(z)\quad \text{holomorphic} \\ &\quad \text{and}\quad \exists q\in \mathbb{R}^+\quad \text{s.t.}\quad A(z)\quad \text{has}\quad \text{order}\quad \alpha(z)\quad \text{with}\quad \mathrm{Re}(\alpha'(0))\leq -q\rangle,\end{aligned} \tag{24.244}$$

be the algebras generated by holomorphic families of classical, respectively, log-polyhomogeneous operators of order $\alpha(z)$ with $\mathrm{Re}(\alpha'(0))\leq -q<0$ for some positive real number q.

Definition 44 *A holomorphic* regularisation *scheme on $C\ell(M,E)$, respectively $C\ell^{*,*}(M,E)$, is a linear map*

$$\begin{aligned}\mathcal{R}: C\ell(M,E) &\to \mathrm{Hol}_0\left(C\ell(M,E)\right)\\ A &\mapsto (z\mapsto A(z))\end{aligned}$$

respectively

$$\begin{aligned}\mathcal{R}: C\ell^{*,*}(M,E) &\to \mathrm{Hol}_0\left(C\ell^{*,*}(M,E)\right)\\ A &\mapsto (z\mapsto A(z))\,,\end{aligned}$$

such that $A(0)=A$ and which preseres the logarithmic type i.e. for any non negative integer k the following implication holds $A\in C\ell^{*,k}(M,E)\Rightarrow A(z)\in C\ell^{*,k}(M,E)$.

Remark 45 *In practice we restrict ourselves to holomorphic regularisations $A\mapsto A(z)$ that send an operator A to a holomorphic family with symbols $z\mapsto \alpha(z)$ affine in z.*

Example 41 *ζ- regularisation*

$$\mathcal{R}^Q: A\mapsto A(z):=A\,Q_\theta^{-z}$$

with Q an admissible operator in $C\ell(M,E)$ with positive order q and spectral cut θ yields typical (and very useful) examples of holomorphic regularisations.

24.2 Regularised traces

On the grounds of Theorem 24, given a holomorphic regularisation $\mathcal{R}$, we can pick the finite part in the Laurent expansion $\mathrm{TR}\left(A(z)\right)$ by means of a regularised evaluator $\mathrm{ev}_0^{\mathrm{reg}}$ and set the following definition.

Definition 45 *A holomorphic regularisation scheme $\mathcal{R}: A\mapsto A(z)$ on $C\ell(M,E)$, respectively on $C\ell^{*,*}(M,E)$, induces a linear map:*

$$\begin{aligned}\mathrm{Tr}^{\mathcal{R}}: \mathbb{C}^{*,*}(M,E) &\to \mathbb{C}\\ A &\mapsto \mathrm{Tr}^{\mathcal{R}}(A):=\mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}\left(A(z)\right)\right),\end{aligned}$$

respectively

$$\begin{array}{rcl} \mathrm{Tr}^{\mathcal{R}} : C\ell^{*,*}(M,E) & \to & \mathbb{C} \\ A & \mapsto & \mathrm{Tr}^{\mathcal{R}}(A) := \mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}\left(A(z)\right)\right), \end{array}$$

which we call $\mathcal{R}$-regularised trace.

Lemma 23 *Let $\mathcal{R} : A \mapsto A(z)$ be a holomorphic regularisation. The linear form $\mathrm{Tr}^{\mathcal{R}}$ extends the usual trace (defined on operators of order $< -d$) as well as the canonical trace TR (defined on non integer order operators) to* ψ**dos** *of all orders. In both cases we have:*

$$\mathrm{Tr}^{\mathcal{R}}(A) = \lim_{z\to 0} \mathrm{TR}\left(A(z)\right) = \mathrm{TR}(A).$$

Proof: For an operator A of order $< -d$, $A(z)$ is also of order $< -d$ in some small neighborhood of 0 so that on that neighborhood, the map $z \mapsto \mathrm{tr}\left(A(z)\right)$ is holomorphic at $z = 0$ and coincides with $z \mapsto \mathrm{TR}\left(A(z)\right)$. Hence,

$$\mathrm{Tr}^{\mathcal{R}}(A) = \lim_{z\to 0} \mathrm{TR}\left(A(z)\right) = \lim_{z\to 0} \mathrm{Tr}\left(A(z)\right) = \mathrm{Tr}(A) = \mathrm{TR}(A).$$

Similarly, given an operator A of non integer order, $A(z)$ is also of non integer order in some small neighborhood of 0 so that on that neighborhood, the map $z \mapsto \mathrm{TR}\left(A(z)\right)$ is holomorphic at $z = 0$. In that case, one can check from the definition of the cut-off regularised integral that $\lim_{z\to 0} ⨍_{\mathbb{R}^d} \mathrm{tr}\left(\sigma(A(z))\right)(x,\xi)\, đ\xi = ⨍_{\mathbb{R}^d} \mathrm{tr}\left(\sigma(A)\right)(x,\xi)\, đ\xi$ so that

$$\begin{array}{rcl} \mathrm{Tr}^{\mathcal{R}}(A) & = & \lim_{z\to 0} \mathrm{TR}\left(A(z)\right) \\ & = & \lim_{z\to 0} \int_M \left(⨍_{T_x^*M} \mathrm{tr}\left(\sigma(A(z))\left(x,\xi\right) đ\xi\right) dx \right. \\ & = & ⨍_{T^*M} \mathrm{tr}\left(\sigma(A)\right)(x,\xi)\, đ\xi\, dx \\ & = & \mathrm{TR}(A). \end{array}$$

□

Regularised traces associated with a ζ-regularisation scheme

$$\mathcal{R}^Q : A \mapsto A\, Q_\theta^{-z},$$

where Q is some admissible operator in $C\ell^{*,*}(M,E)$ with positive order and spectral cut θ, are called **weighted** traces. With the notations of (23.240), the Q-weighted trace of an operator A in $C\ell^{*,*}(M,E)$ is defined by:

$$\mathrm{Tr}_\theta^Q(A) := \mathrm{ev}_0\left(\mathrm{TR}\left(\mathcal{R}^Q(A)(z)\right)\right) = \mathrm{ev}_0\left(\zeta^{\mathrm{mer}}(A,Q)(z)\right). \tag{24.245}$$

These correspond to generalised zeta functions:

$$\zeta_\theta(A,Q)(0) := \mathrm{Tr}_\theta^Q(A); \quad \zeta_{Q,\theta}(0) := \mathrm{Tr}_\theta^Q(I).$$

When A has order with real part $< -d$, then $\mathrm{Tr}^Q(A) = \mathrm{Tr}(A)$ so that the weighted trace coincides with the L^2-trace; thus, weighted traces provide linear extensions of the L^2-trace.
The following theorem provides a useful explicit local formula of the classical operator

$$L(A,B) = \log AB - \log A - \log B.$$

Proposition 67 *Let A and B be two admissible operators with positive orders a and b in $C\ell(M,E)$ such that their product AB is also admissible. We have the following identities for weighted traces:*

$$\frac{d}{dt}_{|t=0} \mathrm{tr}^B(L(A^t, B^\mu)) = 0, \qquad \frac{d}{dt}_{|t=0} \mathrm{tr}^A(L(A^t, B^\mu)) = 0$$

as well as for the noncommutative residue:

$$\frac{d}{dt}_{|t=0} \mathrm{res}(L(A^t, B^\mu)) = 0.$$

Proof: Let us prove the result for the B-weighted trace; a similar proof yields the result for the A-weighted trace. By Proposition 70, weighted traces and the residue commute with differentiation on constant order operator so that

$$\frac{d}{dt}_{|t=0}\mathrm{Tr}^Q\left(L(A^t,B^\mu)\right)=\mathrm{Tr}^Q\left(\frac{d}{dt}_{|t=0}L(A^t,B^\mu)\right)$$

resp.

$$\frac{d}{dt}_{|t=0}\mathrm{res}\left(L(A^t,B^\mu)\right)=\mathrm{res}\left(\frac{d}{dt}_{|t=0}(L(A^t,B^\mu)\right).$$

But

$$\frac{d}{dt}_{|t=0}L(A^t,B^\mu)=\frac{d}{dt}_{|t=0}\log(A^tB^\mu)-\frac{d}{dt}_{|t=0}\log A^t.$$

We therefore apply Lemma **??** to $A_t:=A^tB^\mu$ so that $A_0=B^\mu$, including the case $\mu=0$ for which $A_t=A^t$ and $A_0=I$. Since $\dot{A}_0=\log A\,B^\mu$ and $\dot{A}_0\,A_0^{-1}=\log A$, implementing the weighted trace Tr^B yields

$$\begin{aligned}&\frac{d}{dt}_{|t=0}\mathrm{Tr}^B\left(\log(A^tB^\mu)\right)\\ =\;&\mathrm{Tr}^B(\log A)+\sum_{k=1}^K\frac{(-1)^k}{k+1}\mathrm{tr}^B(\mathrm{ad}^k_{B^\mu}(\log A\,B^\mu)\,B^{-\mu(k+1)})+\mathrm{Tr}^B(R_K(B^\mu,\log A\,B^\mu))\end{aligned}$$

for arbitrary large K, with remainder term

$$\begin{aligned}R_K(B^\mu,\log A\,B^\mu)&=-\frac{d}{dz}\left(\frac{i}{2\pi}\int_{\Gamma_\alpha}\lambda^z\left[(\lambda-B^\mu)^{-1},\mathrm{ad}^K_{B^\mu}(\log A\,B^\mu)\right](\lambda-B^\mu)^{-(K+1)}\,d\lambda\right)_{|z=0}\\ &=-\mathrm{ad}^K_{B^\mu}\left(\frac{d}{dz}\left(\frac{i}{2\pi}\int_{\Gamma_\alpha}\lambda^z\left[(\lambda-B^\mu)^{-1},logA\,B^\mu\right](\lambda-B^\mu)^{-(K+1)}\,d\lambda\right)_{|z=0}\right),\end{aligned}$$

since B commutes with B^μ.
For any positive integer k, by (24.252) we have

$$\begin{aligned}\mathrm{Tr}^B(\mathrm{ad}^k_{B^\mu}(A\,B^\mu)\,B^{-\mu(k+1)})&=\mathrm{Tr}^B\left(\mathrm{ad}_{B^\mu}(\mathrm{ad}^{k-1}_{B^\mu}(A\,B^\mu))\,B^{-\mu(k+1)}\right)\\ &=\mathrm{Tr}^B\left(\mathrm{ad}_{B^\mu}\left(\mathrm{ad}^{k-1}_{B^\mu}(A\,B^\mu)\,B^{-\mu(k+1)}\right)\right)\\ &=-\frac{1}{b}\mathrm{res}\left(\mathrm{ad}^{k-1}_{B^\mu}(A\,B^\mu)\,B^{-\mu(k+1)}\,[B^\mu,\log B]\right)\\ &=0,\end{aligned}$$

since B commutes with $\log B$. A similar computation shows that $\mathrm{Tr}^B(R_K(B^\mu,\log A\,B^\mu))=0$. Thus

$$\frac{d}{dt}_{|t=0}\mathrm{Tr}^B\left(\log(A^tB^\mu)\right)=\mathrm{Tr}^B\left(\log A\right).$$

It follows that $\frac{d}{dt}_{|t=0}\mathrm{Tr}^B\left(\log(A^tB^\mu)\right)=\mathrm{Tr}^B\left(\log A\right)$ independently of μ so that

$$\frac{d}{dt}_{|t=0}\mathrm{Tr}^B\left(L(A^t,B^\mu)\right)=0.$$

Similarly, replacing the weighted trace Tr^B by the noncommutative residue res and using the cyclicity of the noncommutative residue, yields

$$\frac{d}{dt}_{|t=0}\mathrm{res}\left(L(A^t,B^\mu)\right)=0.$$

□

The following local formula for the weighted trace of $L(A,B)$ will later lead to a local formula for the multiplicative anomaly of the zeta determinant. We also show that the residue of $L(A,B)$ vanishes, which will yield back the multiplicativity of the residue determinant derived in [**?**].

Theorem 25 *For two admissible operators $A, B \in C\ell(M,E)$ with positive orders a and b such that their product AB is also admissible, we have*

$$\mathrm{res}(L(A,B)) = 0. \tag{24.246}$$

Moreover, there is an operator

$$W(\tau)(A,B) := \frac{d}{dt}_{|t=0} L(A^t, A^\tau B) \tag{24.247}$$

in $C\ell^0(M,E)$ depending continuously on τ such that

$$\mathrm{Tr}^Q(L(A,B)) = \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b} - \frac{\log Q}{q}\right)\right) d\tau \tag{24.248}$$

where Q is any weight of order q.

Proof: By Proposition 67, we know that $\frac{d}{dt}_{|t=0}\mathrm{res}(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{Tr}^Q(L(A^t,B)) = 0$. We want to compute $\frac{d}{dt}_{|t=\tau}\mathrm{res}(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{res}(L(A^{t+\tau},B))$ and $\frac{d}{dt}_{|t=\tau}\mathrm{Tr}^Q(L(A^t,B)) = \frac{d}{dt}_{|t=0}\mathrm{Tr}^Q(L(A^{t+\tau},B))$. For this we observe that

$$L(AB,D) - L(A,BD) = -\log(AB) - \log(D) + \log A + \log(BD) = L(B,D) - L(A,B)$$

Replacing A by A^t, B by A^τ and D by B, we get

$$L(A^{t+\tau},B) - L(A^t,A^\tau B) = L(A^\tau,B) - L(A^t,A^\tau) = L(A^\tau,B).$$

Implementing the noncommutative residue, by Proposition 67 we have:

$$\begin{aligned} \frac{d}{dt}_{|t=\tau}\mathrm{res}(L(A^t,B)) &= \frac{d}{dt}_{|t=0}\mathrm{res}(L(A^{t+\tau},B)) \\ &= \frac{d}{dt}_{|t=0}\mathrm{res}(L(A^t,A^\tau B)) \\ &= 0. \end{aligned}$$

Hence

$$\mathrm{res}(L(A,B)) = \int_0^1 \frac{d}{dt}_{|t=\tau}\mathrm{res}(L(A^t,B))\, d\tau + \mathrm{res}(L(I,B)) = 0, \tag{24.249}$$

since $L(I,B) = 0$.
If instead we implement the weighted trace Tr^Q, we have:

$$\begin{aligned} \frac{d}{dt}_{|t=\tau}\mathrm{Tr}^Q(L(A^t,B)) &= \frac{d}{dt}_{|t=0}\mathrm{Tr}^Q(L(A^{t+\tau},B)) \\ &= \frac{d}{dt}_{|t=0}\mathrm{Tr}^Q(L(A^t,A^\tau B)). \end{aligned}$$

Since A and B have positive order so has $A^\tau B$, so that applying Proposition 67 with weighted traces $\mathrm{tr}^{A^\tau B}$ yields:

$$\begin{aligned} \frac{d}{dt}_{|t=\tau}\mathrm{Tr}^Q(L(A^t B) &= \frac{d}{dt}_{|t=0}\mathrm{Tr}^Q(L(A^t,A^\tau B)) \\ &= \frac{d}{dt}_{|t=0}\mathrm{tr}^{A^\tau B}(L(A^t,A^\tau B)) \\ &+ \frac{d}{dt}_{|t=0}\left(\mathrm{Tr}^Q(L(A^t,A^\tau B)) - \mathrm{tr}^{A^\tau B}(L(A^t,A^\tau B))\right) \\ &= \frac{d}{dt}_{|t=0}\left(\mathrm{Tr}^Q(L(A^t,A^\tau B)) - \mathrm{tr}^{A^\tau B}(L(A^t,A^\tau B))\right). \end{aligned}$$

Applying (24.251) to $Q_1 = Q$ and $Q_2 = A^\tau B$, we infer that

$$\begin{aligned} & \frac{d}{dt}_{|t=0} \left(\mathrm{Tr}^Q(L(A^t, A^\tau B)) - \mathrm{tr}^{A^\tau B}(L(A^t, A^\tau B)) \right) \\ = \quad & \frac{d}{dt}_{|t=0} \mathrm{res} \left(L(A^t, A^\tau B) \left(\frac{\log(A^\tau B)}{a\tau + b} - \frac{\log Q}{q} \right) \right) \\ = \quad & \mathrm{res} \left(W(\tau)(A, B) \left(\frac{\log(A^\tau B)}{a\tau + b} - \frac{\log Q}{q} \right) \right), \end{aligned}$$

where q is the order of Q and where we have set $W(\tau)(A,B) := \frac{d}{dt}_{|t=0} L(A^t, A^\tau B)$. Since $L(I, B) = 0$, we finally find that

$$\begin{aligned} & \mathrm{Tr}^Q(L(A,B)) = \mathrm{Tr}^Q(L(A^1, B)) - \mathrm{Tr}^Q(L(A^0, B)) \\ = \quad & \int_0^1 \mathrm{res} \left(W(\tau)(A, B) \left(\frac{\log(A^\tau B)}{a\tau + b} - \frac{\log Q}{q} \right) \right) d\tau. \end{aligned} \tag{24.250}$$

□

24.3 Discrepancies

Regularised traces depend on the choice of regularisation and are not cyclic in spite of their name. To simplify the presentation we restrict to classical pseudodifferential operators, but the formulae could be extended to log-polyhomogeneous operators.We also focus on ζ-regularisation schemes.

