A prologomon to renormalisation; analytic aspects

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These lecture present an analytic approach to some of the regularisation (renormalisation) techniques used in quantum field theory, in number theory and in geometry.

1 Two basic tools: regularised evaluators and pseudodifferential symbols

"Evaluating" at zero, meromorphic functions around zero with poles at that point requires a regularisation procedure. The minimal substraction scheme is one way to extend ordinary evaluators at zero to **regularised evaluators** on an algebra of meromorphic functions around zero. On meromorphic functions with simple pole at zero, all other linear extensions of the ordinary evaluation at zero on holomorphic functions, differ from the minimal substraction scheme by a multiple of the residue at zero.
Evaluating meromorphic functions in several variables around zero is not so straightforward, and provides a first glimpse into the difficulties of renormalisation. We use renormalised evaluators, namely linear maps on algebras of meromorphic functions in several variables with linear hyperplanes of poles, that extend the ordinary evaluation a t zero on holomorphic functions and are compatible with the tensor product.

Both Feynman integrals and multiple zeta functions seen as multiple integrals and discrete sums with constraints, involve **pseudodifferential symbols** in their integrands. It turns out that pseudodifferential symbols provide an appropriate analytic tool to interpret regularisation techniques in quantum field theory and number theory.

This first lecture is dedicated to defining regularised evaluators and pseudodifferential symbols.

2 The canonical integral and canonical discrete sum

A first step towards making sense of Feynman integrals is to extend the ordinary integral on L^1-functions to a linear form defined on a larger class of functions, while preserving the desired properties such as translation invariance, covariance and Stokes' type properties. This issue is very much related to extending the discrete sum on $\mathbb{Z}^d$ from L^1 functions on $\mathbb{R}^d$ to a larger class of functions while preserving $\mathbb{Z}^d$-translation invariance.
There are natural and canonical extensions with these properties, which we call the canonical integral or discrete sum, beyond L^1-functions to **non integer order** symbols.

Embedding an integer order symbol in a holomorphic family of symbols, and then implemeting the canonical integral, resp. discrete sum, one builds meromorphic families of integrals, resp. discrete sums of integer order symbols. Regularised evaluators can then be applied to these meromorhpic functions in order to build regularised integrals and discrete sums, which are linear extensions of the ordinary integral and discrete sum. This holomorphic approach provides an interpretations of **dimensional, Riesz and zeta regularisation**.
This lecture is dedicated to the construction of these regularised integrals and discrete sums with examples in physics and number theory.

3 Discrepancies/anomalies in terms of the noncommutative residue

The linear extensions of the ordinary integral and discrete sum on L^1-symbols provided by regularised integrals and discrete sums built in the previous section, do not fulfill the above mentioned properties, namely translation invariance, covariance and Stokes' type properties. These **obstructions/ discrepancies**, which are very much related to anomalies in quantum field theory, can be measured in terms of the noncommutative reisdue.
The **noncommutative residue** is a linear form on the algebra of pseudodifferential symbols with relevant properties such as translation invariance, covariance and Stokes' type properties. As a matter of fact, these properties characterise the noncomutative residue. However, its great drawback is that it vanishes on L^1-symbols and therefore cannot offer a linear extension of the ordinary integral or discrete sum on L^1-symbols. Its asset is its **locality** which we can be read off its very definition and is responsible for the locality of some anomalies in physics.
This lecture is dedicated to the statement and proof of the theorems leading to the local formulae for these discrepancies.

4 Regularised traces; the index as a noncommutative residue

Similar features can be observed on the level of pseudodifferential operators. The ordinary L^2-trace on smoothing operators does not extend to a linear form on the algebra of classical pseudodifferential operators on a closed manifold since linear forms on that algebra (we assume the underlying manifold has dimension > 1) which vanish on brackets, are proportional to the noncommutative residue on operators, obtained as an integral over the manifold of the noncommutative residue on symbols.
But the L^2-trace does have a unique linear extension to the set of **non integer order operators**, which vanishes on non integer order brackets, namely the canonical trace popularised by Kontsevich and Vishik, which can be described in terms of the canonical integral introduced in the first lecture
Regularised traces, which are linear extensions of the L^2-trace to the whole algebra of classical operators are therefore non cyclic. The obstruction to cyclicity and other discrepancies can be measured in terms of the noncommutative residue.
In contrast to the noncommutative residue, regularised traces are not in general since they involve the whole symbol of the operator; a **defect formula** singles out their non locality. In particular, this defect formula shows that the index of a chiral Dirac operator on a closed manifold, seen as a regularised supertrace of the identity operator, is a noncommutative residue and hence local.
This lecture is dedicated to the extension to operators of results derived for symbols in the previous sections, as well as their application to index formulae.

5 Renormalised multiple integrals and discrete sums with constraints

Multiple discrete sums with conical constraints, resp. multiple integrals integrals with affine constraints – both of which have integrands given by tensor products of classical symbols– generalise multiple zeta functions, resp. Feynman integrals.
On the grounds of methods developped by physicists, such sums and integrals (assuming radiality of the symbol in the case of integrals) are shown to be meromorphic in several variables with poles on affine hyperplanes. Interestingly, the hyperplanes of poles passing through zero are the same for both sums with conical constraints and integrals with affine constraints.
On the grounds of this meromorphicity result, we can implement renormalised evaluators at zero described in the first lecture, leading to renormalised multiple integrals with affine constraints, resp. multiple discrete sums with conical constraints, both of which which factorise over disjoint sets of constraints.
This lecture is dedicated to extending regularisation techniques to multiple integrals and discrete sums, for which we use renormalised evaluators introduced in the first lecture.

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