Proposition 68 *1. Given an operator A in $C\ell(M,E)$ and two admissible operators Q_1 and Q_2 in $C\ell(M,E)$ with positive orders q_1 and q_2 and spectral cuts θ_1 and θ_2,*

$$\mathrm{Tr}_{\theta_1}^{Q_1}(A) - \mathrm{Tr}_{\theta_2}^{Q_2}(A) = \mathrm{res}\left(\frac{\log_{\theta_1} Q_1}{q_1} - \frac{\log_{\theta_2} Q_2}{q_2}\right) \tag{24.251}$$

which is a local expression.

2. Given two operators A and B in $C\ell(M,E)$ and an admissible operator Q in $C\ell(M,E)$ with positive orders q and spectral cut θ,

$$\mathrm{Tr}_\theta^Q([A,B]) = \frac{1}{q}\mathrm{res}\left(A\left[B, \log_\theta Q\right]\right). \tag{24.252}$$

Proof: For simplicity we drop the explicit mention of the spectral cuts in the notations.

1. By (22.229), the difference $\frac{\log Q_2}{q_2} - \frac{\log Q_1}{q_1}$ is classical. On the other hand, for any admissible operator Q in $C\ell(M,E)$ with positive order q we have:

$$\mathrm{Tr}^Q(A) = \mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}\left(A\,Q^{-z}\right)\right) = \mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}\left(A\,Q^{-\frac{z}{q}}\right)\right) = \mathrm{TR}^{Q^{\frac{1}{q}}}(A).$$

The result then follows from (23.237) applied to $A(z) = \frac{A\left(Q_1^{-\frac{z}{q_1}} - Q_2^{-\frac{z}{q_2}}\right)}{z}$ which is a holomorphic family of operators with holomorphic order $a - z$ and coincides with $A\left(\frac{\log Q_1}{q_1} - \frac{\log Q_2}{q_2}\right)$ at $z = 0$:

$$\begin{aligned}
\mathrm{Tr}^{Q_1}(A) - \mathrm{Tr}^{Q_2}(A) &= \mathrm{Tr}^{Q_1^{\frac{1}{q_1}}}(A) - \mathrm{Tr}^{Q_2^{\frac{1}{q_2}}}(A)\\
&= \mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}\left(A\left(Q_1^{-\frac{z}{q_1}} - Q_2^{-\frac{z}{q_2}}\right)\right)\right)\\
&= \mathrm{Res}_0\left(\mathrm{TR}\left(z^{-1}\left(A\left(Q_1^{-\frac{z}{q_1}} - Q_2^{-\frac{z}{q_2}}\right)\right)\right)\right)\\
&= \mathrm{res}\left(\frac{\log_{\theta_1} Q_1}{q_1} - \frac{\log_{\theta_2} Q_2}{q_2}\right).
\end{aligned}$$

2. We simplify notations dropping the explicit mention of spectral cut. We first observe that by (22.230) the operator $[B, \log Q]\,A$ is classical. The statement then follows from (23.237) applied to the holomorphic family $A(z) = \frac{[B,Q^{-z}]A}{z}$ of operators of holomorphic order $a + b - qz$ which coincides at $z = 0$ with the operator $-[B, \log Q]\,A$. The cyclicity of the canonical trace on non integer order operators which yields $\mathrm{TR}\left([C(z), D(z)]\right) = 0$ as an identity of meromorphic maps for holomorphic families $C(z)$ and $D(z)$ with order $c - q\,z$, $d - q\,z$ and $q > 0$ leads to the following identities:

$$\begin{aligned}
\mathrm{Tr}^Q([A,B]) &= \mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}\left([A,B]\,Q^{-z}\right)\right)\\
&= \mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}\left(A\,B\,Q^{-z}\right)\right) - \mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}\left(B\,A\,Q^{-z}\right)\right)\\
&= \mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}\left(Q^{-z/2}\,A\,B\,Q^{-z/2}\right)\right) - \mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}\left(Q^{-z/2}\;B\,A\,Q^{-z/2}\right)\right)\\
&= \mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}\left(B\,Q^{-z}\,A - Q^{-z}\,B\,A\right)\right)\\
&= \mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}\left([B, Q^{-z}]\,A\right)\right)\\
&= \mathrm{Res}_0\left(\mathrm{TR}\left(\frac{[B,Q^{-z}]\,A}{z}\right)\right)\\
&= \frac{1}{q}\,\mathrm{res}\left([B, \log Q]\,A\right)\\
&= \frac{1}{q}\,\mathrm{res}\left(A\,[B, \log Q]\right)
\end{aligned}$$

□

Corollary 13 *The logarithm of an admissible operator Q in $C\ell(M,E)$ with positive order q and spectral cut θ has well defined residue and*

$$\zeta_{Q,\theta}(0) = -\frac{1}{q}\mathrm{res}(\log_\theta Q). \tag{24.253}$$

If $\zeta_{Q,\theta}(0) \neq 0$ then Tr_θ^Q is not tracial.

Remark 46 *Formula (24.253) is a special instance of a more general formula to be proven by other means in the next section.*

Proof: Let us prove the first part of the corollary. For simplicity, we drop the explicit mention of choice of spectral cut.
Since the noncommutative residue is local, we can restrict to a local trivialisation around a point x and choose two differential operators $A = x_i \otimes I_V$, resp. $B = \partial_{x_j} \otimes I_V$, where V is the model space for E. Then

$$\mathrm{Tr}^Q([A,B]) = \delta_{ij}\,\mathrm{Tr}^Q(I) = \frac{1}{q}\,\mathrm{res}\left(x_i\,[\partial_{x_j}, \log Q]\right).$$

Since $\sigma(B) = i\xi_j$, it follows that $\sigma([\partial_{x_j}, \log Q]) \sim \partial_{x_j}\sigma(\log Q)$ and $\sigma(A\,[B, \log Q]) \sim x_i\,\partial_{x_j}\sigma(\log Q)$, so that by integration by parts we have:

$$\mathrm{res}\left(A\,[B,\log Q]\right) = \int_{S^*M} \mathrm{tr}\left(\sigma_{-d}(x_i\,[\partial_{x_j}, \log Q])\right)\,dx\,đ_S\xi = -\delta_{ij}\int_{S^*M}\mathrm{tr}\left(\sigma_{-d}(\log Q)\right)\,dx\,đ_S\xi = -\delta_{ij}\mathrm{res}(\log Q).$$

Thus,

$$\mathrm{Tr}^Q(I) = -\frac{1}{q}\mathrm{res}(\log Q).$$

The second part then follows from $\zeta_Q(0) = \mathrm{Tr}^Q(I) = \mathrm{Tr}^Q\left([x_i, \partial_{x_i}]\right)$. □

Example 42 *If M is an odd dimensional Riemannian manifold and Δ_g the corresponding Laplace-Beltrami operator whose orthogonal projection on the kernel is denoted by π_g, then $Q = \Delta_g + \pi_g$ is invertible and (see e.g. [R] Theorem 5.2) $\zeta_{\Delta_g+\pi_g}(0) = 0$, whereas one expects a non vanishing local expression $\zeta_{\Delta_g+\pi_g}(0) = -\frac{1}{2}\,\mathrm{res}(\log(\Delta_g + \pi_g))$ in the even dimensional case.*

Remark 47 *The above constructions extend to weighted supertraces sTr^Q defined similarly to the ordinary weighted traces up to the fact that the ordinary vector bundle E is replaced by a $\mathbb{Z}_2$-graded bundle and the corresponding fibrewise trace by a supertrace on graded operators. As we shall see later on in these notes, if $Q = D^2 + \pi_D$ is the admissible operator built from an odd (for the $\mathbb{Z}_2$-grading) admissible operator D in $C\ell(M,E)$ with order d and the orthogonal projection π_D onto its finite dimensional kernel $\mathrm{Ker}(D)$, then the above formulae read:*

$$ind(D) = s\zeta_{Q,\theta}(0) := \mathrm{sTr}^{D^2+\pi_D}(I) = -\frac{1}{2d}\mathrm{res}(\log(D^2 + \pi_D)),$$

where $ind(D)$ is the index of the operator D.
In particular, if there is an operator D with non zero index then the weighted supertrace $\mathrm{sTr}^{D^2+\pi_D}$ is not cyclic.

25 Defect formulae for regularised traces

We discuss the non locality of weighted traces on the grounds of defect formulae, from which we also derive their continuity on operators with constant order. We end this section with an alternative characterisation of the noncommutative residue.

25.1 Regularised traces; locality versus non locality

In general, the expression of the finite part $\mathrm{ev}_0^{\mathrm{reg}}(\mathrm{TR}(A(z)))$ involves both local and global terms as the following theorem shows.

Theorem 26 *[PS]*

1. *Let, for some non negative integer k, $z \mapsto A(z) \in C\ell^{\alpha(z),k}(M,E)$ be a holomorphic family of log-polyhomogeneous symbols parametrised by $z \in \Omega$, a domain of $\mathbb{C}$ with order $z \mapsto \alpha(z)$ affine in z. Then for any $z_0 \in \Omega$ such that $\alpha'(z_0) \neq 0$ we have*

$$\mathrm{ev}_{z_0}^{\mathrm{reg}}(\mathrm{TR}(A(z))) = \int_M \left(-\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}\left(\sigma(A(z_0))\right)(x,\xi)\, d\!\!\!{}^{-}\xi + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{(\alpha'(z_0))^{l+1}} \mathrm{res}_{x,l}\left(A^{(l+1)}(z_0)\right)\right) dx \tag{25.254}$$

where we have set

$$\mathrm{res}_{x,l}(A) := \int_{S_x^*M} \mathrm{tr}(\sigma_{-d,l}(A))(x,\xi)\, d\!\!\!{}^{-}\xi.$$

2. *In particular, for a holomorphic family $z \mapsto A(z) \in C\ell^{\alpha(z)}(M,E)$ of classical symbols, we have*

$$\mathrm{ev}_{z_0}^{\mathrm{reg}}(\mathrm{TR}(A(z))) = \int_M \left(-\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}\left(\sigma(A(z_0))\right)(x,\xi)\, d\!\!\!{}^{-}\xi - \frac{1}{\alpha'(z_0)} \mathrm{res}_x(A'(z_0))\right) dx \tag{25.255}$$

where we have set

$$\mathrm{res}_x(A) := \int_{S_x^*M} \mathrm{tr}(\sigma_{-d}(A))(x,\xi)\, d\!\!\!{}^{-}\xi.$$

Remark 48 *It is not a priori clear that the r.h.s of (25.254) and (25.255) are well defined; it follows here from the fact that the l.h.s is well defined.*

Proof: The corresponding formulae (9.66) readily derived on the level of symbols applied to $\sigma(z) := \mathrm{tr}(\sigma(A(z)))$ yield

$$\begin{aligned}\mathrm{ev}_{z_0}^{\mathrm{reg}}\left(-\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathrm{tr}(\sigma(A(z)))(x,\xi)\, d\!\!\!{}^{-}\xi \right) &= -\!\!\!\!\!\!\int_{\mathbb{R}^d} \mathrm{tr}\left(\sigma(A(z_0))\right)(x,\xi)\, d\!\!\!{}^{-}\xi \\ &+ \sum_{l=0}^{k} \frac{(-1)^{l+1}}{(\alpha'(z_0))^{l+1}} \mathrm{res}_{x,l}\left(A^{(l+1)}(z_0)\right).\end{aligned}$$

Since the l.h.s integrates over M to a well defined quantity

$$\mathrm{ev}_{z_0}^{\mathrm{reg}}(\mathrm{TR}(A(z))) = \mathrm{ev}_{z_0}^{\mathrm{reg}}\left(\int_M \left(-\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}\left(\sigma(A(z))\right)(x,\xi)\, d\!\!\!{}^{-}\xi \right) dx \right)$$

and the result follows. □

Corollary 14 *Given a holomorphic regularisation scheme $\mathcal{R}: A \mapsto A(z)$ on $C\ell^{*,*}(M,E)$ that sends an operator A to a holomorphic family $A(z)$ with order $\alpha(z)$ affine in z we have*

$$\mathrm{Tr}^{\mathcal{R}}(A) = \int_M \left(-\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}\left(\sigma(A)(x,\xi)\right) d\!\!\!{}^{-}\xi + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{(\alpha'(0))^{l+1}} \mathrm{res}_{x,l}\left(A^{(l+1)}(0)\right)\right) dx \tag{25.256}$$

where we have set

$$\mathrm{res}_{x,l}(B) := \int_{S_x^*M} \mathrm{tr}\left(\sigma_{-d,l}(B)\right)(x,\xi)\, d\!\!\!{}^{-}\xi.$$

In particular, for any holomorphic regularisation $\mathcal{R} : A \mapsto A(z)$ on classical symbols that sends an operator A to a holomorphic family $A(z)$ with order $\alpha(z)$ affine in z, we have

$$\mathrm{Tr}^{\mathcal{R}}(A) \quad = \quad \int_M ⨍_{T_x^*M} \left(\mathrm{tr}\,(\sigma(A))\,(x,\xi)\, đ\xi - \frac{1}{\alpha'(0)}\mathrm{res}_x(A'(0))\right)\,dx, \tag{25.257}$$

where we have set

$$\mathrm{res}_x(B) := \int_{S_x^*M} \mathrm{tr}(\sigma_{-d}(B))(x,\xi)\,đ\xi.$$

Thus $\mathrm{Tr}^{\mathcal{R}}(A)$ is an integral over M of a sum of a local term $\sum_{l=0}^{k} \frac{(-1)^{l+1}}{(\alpha'(z_0))^{l+1}}\mathrm{res}_{x,l}\left(A^{(l+1)}(0)\right)$ involving only a finite number of components of the symbol of A and a global term $⨍_{\mathbb{R}^d} \mathrm{tr}\,(\sigma(A))\,(x,\xi)\,đ\xi$. In some cases $\mathrm{Tr}^{\mathcal{R}}(A)$ reduces either to a local or to a global term.

Corollary 15 *Let $\mathcal{R} : A \mapsto A(z)$ be a holomorphic regularisation on classical pseudodifferential operators that sends an operator A to a holomorphic family $A(z)$ with order $\alpha(z)$ affine in z.*

1. *If $⨍_{\mathbb{R}^d} \mathrm{tr}(\sigma(A))(x,\xi)\,đ\xi = 0$ then $\mathrm{res}\,(A'(0))) := \int_M \mathrm{res}_x\,(A'(0)))\,dx$ is well defined and we have:*

$$\mathrm{Tr}^{\mathcal{R}}(A) = -\frac{1}{\alpha'(0)}\mathrm{res}(A'(0)),$$

   which depends on the first jet of $\mathcal{R}$ at zero.

2. *If $\mathrm{res}_x\,(A'(0))) = 0$ for any x in M then $\mathrm{TR}\,(A) = ⨍_{T^*M} \mathrm{tr}(\sigma(A))(x,\xi)\,đ\xi\,dx$ is well defined and we have*

$$\mathrm{Tr}^{\mathcal{R}}(A) = \mathrm{fp}_{z=0}\mathrm{TR}(A(z)) = \mathrm{TR}(A), \tag{25.258}$$

   independently of the choice of $\mathcal{R}$.

Proof The existence of the various integrals over M together with the identities easily follow from the above corollary. □

25.2 Zeta regularised traces: locality versus non locality

The results of the previous paragraph apply to the ζ-regularisation scheme $\mathcal{R}^Q$ for some admissible operator Q in $C\ell(M,E)$ with spectral cut θ and positive order q.

Theorem 27 *For any $A \in C\ell^{*,k}(M,E)$ and any admissible operator $Q \in C\ell(M,E)$ with spectral cut θ and with positive order q, $\zeta_\theta(A,Q)(z) = \mathrm{TR}(A\,Q_\theta^{-z})$ has poles of order $\leq k+1$ in the discrete set $\{\frac{-a+d-k}{q}, k \in \mathbb{N}_0\}$ and we have*

$$\mathrm{Res}_0^{k+1}\zeta_\theta(A,Q)(z) = q^{k+1}\,\mathrm{res}_k(A). \tag{25.259}$$

The finite part

$$\mathrm{Tr}_\theta^Q(A) := \mathrm{ev}_0^{\mathrm{reg}}\,(\zeta_\theta(A,Q)) = \mathrm{ev}_0^{\mathrm{reg}}\left(\mathrm{TR}(A\,Q_\theta^{-z})\right)$$

reads:

$$\mathrm{Tr}_\theta^Q(A) = \int_M \left(\mathrm{TR}_x(A) + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{q^{l+1}}\mathrm{res}_{x,l}\left(A\,\log_\theta^{k+1} Q\right)\right)\,dx \tag{25.260}$$

where we have set $\mathrm{TR}_x(A) := ⨍_{T_x^*M} \mathrm{tr}_x\sigma_A(x,\xi)\,đ\xi$.
If A is classical this boils down to

$$\mathrm{Tr}_\theta^Q(A) = \int_M \left(\mathrm{TR}_x(A) - \frac{1}{q}\mathrm{res}_x\,(A\,\log_\theta Q)\right).\,dx. \tag{25.261}$$

If A is a differential operator, then $\mathrm{res}_x\,(A\,\log_\theta Q)$ integrates over M to $\mathrm{res}(A\,\log Q) := \int_M \mathrm{res}_x\,(A\,\log_\theta Q)\,dx$ and (25.261) further boils down to a local formula for the weighted trace:

$$\mathrm{Tr}_\theta^Q(A) = -\frac{1}{q}\mathrm{res}\,(A\,\log_\theta Q)\,. \tag{25.262}$$

Corollary 16 *With the above notations*

$$\mathrm{Tr}_\theta^{Q+R}(A) = \mathrm{Tr}_\theta^{Q}(A)$$

for any smoothing operator R.

Proof: It follows from (25.261) that

$$\mathrm{Tr}_\theta^{Q+R}(A) = \mathrm{Tr}_\theta^{Q}(A)$$

since $\log_\theta(Q+R) = \log_\theta Q + R'$ (notice that the spectral cut is unchanged) for some smoothing operator R'. This follows from the very definition of the logarithm using the fact that on the resolvent level we have $(Q+R-\lambda)^{-1} = (Q-\lambda)^{-1} - (Q+R-\lambda)^{-1}\, R\, (Q-\lambda)^{-1}$. □

Formula (25.260) shows how the weighted trace of A splits into local terms involving the noncommutative residue and a global term involving the canonical trace density $\mathrm{TR}_x(A)$.

The following proposition proved in unpublished work with Simon Scott, provides an explicit description of weighted traces of logpolyhomogeneous pseudodifferential operators of the type $A \log^k Q$ for some admissible operator Q with A classical.

Proposition 69 *Let $A \in C\ell(M,E)$ and let Q be a classical admissible pseudodifferential with positive order and spectral cut θ. For any non negative integer k the Q-weighted trace of $A \log_\theta^k Q$ has the following description:*

$$\begin{aligned} &\mathrm{Tr}_\theta^Q\left(A \log_\theta^k Q\right) \\ = \; &\int_M dx \left(\mathrm{TR}_x(A \log_\theta^k Q) + \frac{1}{q}\left(\sum_{l=0}^k \frac{(-1)^{l+1}}{l+1}\right) \mathrm{res}_x(A \log_\theta^{k+1} Q)\right). \end{aligned} \tag{25.263}$$

Proof: We want to apply equation (25.254) to $A(z) = A \log_\theta^k Q\, Q_\theta^{-z}$, the order of which is $\alpha(z) = -q\cdot z + a$ where a is the order of A. With the notation $\sigma \sim \sum_{l=0}^\infty \sigma_{(l)} \log^l |\xi|$, we first compute

$$\begin{aligned} \left(\sigma_{(l)}(A)\right)^{(l+1)}(0) &= \left(\sigma_{(l)}(A \log_\theta^k Q\, Q_\theta^{-z})\right)^{(l+1)}(0) \\ &\sim \sigma_{(l)}(A \log_\theta^k Q) \star \left(\sigma(Q_\theta^{-z})\right)^{(l+1)}(0) \\ &= (-1)^{l+1}\, \sigma_{(l)}(A \log_\theta^k Q) \star \sigma(\log_\theta^{l+1} Q). \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{tr}_\theta^Q\left(A \log_\theta^k Q\right) &= \mathrm{fp}_{z=0} \mathrm{TR}(A(z)) \\ &= \int_M dx \left(\mathrm{TR}_x(A \log_\theta^k Q)\right. \\ &+ \left.\sum_{l=0}^k \frac{(-1)^{l+1}}{q^{l+1}\,(l+1)}\, \mathrm{res}_x \left(\sigma_{(l)}(A \log_\theta^k Q) \star \sigma_(\log_\theta^{l+1} Q)\right)\right). \end{aligned}$$

But

$$\begin{aligned} &\left(\sigma_{A \log_\theta^k Q}\right)_{(l)} \\ \sim \; &\left[\sigma_A \star \left(q \log|\xi| + \left(\sigma_{\log_\theta Q}\right)_{(0)}\right) \star \cdots \star \left(q \log|\xi| + \left(\sigma_{\log_\theta Q}\right)_{(0)}\right)\right]_{(l)} \\ = \; &\sum_{j=0}^k q^j \left[\log^j|\xi|\, \sigma(A) \star \sigma_{(0)}(\log_\theta Q) \star \cdots \star \sigma_{(0)}(\log_\theta Q)\right]_{(l)} \mathbf{q}uad \quad ((k-j) \quad \text{times}) \\ = \; &q^l\, \sigma(A) \star \sigma_{(0)}(\log_\theta Q) \star \cdots \sigma_{(0)}(\log_\theta Q) \qquad ((k-l) \quad \text{times}). \end{aligned}$$

Hence,

$$
\begin{aligned}
& \operatorname{res}_x\left(\sigma_{(l)}(A \log_\theta^k Q) \star \sigma(\log_\theta^{l+1} Q)\right) \\
= \;& q^l \int_{S_x^* M} \left(\sigma_A \star ((\sigma_{\log_\theta Q})_{(0)} \star \cdots \star (\sigma_{\log_\theta Q})_{(0)}) \star \sigma_{\log_\theta^{l+1} Q}\right)_{-d} \qquad (k-l) \quad \text{times} \\
= \;& q^l \int_{S_x^* M} \left(\sigma_A \star \sigma_{\log_\theta Q} \star \cdots \star \sigma_{\log_\theta Q} \star \sigma_{\log_\theta^{l+1} Q}\right)_{-d} \qquad (k-l) \quad \text{times} \\
& \text{since the other terms vanish} \\
= \;& q^l \int_{S_x^* M} \left(\sigma(A) \star \sigma(\log_\theta^{k+1} Q)\right)_{-d} \\
= \;& q^l \operatorname{res}_x\left(A \log_\theta^{k+1} Q\right),
\end{aligned}
$$

which yields (25.263). □

Applying this to $A = I$ and $k = 1$ yields:

Corollary 17

$$
\begin{aligned}
\zeta'_{Q,\theta}(0) &= -\operatorname{Tr}_\theta^Q\left(\log_\theta Q\right) \\
&= -\int_M dx \left(\operatorname{TR}_x(\log_\theta Q) - \frac{1}{2q} \operatorname{res}_x(\log_\theta^2 Q)\right).
\end{aligned}
\tag{25.264}
$$

25.3 Weighted traces of families of operators

The following technical proposition shows that the canonical and weighted traces as well as the non-commutative residue commute with differentiation on families of operators of constant order, a fact that we will use to derive the multiplicative anomaly of determinants. Differentiable families of symbols and operators are defined in the same way as were holomorphic families in Definitions **??** and 43 replacing holomorphic in the parameter z by differentiable in the parameter t.

Proposition 70 *Let A_t be a continuous, resp. differentiable family of $C\ell(M, E)$ of constant order a.*

1. *The map $t \mapsto \operatorname{res}(A_t)$ is continuous, resp. differentiable. When differentiable, the residue commutes with differentiation*
$$\frac{d}{dt}\operatorname{res}(A_t) = \operatorname{res}(\dot{A}_t), \tag{25.265}$$
where we have set $\dot{A}_t = \frac{d}{dt} A_t$.

2. *If the order a is non integer, the map $t \mapsto \operatorname{TR}(A_t)$ is continuous,resp. differentiable. When differentiable, the canonical trace commutes with differentiation*
$$\frac{d}{dt}\operatorname{TR}(A_t) = \operatorname{TR}(\dot{A}_t). \tag{25.266}$$

3. *For any weight Q with order q and spectral cut α, the map $t \mapsto \operatorname{TR}(A_t)$ is continuous,resp. differentiable. When differentiable, the weighted trace commutes with differentiation:*
$$\frac{d}{dt}\operatorname{Tr}_\alpha^Q(A_t) = \operatorname{tr}_\alpha^Q(\dot{A}_t). \tag{25.267}$$

Proof: Using (2.11) we write the symbol $\sigma(A_t)$ of A_t as follows:

$$\sigma(A_t)(x,\xi) = \sum_{j=0}^{N-1} \chi(\xi)\, \sigma_{a-j}(A_t)(x,\xi) + \sigma_{(N)}(A_t)(x,\xi).$$

1. By assumption, the map $t \mapsto \mathrm{tr}\left(\sigma_{-d}(A_t)(x,\cdot)\right)$ is continuous (resp. differentiable) leading to a continuous (resp. differentiable) map $t \mapsto \int_{S_x^*M} \mathrm{tr}_x\left(\sigma_{-d}(A_t)(x,\cdot)\ \right)$ after integration over the compact set S_x^*M with derivative: $t \mapsto \int_{S_x^*M} \mathrm{tr}\left(\dot\sigma_{-d}(A_t)\right)$, where $\dot\sigma_{A_t} = \sigma_{\dot A_t}$ stands for the derivative of σ_{A_t} at t. Thus, the map $t \mapsto \mathrm{res}(A_t)$ is continuous (resp. differentiable). When differentiable, its derivative is given by (25.265).

2. By (??) and (??), in order to check the continuity (reps. differentiability) we need to check the continuity (resp. differentiability) of the map $t \mapsto -\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}\left(\sigma(A_t)\right)(x,\cdot)$. When differentiable, to prove formula (25.266) we need to prove that

$$\frac{d}{dt} -\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}\left(\sigma(A_t)\right)(x,\cdot) = -\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}\left(\dot\sigma(A_t)\right)(x,\cdot).$$

The cut-off integral involves the whole symbol which we denote by $\sigma_t := \sigma(A_t)$ in order to simplify notations. Since the family σ_t has constant order, N can be chosen independently of t in the asymptotic expansion. The corresponding cut-off integral can be computed explicitely (see e.g [PS]):

$$\begin{aligned} -\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}(\sigma_t(x,\xi))\, đ\xi &= \int_{T_x^*M} \mathrm{tr}\left(\left(\sigma_t\right)_{(N)}(x,\xi)\right) đ\xi + \sum_{j=0}^{N-1} \int_{|\xi|\le 1} \chi(\xi)\, \mathrm{tr}\left(\left(\sigma_t\right)_{a-j}(x,\xi)\right) đ\xi \\ &- \sum_{j=0, a-j+n\neq 0}^{N-1} \frac{1}{a-j+n} \int_{|\xi|=1} \mathrm{tr}\left(\left(\sigma_t\right)_{a-j}(x,\omega)\right) đ_S\omega. \end{aligned}$$

The map $t \mapsto \int_{T_x^*M} \mathrm{tr}\left(\left(\sigma_t\right)_{(N)}(x,\xi)\right) đ\xi$ is continuous (resp. differentiable) at any point t_0 since by assumption the maps $t \mapsto \mathrm{tr}\left(\left(\sigma_t\right)_{(N)}(x,\xi)\right)$ are continuous (resp. differentiable) with modulus bounded from above $\left|\mathrm{tr}\left(\left(\dot\sigma_t\right)_{(N)}(x,\xi)\right)\right| \le C|\xi|^{\mathrm{Re}(a)-N}$ by an L^1 function provided N is chosen large enough, where the constant C can be chosen independently of t in a compact neighborhood of t_0. When differentiable, its derivative is given by $t \mapsto \int_{T_x^*M} \mathrm{tr}\left(\left(\dot\sigma_t\right)_{(N)}(x,\xi)\right) đ\xi$. The remaining integrals $\int_{|\xi|\le 1} \chi(\xi)\, \mathrm{tr}\left(\left(\sigma_t\right)_{a-j}(x,\xi)\right) đ\xi$ and $\int_{|\xi|=1} \mathrm{tr}\left(\left(\sigma_t\right)_{a-j}(x,\omega)\right) đ_S\omega$ are also continuous (resp. differentiable) as integrals over compact sets of integrands involving continuous (resp. differentiable) maps $t \mapsto \mathrm{tr}\left(\left(\sigma_t\right)_{a-j}(x,\xi)\right)$. When differentiable, their derivatives are given by $\int_{|\xi|\le 1} \chi(\xi)\, \mathrm{tr}\left(\left(\dot\sigma_t\right)_{a-j}(x,\xi)\right) đ\xi$ and $\int_{|\xi|=1} \mathrm{tr}\left(\left(\dot\sigma_t\right)_{a-j}(x,\omega)\right) đ_S\omega$. Thus, $t \mapsto -\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}(\sigma(A_t)(x,\xi))\, đ\xi$ is continuous (resp. differentiable) with derivative given by $-\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}(\dot\sigma(A_t)(x,\xi))\, đ\xi$.

3. By the defect formula (25.261) we have

$$\mathrm{Tr}_\alpha^Q(A_t) = \int_M dx \left(-\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}(\sigma(A_t))(x,\cdot) - \frac{1}{q} \int_{S_x^*M} \mathrm{tr}\left(\sigma_{-d}(A_t \log_\alpha Q)\right)(x,\cdot) \right)$$

which reduces the proof of the continuity (resp. differentiability) of $t \mapsto \mathrm{Tr}_\alpha^Q(A_t)$ to that of the two maps $t \mapsto -\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}(\sigma(A_t))(x,\cdot)$ and $t \mapsto \int_{S_x^*M} \mathrm{tr}\left(\sigma_{-d}(A_t \log_\alpha Q)\right)(x,\xi)$.
Continuity (resp. differentiability) of the first map was shown in the second item of the proof. Let us first investigate the second map. By (18.193) we have

$$\sigma_{-d}(A_t \log_\alpha Q) = \sum_{|\alpha|+a-j-k=-n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha\, \sigma_{a-j}(A_t)\, \partial_x^\alpha \sigma_{-k}(\log_\alpha Q).$$

By assumption, the maps $t \mapsto \sigma_{a-j}(A_t)$ are continuous (resp. differentiable) so that $t \mapsto \int_{S_x^*M} \mathrm{tr}\left(\sigma_{-d}(A_t \log_\alpha Q)\right)$ is continuous (resp. differentiable). When differentiable, its derivative reads

$$t \mapsto \int_{S_x^*M} \mathrm{tr}\left(\dot\sigma_{-d}(A_t \log_\alpha Q)\right) = \int_{S_x^*M} \mathrm{tr}\left(\sigma_{-d}(\dot A_t \log_\alpha Q)\right).$$

Integrating over the compact manifold M then yields that the map $t \mapsto \mathrm{tr}_\alpha^Q(A_t)$ is continuous (resp. differentiable). When differentiable, its derivative is given by

$$\int_M dx \left(-\!\!\!\!\!\!\int_{T_x^* M} \mathrm{tr}(\sigma(\dot{A}_t)(x,\cdot)) - \frac{1}{q} \int_{S_x^* M} \mathrm{tr}\left(\sigma(\dot{A}_t \, \log_\alpha Q)(x,\cdot)\right) \right) = \mathrm{Tr}_\alpha^Q(\dot{A}_t).$$

25.4 An alternative characterisation of the noncommutative residue

In the previous paragraph, we saw that weighted traces are continuous in the Fréchet topology of operators of constant order. The following result characterises continuous linear forms which vanish on non integer order brackets of operators, in terms of weighted traces and the noncommutative residue.

Proposition 71 *Any continuous* [29] *linear form on $C\ell(M)$ which restricts to a trace*[30] *on $C\ell^{\notin \mathbb{Z}}(M)$, is of the form*

$$c \cdot \mathrm{Tr}_\alpha^Q + \quad \mu \, \mathrm{res}$$

for some complex constants c, μ and an admissible operator Q in $C\ell(M, E)$ with positive order and spectral cut α.

Proof: Let Λ be a continuous linear form on $C\ell^(M)$ which restricts to a trace on $CS_{\mathrm{c.c}}^{\notin \mathbb{Z}}(\mathbb{R}^d)$. By Corollary 12 applied to $\mathcal{D}(M) = \mathbb{C}^{\notin \mathbb{Z}}(M)$, the restriction is proportional to the canonical trace TR:

$$\exists c \in \mathbb{C}, \quad \Lambda_{|_{C\ell^{\notin \mathbb{Z}}(M)}} = c \, \mathrm{TR}.$$

We want to describe all possible linear extensions Λ to classical operators with integer order.
Given an operator $A \in C\ell^{\mathbb{Z}}(M)$ with integer order a, we build a holomorphic family $A(z) = A\, Q_\alpha^{-z}$ using a ζ-holomorphic regularisation scheme. Here α is a spectral cut for the operator Q. In a small neighborhood of zero

$$\Lambda(A(z)) = \mathrm{TR}(A\, Q_\alpha^{-z}).$$

The remaining degree of freedom left to define $\Lambda(A)$ is the choice of a regularised evaluator at $z = 0$. But by Proposition 2 (here $k = 1$), regularised evaluators at zero are of the form $\mathrm{ev}_0^{\mathrm{reg}} + \nu \, \mathrm{Res}_0$, with ν a complex number. Hence,

$$\begin{aligned} \Lambda(A) &= c \, \mathrm{ev}_0^{\mathrm{reg}} \circ \mathrm{TR}\,(A(z)) + \nu \, \mathrm{Res}_0\,(\mathrm{TR}\,(A(z))) \\ &= c \, \mathrm{Tr}_\alpha^Q(A) + \mu \, \mathrm{res}(A), \end{aligned} \tag{25.268}$$

with $\nu = \frac{\mu}{q}$. □

The existence of a weighted trace Tr^Q that does not vanish on brackets, i.e. that does not vanish on brackets combined with Proposition 71 leads to an alternative characterisation of the noncommutative residue similar to the one proved on the symbol level in Theorem (9).

Theorem 28 *The two statements are equivalent:*

1. *Weighted traces Tr^Q do not define traces on $C\ell(M)$.*
2. *Any continuous trace Λ on $C\ell(M)$ is proportional to the noncommutative residue*

$$\Lambda = \mu \, \mathrm{res}, \quad \mu \in \mathbb{C}.$$

Proof: Assume there is an admissible operator in $C\ell(M)$ such that Tr^Q does not define a trace.
By Proposition 71, continuous traces on $C\ell(M)$, which by definition restrict to traces on $C\ell^{\notin \mathbb{Z}}(M)$, are linear combinations of the regularised trace Tr^Q and the noncommutative residue. But since Tr^Q is not a trace, it follows that Λ is proportional to the noncommutative residue.
Conversely, suppose all continuous traces are proportional to the noncommutative residue; then by Proposition 71, Tr^Q which is not proportional to the noncommutative residue, is not a trace. □

[29] For the topology of constant order operators

[30] Meaning by this that it vanishes on brackets in $C\ell^{\notin \mathbb{Z}}(M)$.

26 A local formula for the index in terms of a residue

We express the index of an operator in terms of the residue of its logarithm, thereby providing an a priori local expression for the index since the residue is local. We then use Gilkey's invariant theory to derive the explicit local form of the index. A proof of the Riemann-Roch theorem using the expression of the index as a residue of a logarithm was previously derived by S. Scott and D. Zagier in unpublished work.

26.1 Zeta regularised versus heat-kernel regularised traces

When the weight Q has positive leading symbol $\sigma^L(Q)$, we can choose $\alpha=\pi$ as a spectral cut[31] and the Q-weighted trace Tr^Q relates with the heat-kernel regularised trace we are about to define. Let us first recall some notations from paragraph 1.3.
For any real number b, $\mathcal{F}_0^{b,k}$ (resp. $\mathcal{F}^{b,k}$) stands for the vector space generated by smooth functions on $]0,+\infty[$ with asymptotic behaviour at zero of the type

$$f(\epsilon)\sim_0 \sum_{j=0}^{\infty}\alpha_j\epsilon^{\frac{j-b}{q}}+\sum_{l=0}^{k}\sum_{\frac{j-b}{q}\in\mathbb{Z}}\beta_{j,l}\epsilon^{\frac{j-b}{q}}\log^l\epsilon+\sum_{l=0}^{k}\sum_{j=0}^{\infty}\gamma_{j,l}\epsilon^j\log^l\epsilon \tag{26.269}$$

for some positive q and some real numbers $b,\alpha_j,\beta_{j,l},\gamma_{j,l}$, $j\in\mathbb{N}, l=0,\cdots,k$ (depending on f) (resp. and such that for large enough ϵ,

$$|f(\epsilon)|\leq Ce^{-\epsilon\lambda}$$

for some $\lambda>0$, $C>0$.)
We further set

$$\mathcal{F}_0^k:=\bigcup_{b\in\mathbb{C}}\mathcal{F}_0^{b,k};\mathcal{F}^k:=\bigcup_{b\in\mathbb{C}}\mathcal{F}^{b,k}.$$

We quote the following result from [GS] in the classical case, [L1] in the log-polyhomogeneous case.

Proposition 72 *Let $A\in C\ell^{a,k}(M,E)$ and let $\Delta\in C\ell(M,E)$ be an elliptic operator with positive order q and non negative leading symbol. The map $\epsilon\mapsto\mathrm{tr}\left(A\,e^{-\epsilon\,\Delta}\right)$ which is defined for any $\epsilon>0$, lies in $\mathcal{F}^{n+a,k+1}$. More precisely, it has the following asymptotic behaviour as $\epsilon\to 0$:*

$$\mathrm{Tr}\left(A\,e^{-\epsilon\,\Delta}\right)\sim_{\epsilon\to 0}\sum_{j=0}^{\infty}\epsilon^{\frac{j-n-a}{q}}P_j(\log\epsilon)+\sum_{j=0}^{\infty}\gamma_j\epsilon^j \tag{26.270}$$

where P_j is a polynomial of degree $\leq k$ if $\frac{j-n-a}{q}\notin\mathbb{N}_0$ and $\leq k+1$ if $\frac{j-n-a}{q}\in\mathbb{N}_0$.

We are now ready to introduce the following definition.

Definition 46 *Let $A\in C\ell^{a,k}(M,E)$ and let $\Delta\in C\ell(M,E)$ be an elliptic operator with positive order q and non negative leading symbol. We call the cosntant term in the asymptotic expansion of $\epsilon\mapsto\mathrm{tr}\left(A\,e^{-\epsilon\,\Delta}\right)$ as $\epsilon\to 0$*

$$\mathrm{Tr}^{\mathrm{HK},\Delta}(A):=\mathrm{ev}^{\mathrm{reg}}_{\epsilon=0}\mathrm{Tr}\left(A\,e^{-\epsilon\,\Delta}\right)$$

the heat-kernel regularised trace of A.

Proposition 73 *Let $A\in C\ell(M,E)$ and let $\Delta\in C\ell(M,E)$ be an elliptic (essentially) self-adjoint operator with positive order and non negative leading symbol. Let π_Δ be the orthogonal projection onto the kernel of Δ. Then*

$$\mathrm{Tr}^{\Delta+\pi_\Delta}(A)=\mathrm{Tr}^{\mathrm{HK},\Delta}(A)+\gamma\,\mathrm{res}(A).$$

where γ is the Euler constant.
In particular, if Δ is a differential operator we have:

$$\mathrm{Tr}^{\Delta+\pi_\Delta}(A)=\mathrm{Tr}^{\mathrm{HK},\Delta}(A).$$

[31] We then drop the subscript π for simplicity.

Proof: This follows from the results of Paragraph 1.3 where it was shown that the Mellin transform of a function $f \in \mathcal{F}^0$

$$z \mapsto \mathcal{M}(f)(z) := \frac{1}{\Gamma(z)} \int_0^\infty \epsilon^{z-1} f(\epsilon)\, d\epsilon$$

defines a meromorphic map on the complex plane with poles of order ≤ 1 at 0 and that

$$\mathrm{ev}_{z=0}^{\mathrm{reg}} \mathcal{M}(f)(z) = \mathrm{ev}_{\epsilon=0}^{\mathrm{reg}} f(\epsilon) + \gamma\, \mathrm{Res}_0 \mathcal{M}(f)(z). \tag{26.271}$$

We apply this to the function

$$f_{A,\Delta}(\epsilon) := \mathrm{Tr}\left(A\, e^{-\epsilon\,\Delta}\right) = \mathrm{Tr}\left(A\,\left(e^{-\epsilon\,\Delta'} \oplus \pi_\Delta\right)\right) = \mathrm{Tr}\left(A'\, e^{-\epsilon\,\Delta'}\right) + \mathrm{Tr}\left(\pi_\Delta\, A\, \pi_\Delta\right)$$

where we have set $A' = (1-\pi_\Delta)\, A\, (1-\pi_\Delta)$ using the fact that π_Δ is a projector. Since $(\Delta')^{-z} = \mathcal{M}\left(\epsilon \mapsto e^{-\epsilon\,\Delta'}\right)(z)$, with Δ' the restriction to the orthogonal of the kernel of the operator Δ, we infer thet

$$\begin{aligned}
\mathcal{M}\left(f_{A,\Delta}\right)(z) &= \frac{1}{\Gamma(z)} \int_0^\infty \epsilon^{z-1}\, \mathrm{Tr}\left(A\, e^{-\epsilon\,\Delta'}\right) d\epsilon + \mathrm{Tr}(\pi_\Delta\, A\, \pi_\Delta) \\
&= \mathrm{TR}\left(A\, \frac{1}{\Gamma(z)} \int_0^\infty \epsilon^{z-1}\, e^{-\epsilon\,\Delta'}\, d\epsilon\right) + +\mathrm{Tr}(\pi_\Delta\, A\, \pi_\Delta) \\
&= \mathrm{TR}(A\, (\Delta + \pi_\Delta)^{-z}).
\end{aligned}$$

The second part of the statement then follows from the fact that differenital operators have vanishing noncommutative residue. □

Example 43 *The Laplace-Beltrami operator Δ on the unit circle equipped with the canonical Euclidean metric has real spectrum $\{n^2, n \in \mathbb{Z}\}$, each eigenvalue $n^2, n \neq 0$ with multiplicity 2. It has finite dimensional kernel $\mathrm{Ker}(\Delta)$; let π_Δ denote the orthogonal projection onto this kernel. Applying the above results to $A = \Delta^{a/2}$ for any complex number a and $Q := \Delta + \pi_\Delta$ yields back the following identity.*

$$\begin{aligned}
\mathrm{ev}_{\epsilon=0}^{\mathrm{reg}} \left(\sum_{n=1}^\infty n^a\, e^{-\epsilon n}\right) &= \frac{1}{2}\, \mathrm{ev}_{\epsilon=0}^{\mathrm{reg}} \left(\mathrm{Tr}\left((\Delta')^{a/2}\;\; e^{-\epsilon\,(\Delta')^{1/2}}\right)\right) \\
&= \frac{1}{2}\, \mathrm{ev}_{z=0}^{\mathrm{reg}} \left(\mathrm{TR}\left((\Delta')^{\frac{-z+a}{2}}\right)\right) - \frac{\gamma}{2}\, \mathrm{res}\left((\Delta')^{a/2}\right) \\
&= \frac{1}{2}\, \mathrm{ev}_{z=0}^{\mathrm{reg}} \left(\mathrm{TR}\left((\Delta')^{\frac{-z+a}{2}}\right)\right) - \frac{\gamma}{2}\, \mathrm{res}\left((\Delta')^{a/2}\right) \\
&= \frac{1}{2}\, \mathrm{Tr}^{\sqrt{\Delta}}\left((\Delta')^{a/2}\right) - \frac{\gamma}{2}\, \mathrm{res}((\Delta')^{a/2}) \\
&= \mathrm{ev}_{z=0}^{\mathrm{reg}} \left(\sum_{n=1}^\infty n^{a-z}\right) - \gamma\; \delta_{a+1}.
\end{aligned}$$

When $a \neq -1$, we get back the formula relating cut-off and Riesz regularised sums:

$$-\!\!\!\!\!\!\sum_{n=1}^{\infty, HK} n^a := \mathrm{ev}_{\epsilon=0}^{\mathrm{reg}} \left(\sum_{n=1}^\infty n^a\, e^{-\epsilon n}\right) = \mathrm{ev}_{z=0}^{\mathrm{reg}} \sum_{n=1}^\infty n^{a-z} =: -\!\!\!\!\!\!\sum_{n=1}^{\infty, \mathrm{Riesz}} n^a.$$

26.2 The index as a superresidue

Weighted traces can be extended to weighted supertraces.
Let $E = E_+ \oplus E_-$ be a $\mathbb{Z}_2$-graded vector bundle over a closed manifold M and let $Q_+ \in C\ell(M, E_+)$, $Q_- \in C\ell(M, E_-)$ be two admissible operators with same the spectral cut θ. Setting $Q := Q_+ \oplus Q_-$ we define the weighted supertrace of an even operator $A = A^+ \oplus A^-$, with A^+ in $C\ell(M, E_+)$, A_- in $C\ell(M, E_-)$

$$\mathrm{str}_\theta^Q(A) := \mathrm{tr}_\theta^{Q_+}(A_+) - \mathrm{tr}_\theta^{Q_-}(A_-),$$

which clearly extends the ordinary supertrace $\mathrm{Tr}(A) = \mathrm{Tr}(A_+) - \mathrm{Tr}(A_-)$ on trace-class operators. If $D_+ : C\ell(M, E_+) \to C\ell(M, E_-)$ is an elliptic operator in $C\ell\left(M, (E_+)^* \otimes E_-\right)$ then its (formal) adjoint $D_- := D_+^* : C\ell(M, E_-) \to C\ell(M, E_+)$ is an elliptic operator in $C\ell\left(M, (E_-)^* \otimes E_+\right)$ and $\Delta_+ := D_- D - +$, $\Delta_- := D_+ D_-$ are non negative (formally) self-adjoint elliptic operators.
The following theorem which combines formulae due to McKean and Singer [MS] and to Atiyah and Bott [**?**], expresses the index of D_+:

$$\mathrm{ind}(D^+) := \dim\left(\mathrm{Ker}(D_+)\right) - \dim\left(\mathrm{Ker}(D_-)\right)$$

in terms of the superweighted trace of the identity. On the grounds of (25.262) it then provides a local formula for the index in terms of the noncommutative residue.

Theorem 29 *The superresidue*

$$\mathrm{sres}\left(\log(\Delta + \pi_\Delta)\right) := \int_M \left(\mathrm{res}_x\left(\log \Delta_+ + \pi_{\Delta_+}\right)\right) - \mathrm{res}_x\left(\log \Delta_- + \pi_{\Delta_-}\right)\right)\right)\, dx,$$

where as usual $\mathrm{res}_x(B) = \int_{S_x^* M} \mathrm{tr}\left(\sigma_{-d}(B)(x,\xi)\right) đ\xi$ *is well defined and we have*

$$\mathrm{ind}(D_+) = \mathrm{str}^{\Delta + \pi_\Delta}(I) = \mathrm{str}\left(e^{-\epsilon\,\Delta}\right) = -\frac{1}{2d}\mathrm{sres}\left(\log(\Delta + \pi_\Delta)\right) \quad \forall \epsilon > 0,$$

where π_Δ, π_{Δ_+}, π_{Δ_-} *are the orthogonal projections onto the kernel of* Δ, Δ_+, Δ_- *and* d *is the order of* D*.*

Proof: We first observe that

$$\mathrm{Spec}(\Delta_+) - \{0\} = \mathrm{Spec}(\Delta_-) - \{0\}.$$

Indeed,

$$\Delta_+ u_+ = \lambda^+ u_+ \Rightarrow \Delta_-(D_- u_+) = \lambda^+ D_- u_+ \quad \forall u_+ \in C^\infty(M, E_+)$$

so that an eigenvalue λ^+ of Δ_+ with eigenvector u_+ is an eigenvalue of Δ_- with eigenvector $D_+ u_+$ provided the latter does not vanish. The converse holds similarly.
If we denote by $\{\lambda_n^+, n \in \mathbb{N}\}$ the discrete set of eigenvalues of Δ_+ and by $\{\lambda_n^-, n \in \mathbb{N}\}$ the discrete set of eigenvalues of Δ_- it follows that for any $\epsilon > 0$

$$\begin{aligned}
\mathrm{str}\left(e^{-\epsilon\,\Delta}\right) &= \sum_{n \in \mathbb{N}} e^{-\epsilon\,\lambda_n^+} - \sum_{n \in \mathbb{N}} e^{-\epsilon\,\lambda_n^-} \\
&= \sum_{\lambda_n^+ \neq 0} e^{-\epsilon\,\lambda_n^+} - \sum_{\lambda_n^- \neq 0} e^{-\epsilon\,\lambda_n^-} + \mathrm{ind}(D_+) \\
&= \mathrm{ind}(D_+).
\end{aligned}$$

Taking the limit as $\epsilon \to 0$ leads by Proposition 73 (which easily extends to supertraces) to $\mathrm{str}^\Delta(I) = \mathrm{str}^{\mathrm{HK},\Delta}(I) = \mathrm{ind}(D_+)$ since the noncommutative residue of the identity vanishes. Since the identity is a differential operator, setting $\mathrm{sTR}_x(I) := \mathrm{tr}(I_+) - \mathrm{TR}_x(I_-)$ where I_+ is the identity bundle map on E_+ and I_- is the identity bundle map on E_- and

$$\mathrm{sres}_x\left(\log \Delta\right) := \mathrm{res}_x\left(\log \Delta_+\right) - \mathrm{res}_x\left(\log \Delta_-\right),$$

we have by (25.262) extended to super traces:

$$\begin{aligned}
\mathrm{str}^{\Delta + \pi_\Delta}(I) &= \int_M \left(⨍_{T_x^* M} \mathrm{sTR}_x(I) - \frac{1}{q}\mathrm{sres}_x\left(\log(\Delta + \pi_\Delta)\right)\right) dx \\
&= -\frac{1}{q}\int_M \mathrm{sres}_x\left(\log(\Delta + \pi_\Delta)\right)\, dx \\
&= -\frac{1}{q}\,\mathrm{sres}\left(\log(\Delta + \pi_\Delta)\right),
\end{aligned}$$

which is therefore well defined. □

26.3 Geometric classical pseudodifferential operators

Let $E = S \otimes W$ be a twisted spinor bundle over an n-dimensional closed Riemannian manifold M with auxillary bundle W equipped with a connection ∇^W which we write $d + \omega$ in a local trivialisation. Following Gilkey's notations (see (1.8.17) in [**?**]), we introduce formal variables $g_{ij/\alpha} = \partial_\alpha g_{ij}$ for the partial derivatives of the metric tensor g on M and the connection ω on the external bundle. Let us set

$$\mathrm{ord}(g_{ij/\alpha}) = |\alpha|; \quad \mathrm{ord}(\omega_{i/\beta}) = |\beta|.$$

The following property for operators underlies Gilkey's proof of the Atiyah-Singer theorem [Gi].

Definition 47 *We call a log-polyhomogeneous operator $A \in C\ell^{*,*}(M, E)$ of order a* **geometric**, *if in any local trivialisation, the homogeneous components $\sigma_{a-j}(A)$ are homogeneous of order j in the jets of the metric and of the connection.*
In particular, a differential operator $A = \sum_{|\alpha| \leq a} c_\alpha(x)\, D_x^\alpha \in C\ell(M, E)$ is geometric if $c_\alpha(x)$ is homogeneous of order $j = a - |\alpha|$ in the jets of the metric and of the connection ∇^W.

Example 44 *The Laplace Beltrami operator*

$$\Delta_g = -\frac{1}{\sqrt{g}} \sum_{i=1,j=1}^{n} \partial_i \left(\sqrt{g}\, g^{ij} \partial_j\right)$$

has this property.

Example 45 *More generally, formula (2.4.22) in [Gi] shows that $\Delta_p = d_{p-1}\delta_{p-1} + \delta_p d_p$ on p-forms, where $\delta_k = (-1)^{nk+1} \star_{n-k} d_{n-k-1} \star_{k+1}$, is a geometric operator. Indeed, each derivative applied to $\star$ reduces the order of differentiation by 1 and increases the order in the jets of the metric by 1.*

Example 46 *Let $D_{\mathbb{A}} = \sum_{i=1}^n c(e_i)\, \mathbb{A}_{e_i}$ be the twisted Dirac operator acting on $C^\infty(M, S \otimes W)$, where we have set $\mathbb{A} := \nabla^S \otimes 1 + 1 \otimes \nabla^W$. Then*

$$D_{\mathbb{A}}^2 = -\sum_{ij} g^{ij} \left(\mathbb{A}_i\, \mathbb{A}_j + \sum_k \Gamma_{ij}^k\, \mathbb{A}_k \right) + \sum_{i<j} c(dx^i) c(dx^j) [\, \mathbb{A}_i,\, \mathbb{A}_j]$$

has this property, since locally we have $\mathbb{A}_i u = \partial_i u + \Gamma_{ij} u^j + \Theta_i(u)$ with Γ_{ij} the Christoffel symbols which are homogeneous of degree 1 in the jets of the metric and Θ_i corresponding to the auxillary connection.

Lemma 24 *The derivative $A'(0)$ at zero of a holomorphic germ $A(z) \in C\ell(M, E)$ of geometric operators is also geometric.*

Proof: This follows from $\partial_z \left(\sigma_{\alpha(z)-j}(A(z))\right) = (\sigma(A'(z))_{\alpha(z)-j}$. $\square$

Lemma 25 *The product of two geometric operators $A, B \in C\ell(M, E)$ is again a geometric operator.*

Proof: Since the product $A\,B$ has symbol

$$\sigma(A\,B) \sim \sum_\alpha \frac{(-i)^\alpha}{\alpha!} \partial_\xi^\alpha \sigma(A)\, \partial_x^\alpha \sigma(B),$$

we have

$$\sigma_{a+b-k}(AB) = \sum_{|\alpha|+i+j=k} \frac{(-i)^\alpha}{\alpha!} \partial_\xi^\alpha \sigma_{a-i}(A)\, \partial_x^\alpha \sigma_{b-j}(B)$$

where a is the order of A, b the order of B. Thus, if $\sigma_{a-i}(A)$ and $\sigma_{b-j}(B)$ are homogeneous of degree i and j respectively in the jets of the metric and the connection, $\sigma_{a+b-k}(AB)$ is homogeneous of degree $i + j + |\alpha| = k$. $\square$

The following theorem provides a way to build holomorphic germs of geometric operators.

Theorem 30 *Let $Q \in C\ell(M,E)$ be a geometric admissible (hence invertible elliptic) operator of positive order q with leading symbol $\sigma_q(Q)(x,\xi) = |\xi|^q$.*
Then for any geometric operator A in $C\ell(M,E)$, the family $A(z) := A\,Q^z$ is a holomorphic germ at $z = 0$ of geometric operators.
In particular, $A \log Q$ is geometric.

Proof: By Lemma 25, it is sufficient to prove the result for $A = I$. Furthermore, since $\log Q = (\partial_z Q^z)_{|_{z=0}}$, by the Lemma 24, it suffices to show that Q^z is geometric.
Since

$$Q^z = \frac{1}{2i\pi} \int_\Gamma \lambda^z \, (Q-\lambda)^{-1} \, d\lambda,$$

we need to investigate the resolvent $R(Q,\lambda) = (Q-\lambda)^{-1}$, the homogeneous components $\sigma_{q-j}(R(Q,\lambda))$ of the symbol of which are defined inductively on j by

$$\begin{aligned} \sigma_{-q}(R(Q,\lambda)) &= (\sigma_q(Q) - \lambda)^{-1}, \\ \sigma_{-q-j}(R(Q,\lambda)) &= -\sigma_{-s}(R(Q,\lambda)) \sum_{k+l+|\alpha|=j, l<j} \frac{(-i)^{|\alpha|}}{\alpha!} D_\xi^\alpha \sigma_{q-k}(Q) D_x^\alpha \sigma_{-q-l}(R(Q,\lambda)) \end{aligned} \quad (26.272)$$

To start with, let us compute the first terms $\sigma_{-q-j}(R(Q,\lambda))$ for $j = 0, 1, 2, 3$.

$$\sigma_{-q}(R(Q,\lambda)) = (|\xi|^q - \lambda)^{-1},$$

$$\begin{aligned} \sigma_{-q-1}(R(Q,\lambda)) &= -(|\xi|^q - \lambda)^{-1} \, [(\sigma_{q-1}(Q)\sigma_{-q}(R(Q,\lambda)) - iD_\xi\sigma_q(Q)D_x\sigma_{-q}(R(Q,\lambda))] \\ &= -(|\xi|^q - \lambda)^{-1} \, \sigma_{q-1}(Q) \, \sigma_{-q}(R(Q,\lambda)) \\ &= -(|\xi|^q - \lambda)^{-2} \, \sigma_{q-1}(Q), \end{aligned}$$

$$\begin{aligned} \sigma_{-q-2}(R(Q,\lambda)) &= -(|\xi|^q - \lambda)^{-1} \, [\sigma_{q-2}(Q)\sigma_{-q}(R(Q,\lambda)) + \sigma_{q-1}(Q)\sigma_{-q-1}(R(Q,\lambda)) \\ &\quad -iD_\xi\sigma_{q-1}(Q)D_x\sigma_{-q}(R(Q,\lambda)) - iD_\xi\sigma_q(Q)D_x\sigma_{-q-1}(R(Q,\lambda)) - iD_\xi^2\sigma_q(Q)D_x^2\sigma_{-q}(R(Q,\lambda))] \\ &= -(|\xi|^q - \lambda)^{-1} \, [\sigma_{q-2}(Q)\sigma_{-q}(R(Q,\lambda)) + \sigma_{q-1}(Q)\sigma_{-q-1}(R(Q,\lambda)) - iD_\xi\sigma_q(Q)D_x\sigma_{-q-1}(R(Q,\lambda))] \\ &= -(|\xi|^q - \lambda)^{-2} \, \sigma_{q-2}(Q) + (|\xi|^2 - \lambda)^{-3} \, (\sigma_{q-1}(Q))^2 + i(|\xi|^2 - \lambda)^{-3} \, D_\xi\sigma_a(Q)D_x\sigma_{q-1}(Q), \end{aligned}$$

$$\begin{aligned} \sigma_{-q-3}(R(Q,\lambda)) &= -(|\xi|^q - \lambda)^{-1} \, [\sigma_{q-2}(Q)\sigma_{-q-1}(R(Q,\lambda)) + \sigma_{q-1}(Q)\sigma_{-q-2}(R(Q,\lambda)) \\ &\quad -iD_\xi\sigma_{q-1}(Q)D_x\sigma_{-q-1}(R(Q,\lambda)) - iD_\xi\sigma_q(Q)D_x\sigma_{-q-2}(R(Q,\lambda)) - iD_\xi^2\sigma_q(Q)D_x^2\sigma_{-q-1}(R(Q,\lambda))] \\ &= (|\xi|^q - \lambda)^{-3} \, \sigma_{q-2}(Q)\sigma_{q-1}(Q) \\ &\quad +(|\xi|^q - \lambda)^{-3} \, \sigma_{q-1}(Q)\sigma_{q-2}(Q) - (|\xi|^q - \lambda)^{-4} \, (\sigma_{q-1}(Q))^3 \\ &\quad -i(|\xi|^q - \lambda)^{-4} \sigma_{q-1}(Q) \, D_\xi\sigma_a(Q) \, D_x\sigma_{q-1}(Q) + i(|\xi|^q - \lambda)^{-3} \, D_\xi\sigma_{q-1}(Q)D_x\sigma_{q-1}(Q) \\ &\quad -i(|\xi|^q - \lambda)^{-3} \, D_\xi\sigma_q(Q)D_x \, \sigma_{q-2}(Q) + i(|\xi|^2 - \lambda)^{-4} \, D_\xi\sigma_q(Q) \, D_x \, (\sigma_{q-1}(Q))^2 \\ &\quad -(|\xi|^q - \lambda)^{-4} \, (D_\xi\sigma_q(Q))^2 \, D_x^2\sigma_{q-1}(Q) - i(|\xi|^q - \lambda)^{-3} \, D_\xi^2\sigma_q(Q)D_x^2\sigma_{q-1}(Q). \end{aligned}$$

Using (26.272), one shows by induction on j that $\sigma_{-q-j}(R(Q,\lambda))$ is a finite sum of expressions of the type

$$(-i)^{|\alpha|} \, (|\xi|^q - \lambda)^{-1-k} \, D_\xi^{\alpha_1} D_x^{\beta_1} \sigma_{q-l_1}(Q) \cdots D_\xi^{\alpha_k} D_x^{\beta_k} \sigma_{q-l_k}(Q), \quad |l| + |\alpha| = j, \quad |\alpha| = |\beta|.$$

Inserting this in

$$\sigma_{q\,z-j}(Q^z)(x,\xi) = -\frac{1}{2i\pi} \int_\Gamma \lambda^z \, \sigma_{-q-j}(R(Q,\lambda))(x,\xi) \, d\lambda, \quad (26.273)$$

and applying repeated integrations by parts to compute the Cauchy integrals $-\frac{1}{2i\pi}\int_\Gamma \lambda^z\,(|\xi|^q-\lambda)^{-k-1}$:

$$\begin{aligned}
-\frac{1}{2i\pi}\int_\Gamma \lambda^z\,(|\xi|^q-\lambda)^{-k-1}\,d\lambda &= -\frac{1}{2i\pi}\frac{-z}{k}\int_\Gamma \lambda^{z-1}\,(|\xi|^q-\lambda)^{-k}\,d\lambda\\
&= -\frac{1}{2i\pi}\frac{-z(-z-1)\cdots(-z-(k-1))}{k(k-1)\cdots 1}\int_\Gamma \lambda^{z-k}\,(|\xi|^q-\lambda)^{-1}\,d\lambda\\
&= -\frac{(-1)^k\,z(z+1)\cdots(z+(k-1))}{k!}\frac{1}{2i\pi}\int_\Gamma \lambda^{z-k}\,(|\xi|^q-\lambda)^{-1}\,d\lambda\\
&= (-1)^{k-1}\,\frac{z(z+1)\cdots(z+(k-1))}{k!}\,|\xi|^{q(z-k)},
\end{aligned}$$

shows that $\sigma_{qz-j}(Q^z)(x,\xi)$ is a linear combination of symbols of the type

$$|\xi|^{q(z-k)}D_\xi^{\alpha_1}D_x^{\beta_1}\sigma_{q-l_1}(Q)(x,\xi)\cdots D_\xi^{\alpha_k}D_x^{\beta_k}\sigma_{q-l_k}(Q)(x,\xi),\quad |l|+|\alpha|=j,\quad |\alpha|=|\beta|.$$

Since $\sigma_{q-l}(Q)$ is homogeneous of order l in the jets of the metric and the connection, it follows that for any complex number z, the symbol $\sigma_{qz-j}(Q^z)$ is homogeneous of order j in the jets of the metric and the connection. □

26.4 The noncommutative residue density as a form valued invariant polynomial

Let us as in Section 1 consider a twisted spinor bundle $E=S\otimes W$ over an n-dimensional closed Riemannian manifold M with auxillary bundle W equipped with a connection ∇^W.
Adopting Gilkey's notations [Gi] par. 2.4, let us denote by $\mathcal{P}_{n,k,p}^{g,\nabla^W}$ with ∇^W the connection on the auxillary bundle W, which we write $\mathcal{P}_{n,k,p}^{g}$ if $E=S$, the linear space consisting of p- form valued invariant[32] polynomials that are homogeneous of order k in the jets of the metric[33] and of the connection ∇^W.

Example 47 *The scalar curvature r_M belongs to $\mathcal{P}_{n,2,0}^{g}$ since it reads:*

$$r_M=2\sum_{i,j}\left(\partial_{i,j}^2 g_{ij}-\partial_{i,i}^2 g_{jj}\right)$$

in a normalised coordinate system.

Theorem 31 *Let $A(z)\in C\ell(M,E)$ be a holomorphic germ at 0 of* **geometric** *operators with order $\alpha(z)$ such that $\alpha'(0)\neq 0$.*

1. *If $A(0)$ is a differential operator, the noncommutative residue density $\mathrm{res}_x(A'(0))\,dx$ lies in $\mathcal{P}_{n,\alpha(0)+n,n}^{g,\nabla^W}$. It is spanned by expressions of order n in the covariant derivatives of the curvature tensors.*

2. *If moreover $A(0)$ is a multiplication operator, then the residue density $\mathrm{res}_x(A'(0))\,dx$ is generated by Pontryjagin forms of the tangent bundle and by the Chern forms of W.*

Remark 49 1. *The assumption that $A(0)$ be a multiplication operator follows from combining the fact that it is a differential operator and that it has order 0.*

2. *If there is no dependence in ∇^W, then $\mathrm{res}_x(A'(0))\,dx$ is generated by Pontryjagin forms of the tangent bundle.*

Proof:

[32] By invariant we mean that they agree in any coordinate system around x_0 which is normalised w.r. to the point x_0, i.e. such that $g_{ij}(x_0)=\delta_{ij}$ and $\partial_k g_{ij}(x_0)=0$.
[33] The order in the jets of the metric is defined by $\mathrm{ord}(\partial_x^\alpha g_{ij})=|\alpha|$.

1. By Lemma 24, the derivative $A'(0)$ is a geometric operator. In particular, since $A'(0)$ is like $A(0)$, of order $\alpha(0)$, the homogeneous component $\sigma_{-n}(A'(0))$ is homogeneous of order $\alpha(0)+n$ in the jets of the metric and the connection. On the other hand, by Theorem **??**, if $A(0)$ is a differential operator, $\mathrm{res}_x\,(A'(0))_{-n}\,dx$ defines a global density so that it lies in $\mathcal{P}^{g,\nabla^W}_{n,\alpha(0)+n,n}$. By Weyl's description of a spanning set for $\mathcal{P}^{g,\nabla^W}_{n,\alpha(0)+n,n}$ (see [Gi] Lemma 2.4.4), we know that $\mathrm{res}_x\,(A'(0))_{-n}\,dx$ is spanned by expressions of order n in the covariant derivatives of the curvature tensors.

2. The second part of the statement then follows from Theorem 2.6.2 in [Gi] which says that when the degree of the form coincides with the degree of homogeneity, then only two jets of the metric and connection come into play (so only curvatures not their covariant derivatives), an observation which also played a role in Atiyah, Bott and Patodi's proof. More precisely, the direct sum $\oplus_p\mathcal{P}^{g,\nabla^W}_{n,p,p}$ is generated by Pontryjagin forms of the tangent bundle and by the Chern forms of W. If there is no dependence in ∇^W, then $\oplus_p\mathcal{P}^{g}_{n,p,p}$ is generated by Pontryjagin forms of the tangent bundle.

□

Combining Theorem 30 with Theorem 31 applied to $A(z)=A\,Q^z$, immediately leads to the following result.

Corollary 18 *Let $Q\in C\ell(M,E)$ be a* **geometric** *admissible (and hence invertible elliptic) classical pseudodifferential operator of positive order with leading symbol $\sigma_q(Q)(x,\xi)=|\xi|^q$.*

1. *For any* **geometric** *differential operator $A\in C\ell(M,E)$ of order a, the residue density $\mathrm{res}_x\,(A\,\log Q)_{-n}\,dx$ lies in $\mathcal{P}^{g,\nabla^W}_{n,a+n,n}$.*

2. *If moreover A is a multiplication operator then then the residue density $\mathrm{res}_x(A\,\log Q)\,dx$ is generated by Pontryjagin forms of the tangent bundle and by the Chern forms of W.*
   In particular, $\mathrm{res}_x\,(\log Q)_{-n}\,dx$ is generated by Pontryjagin forms of the tangent bundle and by the Chern forms of W.

Example 48 *With the notations of the first section, we have that $\mathrm{res}_x\left(\log(\Delta_g+\pi_{\Delta_g})\right)_{-n}dx$ lies in $\mathcal{P}^{g}_{n,n,n}$ and $\mathrm{res}_x\left(\log(D^2_{\not{A}}+\pi_{D^2_{\not{A}}})\right)_{-n}dx$ lies in $\mathcal{P}^{g,\nabla^W}_{n,n,n}$.*
*It follows from (**??**) that the index $ind(D_{\not{A}})$ is the integral over M of a form valued invariant polynomial in $\mathcal{P}^{g,\nabla^W}_{n,n,n}$.*

26.5 The Atiyah-Singer index theorem revisited

As in the previous sections, $E=S\otimes W$ is a twisted spinor bundle over an n-dimensional closed Riemannian manifold M with auxillary bundle W equipped with a connection ∇^W.
Let us state a few functorial properties of the residue density.
Let E_1 and E_2 be two vector bundles over the same manifold M, and $A_i(z)\in C\ell(M,E_i)$ two holomorphic germs at $z=0$ with holomorphic order $\alpha_i(z)$ such that $A_i(0)$ is a differential operator of order $a=\alpha_1(0)=\alpha_2(0)$ and $\alpha_i'(0)\neq 0$. Then $A_1(z)\otimes A_2(z)$ is a holomorphic germ at $z=0$ with holomorphic order $\alpha_1(z)\,\alpha_2(z)$ and we have $(A_1\otimes A_2)'(0)=A_1'(0)\otimes A_2'(0)$ so that

$$\sigma_{a-j}\left((A_1\otimes A_2)'(0)\right)=\sigma_{a-j}(A_1'(0))+\sigma_{a-j}(A_2'(0)).$$

It follows that for any point $x\in M$

$$\mathrm{res}_x\left((A_1\otimes A_2)'(0)\right)=\mathrm{res}_x(A_1'(0))+\mathrm{res}_x(A_2'(0)). \qquad (26.274)$$

If now $M=M_1\times M_2$ and $E=E_1\boxtimes E_2$, where E_i is a vector bundle over M_i and if $A_i(z)\in C\ell(M_i,E_i), i=1,2$ are two holomorphic germs at 0,with holomorphic order $\alpha_i(z)$ such that $A_i(0)$ is a differential operator of order $a=\alpha_1(0)=\alpha_2(0)$ and $\alpha_i'(0)\neq 0$. Then $A_1(z)\boxtimes A_2(z)$ is a holomorphic germ at 0 and

$$\sigma_{a-j}(\log((A_1\boxtimes A_2)'(0)))=\sum_{p+q=j}\sigma_{a-p}(A_1'(0))\,\sigma_{a-q}(A_2'(0)).$$

When $a=0$ it follows that

$$\sigma_{-n}\left(\log((A_1 \boxtimes A_2)'(0))\right) = \sum_{p+q=n} \sigma_{-p}(A_1'(0))\,\sigma_{-q}(A_2'(0)). \tag{26.275}$$

From Theorem 31 combined with the functorial properties of the residue densities, we infer the following statement.

Theorem 32 *Let $A(z) \in C\ell(M,E)$ be a holomorphic germ at 0 of* **geometric** *operators with order $\alpha(z)$ such that $A(0)$ is a multiplication operator and $\alpha'(0) \neq 0$.*
There exist Pontryjagin forms $\alpha_j \in \mathcal{P}_{2j}(O(\mathbb{R}^n))$ of form degree $4j$ and functions $C_{j,k}(x)$ such that

$$\mathrm{sres}_x(A'(0))\,dx = \sum_{4j+2k=n} C_{j,k}(x)\ \alpha_j(g)(x) \wedge \mathrm{str}_x\left(\mathrm{ch}_k(\nabla^W)(x)\right),$$

where str_x stands for the supertrace on the fibre E_x above $x \in M$.

Proof: We borrow arguments used by Gilkey [Gi] in his proof of the Atiyah-Singer index theorem. By Theorem 31 combined with the multiplicative property (26.275), there exist Pontryjagin forms $\alpha_j \in \mathcal{P}_{2j}(O(\mathbb{R}^n))$ of degree $4j$, Chern forms $\beta_j \in \mathcal{P}_{2j}(\mathrm{gl}(W))$ of degree $2k$ and functions $C_{p,q}(x)$ such that

$$\mathrm{sres}_x(A'(0))\,dx = \sum_{4p+2q=n} C_{p,q}(x)\ \alpha_p(g)(x) \wedge \mathrm{str}_x\left(\beta_q(\nabla^W)(x)\right).$$

The additivity property (26.274) then imposes $\beta_k(\nabla^W)$ to be proportional to $\mathrm{ch}_k(\nabla^W)$ since the Chern character of degree k is the only characteristic $2k$-form which is additive w.r. to sums. □

Applying this to a holomorphic germ $A(z) = A\,Q^z$ leads to the folllowing statement.

Corollary 19 *Let $Q \in C\ell(M,E)$ be a* **geometric** *admissible (and hence invertible elliptic) classical pseudodifferential operator of positive order with leading symbol $\sigma_q(Q)(x,\xi) = |\xi|^q$.*
For any multiplication operator $A \in C\ell(M,E)$, there exist Pontryjagin forms $\alpha_j \in \mathcal{P}_{2j}(O(\mathbb{R}^n))$ of form degree $4j$ and functions $C_{j,k}(x)$ such that

$$\mathrm{sres}_x(A\,\log Q)\,dx = \sum_{4j+2k=n} C_{j,k}(x)\ \alpha_j(g)(x) \wedge \mathrm{str}_x\left(\mathrm{ch}_k(\nabla^W)(x)\right).$$

Applying this to $Q := D^2 + \pi_{D^2}$ and replacing residues by super residues leads to the following corollary.

Corollary 20 *Given any superconnection ∇ and the corresponding twisted Dirac operator $D_{\mathbb{A}}$, there exist Pontryjagin forms $\alpha_j \in \mathcal{P}_{2j}(O(\mathbb{R}^n))$ of form degree $4j$ and functions $C_{j,k}(x)$ such that*

$$\mathrm{sres}_x(\log(D^2 + \pi_{D^2}))\,dx = \sum_{4j+2k=n} C_{j,k}(x)\ \alpha_j(g)(x) \wedge \mathrm{str}_x\left(\mathrm{ch}_k(\nabla^W)(x)\right),$$

and hence

$$\mathrm{ind}(D) = -\frac{1}{2} \sum_{4j+2k=n} \int_M C_{j,k}(x)\ \alpha_j(g)(x) \wedge \mathrm{str}_x\left(\mathrm{ch}_k(\nabla^W)(x)\right).$$

□

27 Multiplicative anomaly of regularised determinants

We describe two types of regularised determinants, weighted determinants and the ζ-determinant, which relate by a local formula. Weighted and ζ-determinants are not multiplicative but, as it is well-known since the work of Okikiolu on the one hand and Kontsevich and Vishik on the other hand, the corresponding multiplicative anomaly which measures the obstruction to the multiplicativity is local in a sense we make precise. This chapter is based on [OP].

27.1 Weighted and zeta determinants

An admissible operator $A \in C\ell(M,E)$ with spectral cut θ and positive order has well defined Q-weighted determinant [D] (see also [FrG]) where $Q \in C\ell(M,E)$ is a weight with spectral cut α:

$$\mathrm{Det}_\alpha^Q(A) := e^{\mathrm{Tr}_\alpha^Q(\log_\theta A)}.$$

Since the weighted traces restrict to the ordinary trace on trace-class operators, this determinant extends the ordinary determinant on operators in the determinant class.
The weighted determinant, as well as being dependent on the choice of spectral cut θ, also depends on the choice of spectral cut α.

Proposition 74 *Let $0 \leq \theta < \phi < 2\pi$ be two spectral cuts for the admissible operator A. If there is a cone $\Lambda_{\theta,\phi}$ (see 22.223) which does not intersect the spectrum of the leading symbol of A then*

$$\mathrm{Det}_\theta^Q(A) = \mathrm{Det}_\phi^Q(A).$$

Proof: Under the assumptions of the proposition,the cone $\Lambda_{\phi,\theta}$ defined as in Proposition 61, contains only a finite number of points in the spectrum of A so that $\log_\phi A - \log_\theta A = 2i\pi\Pi_{\theta,\phi}(A)$ is a finite rank operator and hence smoothing. Hence,

$$\begin{aligned}\frac{\mathrm{Det}_\phi^Q(A)}{\mathrm{Det}_\theta^Q(A)} &= e^{\mathrm{Tr}^Q\left(\log_\phi A - \log_\theta A\right)} = e^{\mathrm{Tr}^Q(2i\pi\,\Pi_{\theta,\phi}(A))}\\ &= e^{2i\pi\,\mathrm{tr}(\Pi_{\theta,\phi}(A))} = e^{2i\pi\,\mathrm{rk}(\Pi_{\theta,\phi}(A))}\\ &= 1,\end{aligned}$$

where rk stands for the rank. □

An admissible operator $A \in C\ell(M,E)$ with spectral cut θ and positive order has well defined ζ-determinant:

$$\mathrm{Det}_{\zeta,\theta}(A) := e^{-\zeta'_{A,\theta}(0)} = e^{\mathrm{tr}_\theta^A(\log_\theta A)}$$

since $\zeta_{A,\theta}(z) := \mathrm{TR}(A_\theta^{-z})$ is holomorphic at $z = 0$. In the second equality, the weighted trace has been extended to logarithms as before, picking out the constant term of the meromorphic map $z \mapsto \mathrm{TR}(\log_\theta A\, Q^{-z})$.
Recall from formula (25.264) that

$$\log \mathrm{Det}_{\zeta,\theta}(A) = \int_M dx\,\left[\mathrm{TR}_x(\log_\theta A) - \frac{1}{2a}\mathrm{res}_x(\log_\theta^2 A)\right] \tag{27.276}$$

where a is the order of A and where res_x is the noncommutative residue density extended to log-polyhomogeneous operators defined previously. This expression corresponds to minus the coefficient in z of the Laurent expansion of $\mathrm{TR}(A^{-z})$.

The ζ-determinant generally depends on the choice of spectral cut. However, it is invariant under mild changes of spectral cut in the following sense.

Proposition 75 *Let $0 \leq \theta < \phi < 2\pi$ be two spectral cuts for the admissible operator A. If there is a cone $\Lambda_{\theta,\phi}$ (see 22.223) which does not intersect the spectrum of the leading symbol of A then*

$$\mathrm{Det}_{\zeta,\theta}(A) = \mathrm{Det}_{\zeta,\phi}(A).$$

Proof: By (27.276), and since $\log_\phi A - \log_\theta A = 2i\pi\Pi_{\theta,\phi}(A)$ is a finite rank operator and hence smoothing under the assumptions of the proposition, we have

$$\begin{aligned}
\frac{\mathrm{Det}_{\zeta,\phi}(A)}{\mathrm{Det}_{\zeta,\theta}(A)} &= e^{\int_M dx\,\left[\mathrm{TR}_x(\log_\phi A) - \frac{1}{2a}\mathrm{res}_x(\log_\phi^2 A)\right] - \int_M dx\,\left[\mathrm{TR}_x(\log_\theta A) - \frac{1}{2a}\mathrm{res}_x(\log_\theta^2 A)\right]} \\
&= e^{\int_M dx\,\left[\mathrm{TR}_x(\log_\phi A - \log_\theta A) - \frac{1}{2a}\mathrm{res}_x(\log_\phi^2 A - \log_\theta^2 A)\right]} \\
&= e^{\int_M dx\,\left[\mathrm{TR}_x(2i\pi\,\Pi_{\theta,\phi}(A)) - \frac{1}{2a}\mathrm{res}_x\left((\log_\phi A + \log_\theta A)\,2i\pi\,\Pi_{\theta,\phi}(A)\right)\right]} \\
&= e^{2i\pi\,\mathrm{tr}(\Pi_{\theta,\phi}(A)) - \frac{2i\pi}{2a}\mathrm{res}\left((\log_\phi A + \log_\theta A)\,\Pi_{\theta,\phi}(A)\right)} \\
&= e^{2i\pi\,\mathrm{rk}(\Pi_{\theta,\phi}(A))} \\
&= 1,
\end{aligned}$$

where we have used the fact that the noncommutative residue vanishes on smoothing operators on which the canonical trace coincides with the usual trace on smoothing operators. □

27.2 Multiplicative anomaly of weighted determinants

Unlike ordinary matrix determinants, weighted determinants are not multiplicative. The multiplicative anomaly for Q-weighted determinants of two admissible operators A, B with spectral cuts θ, ϕ such that AB has spectral cut ψ is defined by:

$$\mathcal{M}^Q_{\theta,\phi,\psi}(A,B) := \frac{\mathrm{Det}^Q_\psi(AB)}{\mathrm{Det}^Q_\theta(A)\,\mathrm{Det}^Q_\phi(B)},$$

which we write $\mathcal{M}^Q(A,B)$ to simplify notations.

Proposition 76 *Let A and B be two admissible operators with spectral cuts θ and ϕ in $[0, 2\pi[$ such that there is a cone delimited by the rays L_θ and L_ϕ which does not intersect the spectra of the leading symbols of A, B and AB. Then the product AB is admissible with a spectral cut ψ inside that cone and for any weight Q with spectral cut, dropping the explicit mention of the spectral cuts we have:*

$$\log \mathcal{M}^Q(A,B) = \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau + b} - \frac{\log_\alpha Q}{q}\right)\right) d\tau. \tag{27.277}$$

Weighted determinants are multiplicative on commuting operators.

Proof: Since the leading symbol of the product AB has spectrum which does not intersect the cone delimited by L_θ and L_ϕ, the operator AB only has a finite number of eigenvalues inside that cone. We can therefore choose a ray ψ which avoids both the spectrum of the leading symbol of AB and the eigenvalues of AB in which case the weighted determinants $\mathrm{Det}^Q_\theta(A)$, $\mathrm{Det}^Q_\phi(B)$ and $\mathrm{Det}^Q_\psi(AB)$ do not depend on the choices of spectral cuts satisfying the requirements of the proposition.
Since

$$\log \mathcal{M}^Q(A,B) = \log \mathrm{Det}^Q(AB) - \log \mathrm{Det}^Q(A) - \log \mathrm{Det}^Q(B) = \mathrm{Tr}^Q(L(A,B)),$$

the logarithm of the multiplicative anomaly for weighted determinants is a local quantity (24.248) derived in Theorem 25.
To prove the second part of the statement we observe that

$$[A,B] = 0 \Longrightarrow L(A,B) = 0. \tag{27.278}$$

Indeed, let Γ be a contour as in formula (25.267) along a spectral ray around the spectrum of $A^{t_0}B$ for some fixed t_0, then

$$\begin{aligned}
\frac{d}{dt}_{|t=t_0}\log(A^tB) &= \frac{i}{2\pi}\int_\Gamma \log\lambda\,\frac{d}{dt}_{|t=t_0}(A^tB-\lambda)^{-1}\,d\lambda\\
&= \frac{i}{2\pi}\int_\Gamma \log\lambda\,(A^{t_0}B-\lambda)^{-1}\,\log A\,A^{t_0}B\,(A^{t_0}B-\lambda)^{-1}\,d\lambda\\
&= \log A\,A^{t_0}B\,\frac{i}{2\pi}\int_\Gamma \log\lambda\,(A^{t_0}B-\lambda)^{-2}\,d\lambda \quad \text{since} \quad [A,B]=0\\
&= -\log A\,A^{t_0}B\,\frac{i}{2\pi}\int_\Gamma \lambda^{-1}(A^{t_0}B-\lambda)^{-1}\,d\lambda \quad \text{by integration by parts}\\
&= -\log A\; A^{t_0}B\,(A^{t_0}B)^{-1}\\
&= -\log A.
\end{aligned}$$

Similarly, we have $\frac{d}{dt}_{|t=t_0}\log(A^t) = -\log A$ so that finally $\frac{d}{dt}_{|t=t_0}L(A^t,B) = \frac{d}{dt}_{|t=t_0}\log(A^tB) - \frac{d}{dt}_{|t=t_0}\log(A^t)$ vanishes. It follows that $L(A,B)=\int_0^1 \frac{d}{dt}_{|t=\tau}L(A^t,B)\,d\tau = 0$.
Since $L(A,B)$ vanishes when A and B commute, weighted determinants are multiplicative on commuting operators. □

27.3 The multiplicative anomaly of the zeta determinant

The ζ-determinant is not multiplicative [34]. Indeed, let A and B be two admissible operators with positive order and spectral cuts θ and ϕ and such that AB is also admissible with spectral cut ψ. The multiplicative anomaly

$$\mathcal{M}_\zeta^{\theta,\phi,\psi}(A,B) := \frac{\mathrm{Det}_{\zeta,\psi}(AB)}{\mathrm{Det}_{\zeta,\theta}(A)\,\mathrm{Det}_{\zeta,\phi}(B)},$$

was proved to be local, independently by Okikiolu [O2] for operators with scalar leading symbol and by Kontsevich and Vishik [KV] for operators "close to identity".
For simplicity, we drop the explicit mention of θ,ϕ,ψ and write $\mathcal{M}_\zeta(A,B)$.

By Proposition 65, the operators $L(A,B)\,\frac{\log A}{a} - K(A,B)$ and $L(A,B)\,\frac{\log B}{b} - K(A,B)$ are classical operators of zero order with $K(A,B) := \frac{1}{2(a+b)}\log^2_\psi A\,B - \frac{1}{2a}\log^2_\theta A - \frac{1}{2b}\log^2_\phi B$.

The following theorem provides a local formula for the multiplicative anomaly independently of Okikiolu's assumption that the leading symbols be scalar.

Theorem 33 *Let A and B be two admissible operators in $C\ell(M,E)$ with positive orders a,b and with spectral cuts θ and ϕ in $[0,2\pi[$ such that there is a cone delimited by the rays L_θ and L_ϕ which does not intersect the spectra of the leading symbols of A, B and AB. Then the product AB is admissible with a spectral cut ψ inside that cone and the multiplicative anomaly $\mathcal{M}_\zeta^{\theta,\phi,\psi}(A,B)$ is local as a noncommutative residue, independently of the choices of θ,ϕ, and ψ satisfying the above requirements. Explicitly, and dropping the explicit mention of the spectral cuts, there is a classical operator $W(\tau)(A,B)$ given by (24.247) of order zero depending continuously on τ such that:*

$$\begin{aligned}
&\log\mathcal{M}_\zeta(A,B)\\
&= \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b} - \frac{\log B}{b}\right)\right)d\tau\\
&+ \mathrm{res}\left(\frac{L(A,B)\,\log B}{b} - \frac{\log^2 A\,B}{2(a+b)} + \frac{\log^2 A}{2a} + \frac{\log^2 B}{2b}\right)\\
&= \int_0^1 \mathrm{res}\left(W(\tau)(A,B)\left(\frac{\log(A^\tau B)}{a\tau+b} - \frac{\log A}{a}\right)\right)d\tau\\
&+ \mathrm{res}\left(\frac{L(A,B)\,\log A}{a} - \frac{\log^2 A\,B}{2(a+b)} + \frac{\log^2 A}{2a} + \frac{\log^2 B}{2b}\right) \qquad (27.279)
\end{aligned}$$

[34] It was shown in [LP] that all multiplicative determinants on elliptic operators can be built from two basic types of determinants; they do not include the ζ-determinant.

When A and B commute the multiplicative anomaly reduces to:

$$\begin{aligned}\log \mathcal{M}_\zeta(A,B) &= -\mathrm{res}\left(\frac{1}{2(a+b)}\log^2(A\,B) - \frac{1}{2a}\log^2 A - \frac{1}{2b}\log^2 B\right)\\ &= \frac{ab}{2(a+b)}\mathrm{res}\left[\left(\frac{\log A}{a} - \frac{\log B}{b}\right)^2\right]. \qquad (27.280)\end{aligned}$$

Remark 50 For commuting operators, (27.280) gives back the results of Wodzicki as well as formula (III.3) in [D]:

$$\log \mathcal{M}_\zeta(A,B) = \frac{\mathrm{res}\left(\log^2(A^b B^{-a})\right)}{2ab(a+b)}.$$

Proof: As in the proof of the locality of the multiplicative anomaly for weighted determinants (see Proposition 76), the independence of the choice of spectral cuts satisfying the requirements of the theorem follows from Lemma **??**.
Combining equations (27.276), the defect formula (25.261) applied to the operator $L(A,B)$ and weight B with equation (24.248) applied to $Q = B$ we write:

$$\begin{aligned}&\log \mathcal{M}_\zeta(A,B)\\ =\ &\log \mathrm{Det}_\zeta(AB) - \log \mathrm{Det}_\zeta(A) - \log \mathrm{Det}_\zeta(B)\\ =\ &\int_M dx\ [\mathrm{TR}_x(L(A,B))\\ &-\left(\frac{1}{2(a+b)}\mathrm{res}_x(\log^2 AB) - \frac{1}{2a}\mathrm{res}_x(\log^2 A) - \frac{1}{2b}\mathrm{res}_x(\log^2 B)\right)\Big]\\ =\ &\mathrm{tr}^B(L(A,B)) + \int_M dx\ \left[\frac{1}{b}\mathrm{res}_x\left(L(A,B)\,\log B\right)\right.\\ &\left.-\left(\frac{1}{2(a+b)}\mathrm{res}_x\left(\log^2 A\,B\right) - \frac{1}{2a}\mathrm{res}_x(\log^2 A) - \frac{1}{2b}\mathrm{res}_x(\log^2 B)\right)\right] \qquad (27.281)\\ =\ &\int_0^1 \mathrm{res}\left(W(\tau)(A,B)\ \left(\frac{\log(A^\tau B)}{a\tau+b} - \frac{\log B}{b}\right)\right)d\tau\\ +\ &\mathrm{res}\left(\frac{L(A,B)\,\log B}{b} - \frac{\log^2 A\,B}{2(a+b)} + \frac{\log^2 A}{2a} + \frac{\log^2 B}{2b}\right),\end{aligned}$$

which proves the first equality in (27.279). The second one can be derived similarly exchanging the roles of A and B.
When A and B commute, by (27.278), the operator $L(A,B)$ vanishes so that (27.281) reduces to:

$$\begin{aligned}\log \mathcal{M}_\zeta(A,B) &= \mathrm{tr}^B(L(A,B)) + \int_M dx\ \left[\frac{1}{b}\mathrm{res}_x\left(L(A,B)\,\log B\right)\right.\\ &\left.-\left(\frac{1}{2(a+b)}\mathrm{res}_x\left(\log^2 A\,B\right) - \frac{1}{2a}\mathrm{res}_x(\log^2 A) - \frac{1}{2b}\mathrm{res}_x(\log^2 B)\right)\right]\\ &= -\mathrm{res}\left(\frac{\log^2 A\,B}{2(a+b)} - \frac{\log^2 A}{2a} - \frac{\log^2 B}{2b}\right)\\ &= \frac{ab}{2(a+b)}\mathrm{res}\left[\left(\frac{\log A}{a} - \frac{\log B}{b}\right)^2\right].\end{aligned}$$

□

28 Conformal anomaly of the ζ-determinant

Conformal anomalies arise naturally in quantum field theory. A conformally invariant classical action $\mathcal{A}(g)$ in a background metric g, for example the string theory (described previously by the classical action $\mathcal{A}(X, g)$) or nonlinear sigma model action, does not usually lead to a conformally invariant effective action $\mathcal{W}(g)$, since the quantization procedure breaks the conformal invariance and hence gives rise to a conformal anomaly. In particular, in string theory the conformal invariance persists after quantization only in specific critical dimensions. This chapter is based on [PayR2].

28.1 Conformally covariant operators

We view the Laplace-Beltrami operator Δ_g associated with a Riemannian metric g as an example of a more general class of conformally covariant operators.
Given a vector bundle E over a closed manifold M, le us consider maps

$$\begin{array}{rcl} \mathrm{Met}(M) & \to & Cl(M, E) \\ g & \mapsto & A_g, \end{array}$$

where $\mathrm{Met}(M)$ denotes the space of Riemannian metrics on M.

Definition 48 *The operator $A_g \in Cl(M, E)$ associated to a Riemannian metric g is* **conformally covariant** *of* bidegree (a, b) *if the pointwise scaling of the metric $\bar{g} = e^{2f} g$, for $f \in C^\infty(M, \mathbb{R})$ yields*

$$A_{\bar{g}} = e^{-bf} A_g e^{af} = e^{(a-b)f} A'_g, \quad \text{for } A'_g := e^{-af} A_g\, e^{af}, \tag{28.282}$$

for constants $a, b \in \mathbb{R}$.

We survey known conformally covariant differential and pseudodifferntial operators; more details are in Chang [Ch].
Operators of order 1. (Hitchin [Hi]) For M^n spin, the Dirac operator $D_g := \gamma^i \cdot \nabla_i^g$ is a conformally covariant operator of bidegree $\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$.

Operators of order 2. If $\dim(M) = 2$, the Laplace-Beltrami operator Δ_g is conformally covariant of bidegree $(0, 2)$. It is well known that in dimension two

$$R_{\bar{g}} = e^{-2f} \left(R_g + 2\Delta_g f\right), \tag{28.283}$$

where R_g is the scalar curvature, and by the Gauss-Bonnet theorem see e.g. [R])

$$\int_M R_g \,\mathrm{dvol}_g = 2\pi\chi(M), \tag{28.284}$$

with the Euler characteristic $\chi(M)$ (much more than) a conformal invariant.

On a Riemannian manifold of dimension n, the Yamabe operator, also called the conformal Laplacian,

$$L_g := \Delta_g + c_n\, R_g,$$

is a conformally covariant operator of bidegree $\left(\frac{n-2}{2}, \frac{n+2}{2}\right)$, where $c_n := \frac{n-2}{4(n-1)}$.
Operators of order 4. (Paneitz [Pan, BO]) In dimension n, the Paneitz operators

$$P_g^n := \tilde{P}_g^n + (n-4) Q_g^n$$

are conformally covariant scalar operators of bidegree $\left(\frac{n-4}{2}, \frac{n+4}{2}\right)$. Here $\tilde{P}_g^n := \Delta_g^2 + d^* \left((n-2)\, J_g\, g - 4A_g \cdot\right) d$ with

$$J_g := \frac{R_g}{2(n-1)}, A_g = \frac{Ric_g - \frac{R_g}{n}\, g}{n-2} + \frac{J_g}{n} g,$$

$A_g \cdot$ the homomorphism on T^*M given by $\phi = (\phi_i) \mapsto (A_g)_i^j\, \phi_j$, and $Q_g^n := \frac{n\, J_g^2 - 4|A_g|^2 + 2\Delta_g\, J_g}{4}$ is Branson's Q-curvature [B1], a local scalar invariant that is a polynomial in the coefficients of the metric tensor and its inverse, the scalar curvature and the Christoffel symbols. Note that $A_g = \frac{1}{n}J_g\, g$ precisely when g is Einstein.

The Q-curvature generalizes the scalar curvature R_g in the following sense. On a 4-manifold, we have

$$Q^4_{\bar{g}} = e^{-4f}\left(Q^4_g + \frac{1}{2}P^4_g f\right)$$

(cf. (28.283)), and $\int_M Q^4_g d\mathrm{vol}_g$ is a conformal invariant (cf. (28.284)), as is $\int_M Q^n_g d\mathrm{vol}_g$ in even dimensions [**?**].

Operators of order $2k$**.** (Graham, Jenne, Mason and Sparling [GJMS]) Fix $k \in \mathbb{Z}^+$ and assume either n is odd or $k \leq n$. There are conformally covariant (self-adjoint) scalar differential operators $P^n_{g,k}$ of bidegree $\left(\frac{n-2k}{2}, \frac{n+2k}{2}\right)$ such that the leading part of $P^n_{g,k}$ is Δ^k_g and such that $P^n_{g,k} = \Delta^k_g$ on $\mathbb{R}^n$ with the Euclidean metric.

$P^n_{g,k}$ generalizes P^n_g, since $P^n_g = P^n_{g,2}$, and satisfies

$$P^n_{g,k} = \tilde{P}^n_g + \frac{n-2k}{2}Q^n_g$$

where $\tilde{P}^n_g = d^* S^n_g d$ for a natural differential operator S^n_g on 1-forms.

Note that $P^n_{g,k}$ has bidegree (a, b) with $b - a = 2k$ independent of the dimension and in particular has bidegree $(0, 2k)$ in dimension $2k$.

Pseudodifferential Operators. (Branson and Gover [BG], Petersen [**?**]) Peterson has constructed ψ**dos** , $P^n_{g,k}, k \in \mathbb{C}$, of order $2\mathrm{Re}(k)$ and bidegree $((n-2k)/2, (n+2k)/2)$ on manifolds of dimension $n \geq 3$ with the property that $P^n_{\bar{g},k} - e^{-bf}P^n_{g,k}e^{af}$ is a smoothing operator. Thus any conformal covariant built from the total symbol of $P^n_{g,k}$ is a conformal covariant of $P^n_{g,k}$ itself. The family $P^n_{g,k}$ contains the previously discovered conformally covariant pseudodifferential operators associated to conformal boundary value problems [**?**].

28.2 Conformal anomalies

Let M be a closed Riemannian manifold and $\mathrm{Met}(M)$ denote the space of Riemannian metrics on M. $\mathrm{Met}(M)$ is trivially a Fréchet manifold as the open cone of positive definite symmetric (covariant) two-tensors inside the Fréchet space

$$C^\infty(T^*M \otimes_s T^*M) := \{h \in C^\infty(T^*M \otimes T^*M) : h_{ab} = h_{ba}\}$$

of all smooth symmetric two-tensors. The Weyl group $W(M) := \{e^f : f \in C^\infty(M)\}$ acts smoothly on $\mathrm{Met}(M)$ by Weyl transformations

$$W(g, f) = \bar{g} := e^{2f}g,$$

and given a reference metric $g \in \mathrm{Met}(M)$, a functional $\mathcal{F} : \mathrm{Met}(M) \to \mathbb{C}$ induces a map

$$\begin{aligned} \mathcal{F}_g = \mathcal{F} \circ W(g, \cdot) : C^\infty(M) &\to \mathbb{C}, \\ f &\mapsto \mathcal{F}(e^{2f}g). \end{aligned}$$

Definition 49 *A functional* $\mathcal{F}$ *on* $\mathrm{Met}(M)$ *is conformally invariant for a reference metric* g *if* $\mathcal{F}_g$ *is constant on a conformal class, i.e.*

$$\mathcal{F}(e^{2f}g) = \mathcal{F}(g) \quad \forall f \in C^\infty(M).$$

A functional $\mathcal{F}$ *on* $\mathrm{Met}(M)$ *is conformally invariant if it is conformally invariant for all reference metrics. A functional* $\mathcal{F} : \mathrm{Met}(M) \times M \to \mathbb{C}$ *is called a pointwise conformal covariant of weight* w *if*

$$\mathcal{F}(e^{2f}g, x) = w \cdot f(x)\mathcal{F}(g, x) \quad \forall f \in C^\infty(M), \quad \forall x \in M.$$

A functional $\mathcal{F} : \mathrm{Met}(M) \to \mathbb{C}$ which is Fréchet differentiable has a differential

$$d\mathcal{F}(g) : T_g\mathrm{Met(M)} = \mathrm{C^\infty(T^*M \otimes_s T^*M)} \to \mathbb{C},$$

$$d\mathcal{F}(g).h := \frac{d}{dt}\Big|_{t=0} \frac{\mathcal{F}(g+th) - \mathcal{F}(g)}{t}.$$

For such an $\mathcal{F}$, the differentiability of the Weyl map implies that the composition $\mathcal{F}_g : C^\infty(M) \to \mathbb{C}$ is differentiable at 0 with differential $d\mathcal{F}_g(0) : T_0C^\infty(M) = C^\infty(M) \to \mathbb{C}$.

Definition 50 *The conformal anomaly for the reference metric g of a differentiable functional $\mathcal{F}$ on $\mathrm{Met}(M)$ is $d\mathcal{F}_g(0)$. In physics notation, the conformal anomaly in the direction $f \in C^\infty(M)$ is*

$$\begin{aligned} \delta_f \mathcal{F}_g &:= d\mathcal{F}_g(0).f = d\mathcal{F}(g).2f\,g \\ &= \lim_{t=0} \frac{\mathcal{F}(g+2tfg) - \mathcal{F}(g)}{t} = \frac{d}{dt}\Big|_{t=0} \mathcal{F}(e^{2tf}g). \end{aligned}$$

Remark 51 *$\mathcal{F}$ is conformally invariant if and only if $d\mathcal{F}_g(0).f = 0$ for all $g \in \mathrm{Met}(M), f \in C^\infty(M)$.*

For a fixed Riemannian metric $g = (g_{ab})$, we equip $C^\infty(M)$ with the L^2 metric

$$(f, \tilde{f})_g = \int_M f(x)\tilde{f}(x)\mathrm{dvol}_g(x).$$

We define an L^2 metric on $\mathrm{Met}(M)$ by

$$\langle h, k\rangle_g := \int_M g^{ac}(x)g^{bd}(x)\, h_{ab}(x)\, k_{cd}(x)\, \mathrm{dvol}_g(x) = \int_M h^{cd}(x)\, k_{cd}(x)\, \mathrm{dvol}_g(x) \tag{28.285}$$

with $(g^{ab}) = (g_{ab})^{-1}$ and $h^{ab}(x) := g^{ac}(x)g^{bd}(x)\, h_{cd}(x)$. The L^2 metric induces a weak L^2-topology on $\mathrm{Met}(M)$, and $L^2(T^*M \otimes_s T^*M)$, the L^2-closure of $C^\infty(T^*M \otimes_s T^*M)$ with respect to $\langle\ ,\ \rangle_g$, is independent of the choice of g up to Hilbert space isomorphism. The choice of a reference metric yields the inner product (28.285) on the tangent space $T_g\mathrm{Met}(M) = C^\infty(T^*M \otimes_s T^*M)$, giving the weak L^2 Riemannian metric on $\mathrm{Met}(M)$, and forming the completion of each tangent space.

The various inner products are related as follows:

Lemma 26 *For $g \in \mathrm{Met}(M), h \in C^\infty(T^*M \otimes_s T^*M)$ and $f \in C^\infty(M)$, we have*

$$\langle h, f\, g\rangle_g = (\mathrm{tr}_g(h), f)_g$$

where we have set: $\mathrm{tr}_g(h) := h^b_b = g^{ab}h_{ab}$.

Proof: We have

$$\begin{aligned} \langle h, f\, g\rangle_g &= \int_M f(x)\, g^{ac}(x)g^{bd}(x)\, h_{ab}(x)\, g_{cd}(x)\, \mathrm{dvol}_g(x) \\ &= \int_M f(x)\, g^{ab}(x)\, h_{ab}(x)\, \mathrm{dvol}_g(x) \\ &= (\mathrm{tr}_g(h), f)_g\,. \end{aligned}$$ □

□

Definition 51 *If the differential $d\mathcal{F}(g) : C^\infty(T^*M \otimes_s T^*M) \to \mathbb{C}$ extends to a continuous functional $\overline{d\mathcal{F}(g)} : L^2(T^*M \otimes_s T^*M) \to \mathbb{C}$, then by Riesz's lemma there is a unique two-tensor $T_g(\mathcal{F})$ with*

$$\overline{d\mathcal{F}(g)}.h = \langle h, T_g(\mathcal{F})\rangle_g, \quad \forall h \in L^2(T^*M \otimes_s T^*M).$$

$T_g(\mathcal{F})$ is precisely the L^2 gradient of $\mathcal{F}$ at g.

Proposition 77 *Let $\mathcal{F}$ be a functional on* $\mathrm{Met}(M)$ *which is differentiable at the metric g and whose differential $d\mathcal{F}(g)$ extends to a continuous functional $\overline{d\mathcal{F}(g)} : L^2(T^*M \otimes_s T^*M) \to \mathbb{C}$. Then the differential $d\mathcal{F}_g(0)$ also extends to a continuous functional $\overline{d\mathcal{F}_g(0)} : L^2(M) \to \mathbb{C}$. Identifying the conformal anomaly at g with a function in $L^2(M)$, we have*

$$\overline{d\mathcal{F}_g(0)} = 2\,\mathrm{tr}_g\left(T_g(\mathcal{F})\right).$$

In particular, the functional $\mathcal{F}$ is conformally invariant iff $\mathrm{tr}_g\left(T_g(\mathcal{F})\right) = 0$ for all metrics g.

Proof: The differential $d(\mathcal{F}_g)_0$ extends to a continuous functional because

$$d\mathcal{F}_g(0).f = d\mathcal{F}(g)(2f\ g) \Rightarrow \overline{d\mathcal{F}_g(0)}.f = \overline{d\mathcal{F}(g)}.2f\ g$$

By Lemma 26,

$$\overline{d\mathcal{F}_g(0)}.f = \overline{d\mathcal{F}(g)}.2f\ g = \langle T_g(\mathcal{F}), 2f\,g\rangle_g = 2\left(\mathrm{tr}_g(T_g(\mathcal{F})), f\right)_g,$$

as desired. $\square$

Definition 52 *Under the assumptions of the Proposition, the function*

$$x \mapsto \delta_x \mathcal{F}_g := 2\mathrm{tr}_g\left(T_g(\mathcal{F})\right)(x)$$

is called the local anomaly of the functional $\mathcal{F}$ at the reference metric g.

Example 49 *This example is taken from bosonic string theory. Let (M, g) be a closed Riemannian surface and $X : M \to \mathbb{R}^d$ a smooth map. Let*

$$\Delta_g := -\frac{1}{\sqrt{\mathrm{det}g}}\partial_i\sqrt{\mathrm{det}g}g_{ij}\partial_j$$

(where as before $\mathrm{det}g$ *stands for the determinant of the metric matrix (g_{ij}) and (g^{ij}) its inverse) denote the Laplace-Beltrami operator on M. The classical Polyakov action [Po] (see also [AJPS] and references therein for a review) for bosonic string*

$$\mathcal{A}(X, g) := \langle \Delta_g X, X\rangle_g$$

yields a conformal invariant (depending on X) since

$$\begin{aligned} \mathcal{A}(X, e^{2f}\,g) &:= \langle \Delta_{e^{2f}g} X, X\rangle_{e^{2f}g} \\ &= \langle e^{-2f}\Delta_g X, X\rangle_{e^{2f}g} \\ &= \langle \Delta_g X, X\rangle_g. \end{aligned}$$

For any $h \in C^\infty(M, T^*M \otimes_s T^*M)$, since $d\sqrt{\mathrm{det}g}.h = \frac{1}{2}\sqrt{\mathrm{det}g}\,g^{ij}h_{ij} = \frac{1}{2}\sqrt{\mathrm{det}g}\,\mathrm{tr}^g(h)$ and since $d\,g^{-1}.h = -g^{-1}\,h\,g^{-1}$ we have

$$\begin{aligned} d\mathcal{A}(X,\cdot)(g)\cdot h &= d\left(\int_M X^\mu\,\partial_i\sqrt{\mathrm{det}g}g^{ij}\partial_j X^\mu\right).h \\ &= \frac{1}{2}\left(\int_M X^\mu\,\partial_i\sqrt{\mathrm{det}g}\,\mathrm{tr}_g(h)g^{ij}\partial_j X^\mu\right) - \int_M \partial_i\sqrt{\mathrm{det}g}\,g^{ij}h_{kl}g^{kl}\partial_j X^\mu\,X^\mu \\ &= -\frac{1}{2}\int_M \sqrt{\mathrm{det}g}\,\mathrm{tr}_g(h)g^{ij}\partial_i X^\mu\partial_j X^\mu + \int_M \sqrt{\mathrm{det}g}\,g^{ij}h_{kl}g^{kl}\,\partial_i X^\mu\,\partial_j X^\mu \\ &= -\frac{1}{2}\int_M \sqrt{\mathrm{det}g}\,\mathrm{tr}_g(h)\mathrm{tr}_g(\partial_i X^\mu\partial_j X^\mu) - \int_M \sqrt{\mathrm{det}g}\,h^{ij}\,\partial_i X^\mu\,\partial_j X^\mu \\ &= -\frac{1}{2}\langle \mathrm{tr}_g(\partial_i X^\mu\partial_j X^\mu), \mathrm{tr}_g(h)\rangle_g + \langle h^{ij}, \partial_i X^\mu\,\partial_j X^\mu\rangle_g \\ &= -\langle\frac{1}{2}\mathrm{tr}_g(\partial_i X^\mu\partial_j X^\mu)\,g + \partial_i X^\mu\,\partial_j X^\mu, h\rangle_g \\ &= \langle h, T_g(x)\rangle_g \end{aligned}$$

so that with the above notations we have $\overline{d\mathcal{A}(X,\cdot)(g)} = T_g$ where T_g is the two covariant tensor $T_{ij} := \partial_i X^\mu\,\partial_j X^\mu - \frac{1}{2}\mathrm{tr}_g(\partial_i X^\mu\partial_j X^\mu)\,g$ called the energy-momentum tensor. Note that $\mathrm{tr}_g(T_g) = 0$ as expected since $\mathcal{A}(X,\cdot)$ is conformally invariant.

28.3 Conformal anomaly

From a path integral point of view, the conformal anomaly of the quantized action is often said to arise from a lack of conformal invariance of the formal measure on the configuration space of the QFT. Whatever this means, we can detect the source of the conformal anomaly in the quantization procedure. In order to formally reduce the path integral to a Gaussian integral, one writes the classical action as a quadratic expression $\mathcal{A}(g)(\phi) = \langle A_g\phi, \phi\rangle_g$ where ϕ is a field, typically a tensor on M, A_g a differential operator on tensors and $\langle\cdot,\cdot\rangle_g$ the inner product induced by g. Because this inner product is not conformally invariant, the conformal invariance of $\mathcal{A}(g)$ usually translates to a conformal covariance of the operator A_g. Thus this first step, which turns a conformally invariant quantity (the classical action) to a conformally covariant operator, already breaks the conformal invariance.
The second step in the computation of the path integral uses an Ansatz to give a meaning to the formal determinants that arise from the Gaussian integration. Mimicking finite dimensional computations, the effective action derived from a formal integration over the configuration space $\mathcal{C}$ is

$$e^{-\frac{1}{2}\mathcal{W}(g)} := \int_{\mathcal{C}} e^{-\frac{1}{2}\mathcal{A}(g)(\phi)}\, \mathcal{D}\phi = \text{“det”}\,(A_g)^{-\frac{1}{2}}.$$

If there were a well defined determinant “det” on differential operators with the usual properties, (**??**) would yield for a conformally covariant operator of bidegree (a, b)

$$\begin{aligned}\text{“det”}\,(A_{e^{2f}g}) &= \text{“det”}\,(e^{-b\,f}A_g e^{a\,f}) \\ &= \text{“det”}\,(e^{-b\,f})\,\text{“det”}\,(A_g)\,\text{“det”}\,(e^{a\,f}) \\ &= \text{“det”}\,(e^{(a-b)\,f})\,\text{“det”}\,(A_g),\end{aligned}$$

where $e^{c\,f}$ is treated as a multiplication operator for $c \in \mathbb{R}$. Hence, even if a “good” determinant exists, the effective action $\mathcal{W}(g)$ would still suffer a conformal anomaly, since A_g is only conformally covariant:

$$\delta_f\mathcal{W}(g) = \delta_f \log\text{“det”}\,(A_g) = \delta_f\log\text{“det”}\,(e^{(a-b)\,f}) = (a-b)\,\text{“tr”}\,(f),$$

where “tr” is a hypothetical trace associated to “det”.

The ζ-determinant Det_ζ on operators is used by both physicists and mathematicans as an Ersatz for the usual determinant on matrices. The following well-known result shows that the above heuristic derivation holds replacing the trace tr on matrices by a weighted trace tr^{A_g}.

Theorem 34 *[BO], [**?**], [**?**] Let $A_g \in C\ell(M, E)$ be a conformally covariant admisible elliptic operator with positive order α independent of g. Then the conformal anomaly of the ζ-determinant of A_g is a local expression given by:*

$$\delta_f\ \log\mathrm{Det}_\zeta(A_g) = (a-b)\,\mathrm{tr}^{A_g}(f) = -\frac{1}{\alpha}\mathrm{res}\,(f\ \log A_g)\,.$$

Remark 52 *The regularisation procedures involved in the ζ-determinant and the finite part of the heat-operator expansion are* not *responsible for the conformal anomaly of the effective action $\mathcal{W}(g)$; the conformal anomaly appears as soon as one uses the conformally covariant operator A_g associated to the originally conformally invariant action $\mathcal{A}(g)$. This conformal anomaly therefore has nothing to do with the multiplicative anomaly investigated in the previous section.*

Proof: We use Proposition **??** combined with the fact that $\delta_f A_g := (a-b)f\,A_g$ to write

$$\begin{aligned}\delta_f \log\,\mathrm{Det}_{\zeta,\theta}(A_g) &= \left(\frac{d}{dt}\log\,\mathrm{Det}_{\zeta,\theta}(A_{e^{2tf}g})\right)_{|t=0} \\ &= \mathrm{tr}^{A_g}\left(A_g^{-1}\frac{d}{dt}A_{e^{2tf}g}\right)_{|t=0} \\ &= (a-b)\,\mathrm{tr}^{A_g}(f).\end{aligned}$$

Since the multiplication operator by f is a differential operator, we can write $\mathrm{tr}^{A_g}(f) = -\frac{1}{\alpha}\mathrm{res}\,(\log A_g)\,.$
□

Remark 53 *In [**?**], we use the Kontsevich-Vishik canonical trace to produce a series of conformal spectral invariants (or covariants or anomalies) associated to conformally covariant pseudodifferential operators. Although only one covariant is new, the use of canonical traces provides a systematic treatment of these covariants.*

Example 50 *One can show that for the conformal anomaly of the* ζ*-determinant of the conformally covariant operator* Δ_g *on a Riemann surface* (M, g) *reads:*

$$\delta_f(\mathrm{Det}_\zeta(\Delta_g)) = \frac{1}{24\pi}\left(\langle f, \Delta_g f\rangle_g + 2\langle R_g, f\rangle_g\right)$$

where as before R_g *is the scalar curvature. It contributes to the conformal anomaly of the partition function for bosonic strings [AJPS].*

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